Stable points of a polyhedron

by

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DEFINITION 1. A point a of a space M is *labile* (see [3], p. 159, definition of a *homotopically labile* point) if for each neighbourhood U of a there exists a continuous function $h: M\times\langle 0,1\rangle\to M$ (a homotopy) which satisfies the following four conditions:

(1) $h(x,0)=x$ for every $x\in M$;

(2) $h(x,t)=x$ for every $x\notin U$, $t\in\langle 0,1\rangle$;

(3) $h(x,t)\in U$ for every $x\in U$, $t\in\langle 0,1\rangle$;

(4) $h(x,1)\neq a$ for every $x\in M$.

A point a is *stable* if it is not labile.

The term "lability" is used in paper [4] in an other sense.

DEFINITION 2. A point $a\in M$ is *almost-labile* (see [4], definition of a *labile* point) if it satisfies Definition 1, condition (4) being replaced by

(4′) there exists an $a'\in U$ such that $h(x,1)\neq a'$ for every $x\in M$.

It is obvious that if a is a labile point, then it is almost-labile, but not conversely; the set of all almost-labile points in M is closed; on the other hand, the set of all labile points in M is not necessarily closed.

Let P be a polyhedron, T its triangulation. The simplex $\Delta\in T$ is said to be a *free face* of P if there exists exactly one simplex Δ' in T such that Δ is a proper face of Δ'. It is easy to see that the property "to have a free face" does not depend on the triangulation; it is also clear that if a polyhedron has a free face, then it contains a labile point. Moreover, if the dimension of the free face is >0, then all the points lying in the interior of that free face are labile.

H. Hopf and E. Pannwitz showed (see [4], p. 446) that for every integer $n\geqslant 2$ there exists a homogeneously n-dimensional polyhedron which has no free faces but contains two almost-labile points (certainly, for $n<2$ it is impossible). However, by Theorem 1 (see § 1 of this paper) it follows that the examples given by Hopf and Pannwitz contain no labile points.

In § 1 of this paper it is shown that for every integer $n\geqslant 3$ there exists a homogeneously n-dimensional polyhedron which has no free faces but contains (at least) two labile points, and that for $n < 3$ such a situation is impossible.

In paper [3] K. Borsuk and J. W. Jaworowski raised the following question: "Is the stability of points invariant under Cartesian multiplication?" In § 2 a negative answer to this question is given; namely two polyhedra A and B are constructed such that the Cartesian product $A \times B$ contains a labile point the coordinates of which are both stable in the corresponding factors; in that example B is a segment.

§ 1. An example of a homogeneously n-dimensional polyhedron ($n \geqslant 3$) without free faces but with labile points.

Let $\mathrm{C}X$ be the *cone* over a space X, i.e. the factor-space $X \times \langle 0, 1 \rangle / X \times \{0\}$. Each point $[(x, t)] \in \mathrm{C}X$, where $t \neq 0$ may be identified with the point $(x, t) \in X \times (0, 1\rangle$, whence we can write $X \times (0, 1\rangle \subset \mathrm{C}X$.

The point $c = [X \times \{0\}] = [(x, 0)]$ is called the *vertex of the cone*, the set $X \times \{1\}$ is said to be the *basis of the cone*.

The space X is called *contractible* if there exists a retraction of the cone over X onto its basis.

LEMMA 1. *The vertex of the cone $\mathrm{C}X$ over a compact space X is labile if and only if X is contractible.*

Proof. 1. Let the vertex $c \in \mathrm{C}X$ be labile. Since the set $U = \mathrm{C}X - X \times \{1\}$ is a neighbourhood of c, there exists a homotopy $h \colon \mathrm{C}X \times \langle 0, 1 \rangle \to \mathrm{C}X$ satisfying conditions (1)-(4) (Definition 1) with respect to c. Let us write $f(z) = h(z, 1)$ for each $z \in \mathrm{C}X$. The function f has two properties:

(*) $f(z) \neq c$ for every $z \in \mathrm{C}X$ (this follows from (4));

(**) $f(z) = z$ for every $z \in X \times \{1\}$ (this follows from (2)).

By (*) we can write $f \colon \mathrm{C}X \to \mathrm{C}X - \{c\}$. Now, let $\pi \colon \mathrm{C}X - \{c\} \to X \times \{1\}$ be the projection given by $\pi(x, t) = (x, 1)$. By (**) we infer that the composition $\pi \circ f \colon \mathrm{C}X \to X \times \{1\}$ is a retraction; consequently X is contractible.

2. Now, let X be contractible; this means that there exists a retraction $r \colon \mathrm{C}X \to X \times \{1\}$. Let U be a neighbourhood of the vertex c of $\mathrm{C}X$.

Let U_ε (for every positive real number $\varepsilon \leqslant 1$) denote the set $\{[(x, t)] \in \mathrm{C}X \colon t \leqslant \varepsilon\}$ and let $\varphi_\varepsilon \colon \mathrm{C}X \to U_\varepsilon$ be the homeomorphism given by the formula $\varphi_\varepsilon[(x, s)] = [(x, \varepsilon \cdot s)]$. Furthermore, let U_0 be the empty set. Since the space X is compact, there exists a $\lambda > 0$ such that $U_\lambda \subset U$. This implies that the homotopy $h \colon \mathrm{C}X \times \langle 0, 1 \rangle \to \mathrm{C}X$ given by the formula

$$h(z, t) = \begin{cases} z & \text{for every} \quad z \notin U_{\lambda \cdot t}, \\ \varphi_{\lambda \cdot t} \circ r \circ \varphi_{\lambda \cdot t}^{-1}(z) & \text{for every} \quad z \in U_{\lambda \cdot t} \end{cases}$$

satisfies conditions (1)-(4) with respect to the vertex c; consequently it is labile.

Remark. The lability (and almost-lability) is a local property, i.e. if U is a neighbourhood of a point $x \in X$, V is a neighbourhood of $y \in Y$ and there exists a homeomorphism of U onto V which sends x onto y, then the lability (or almost-lability) of x in X implies the lability (or almost-lability) of y in Y. Hence if a point $x \in X$ has a neighbourhood U which is (homeomorphic to) a cone with the vertex x over a compact set C, then x is labile if and only if C is contractible. Now, let us suppose that P is a polyhedron, and x an (arbitrary) point of it. Let us choose a triangulation T of P such that x is a vertex. Let S_x be the star of the vertex x, i.e. the union of all simplexes $\Delta \in T$ such that $x \in \Delta$; let $S_x^\bullet$ be the boundary of S_x, i.e. the union of all simplexes $\Delta \in T$ such that $\Delta \subset S_x$ and $x \notin \Delta$. Let us remark that S_x is a neighbourhood of x and that S_x is a cone with the vertex x over $S_x^\bullet$.

The above remark and Lemma 1 imply:

THEOREM 1. *A vertex x of a triangulation T of a polyhedron P is labile if and only if the boundary of the star of x is contractible.*

Let $\mathrm{S}X$ be the *suspension* of a space X, i.e. the factor-space $(X \times \langle -1, 1 \rangle / X \times \{-1\}) / X \times \{1\}$. The space $\mathrm{S}X$ is homeomorphic to the space formed by two cones X disjoint apart from the common basis. The points $[X \times \{-1\}]$ and $[X \times \{1\}]$ are called the *vertices of the suspension*; the set $X \times \{0\}$ is called the *basis of the suspension*.

Let us observe the following four simple properties of the operation of suspension:

(i) If X is contractible then so is $\mathrm{S}X$.

(ii) The vertices of the suspension $\mathrm{S}X$ are labile if and only if X is contractible (see Lemma 1).

(iii) Suspension of a polyhedron without free faces is also a polyhedron without free faces.

(iv) Suspension of a homogeneously n-dimensional polyhedron is a homogeneously $(n+1)$-dimensional polyhedron.

COROLLARY 1. *There are no labile points in the polyhedra constructed by Hopf and Pannwitz in* [4] *on page* 446.

Proof. The examples given by Hopf and Pannwitz are constructed in the following manner: Let Q^n be the n-cube and S^{n-1} its boundary; let p_0 be a fixed point of S^1. Let B^{n-1} be the subset $(S^{n-2} \times S^1) \cup (Q^{n-1} \times \{p_0\})$ of the product $Q^{n-1} \times S^1$ (for $n \geqslant 2$) and let $P^n = \mathrm{S}B^{n-1}$ (the suspension of B^{n-1}). The polyhedron P^n is homogeneously n-dimensional, it has no free faces and the vertices of the suspension are almost-labile (see [4], p. 446). But B^{n-1} is not contractible which implies by (ii) that the vertices of the suspension are stable. Now, let x be an other point in P^n. If $x \in S(S^{n-2} \times \{p_0\})$ then $S_x^\bullet$ is formed by three $(n-1)$-cubes which are disjoint apart from the common boundary; such a polyhedron is not contractible. If $x \notin S(S^{n-2} \times \{p_0\})$ then $S_x^\bullet$ is an $(n-1)$-sphere.

Therefore, by Theorem 1, P^n contains no labile points.

COROLLARY 2. *If P is a polyhedron of dimension at most 2 without free faces then each point of it is stable.*

Proof. On the contrary, let us suppose that $v \in P$ is labile.

We can suppose that v is a vertex, changing the triangulation if needed. Then $S_v^\bullet$, i.e. the boundary of the star S_v of v, is a contractible (in particular non-empty) polyhedron of dimension at most 1.

In fact: $\dim S_v^\bullet = \dim S_v - 1 \leqslant \dim P - 1 = 1$. Hence $S_v^\bullet$ is a single point or a tree, depending of whether the dimension of P at the point v is equal to 1 or 2. If $S_v^\bullet$ is a point then S_v is the only 1-simplex for which v is an end-point. In this case the 0-simplex v is a free face of P. If $S_v^\bullet$ is a tree then it contains a point p which is a free face in $S_v^\bullet$ (namely one of the points of order 1 in $S_v^\bullet$) and then the 1-simplex which joins the points p and v is a free face in P; therefore in both cases we obtain a contradiction.

COROLLARY 3. *For every integer $n \geqslant 3$ there exists a homogeneously n-dimensional polyhedron without free faces but with labile points.*

Proof. Let P_2 be an arbitrary homogeneously 2-dimensional contractible polyhedron which contains no free faces (for example the homogeneously 2-dimensional contractible polyhedron which is not a union of two contractible polyhedra different from it; the construction of such an example was given by K. Borsuk in [2]).

Now we define P_n by induction as the suspension of P_{n-1} (for $n \geqslant 3$).

Let p_n and q_n denote the vertices of the suspension $SP_{n-1} = P_n$. P_2 being contractible, by (i) P_3 is also contractible; by induction all the P_n are contractible. Thus by (ii) the points p_n and q_n are labile in P_n (for each $n \geqslant 3$). P_2 contains no free faces, and therefore by (iii) P_3 and further all the P_n contain no free faces. Since P_2 is homogeneously 2-dimensional, each P_n (for $n \geqslant 2$) is by (iv) and by induction homogeneously n-dimensional, which completes the proof.

§ 2. A Cartesian product of two polyhedra which contains a labile point with both coordinates stable in the correspondent factors. By the *join* $X \wedge Y$ of two spaces X and Y we mean the subset

$$CX \times (Y \times \{1\}) \cup (X \times \{1\}) \times CY$$

of the product $CX \times CY$, where CX and CY are cones over X and Y respectively. If Y is a two-point-space, then $X \wedge Y$ is homeomorphic to the suspension SX of the space X.

LEMMA 2. *The spaces $CX \times CY$ and $C(X \wedge Y)$ are homeomorphic; moreover, there exists a homeomorphism φ: $CX \times CY \to C(X \wedge Y)$ such that*

$$\varphi([X \times \{0\}], [Y \times \{0\}]) = [(X \wedge Y) \times \{0\}] .$$

Proof. If $z \in CX \times CY$ then z is a pair (u, v), where $u \in CX$, $v \in CY$; further $u = [(x, t)]$, $v = [(y, s)]$, where $x \in X$, $y \in Y$, s and $t \in \langle 0, 1 \rangle$.

The function φ: $CX \times CY \to C(X \wedge Y)$ obtained by the formula

$$\varphi(z) = \begin{cases} [(X \wedge Y) \times \{0\}] & \text{for} \quad s = t = 0, \\ [([x, t/s], [y, 1]), s] & \text{for} \quad t \leqslant s \neq 0, \\ [([x, 1], [y, s/t]), t] & \text{for} \quad s \leqslant t \neq 0 \end{cases}$$

is the required homeomorphism.

THEOREM 2. *There exists a polyhedron A containing a stable point a such that pair $(a, 0)$ is a labile point in the product $A \times \langle -1, 1 \rangle$ of A and the closed interval $\langle -1, 1 \rangle$.*

Proof. Let K be an arbitrary polyhedron which satisfies the following two conditions:

(I) K is not contractible;

(II) the suspension of K is contractible.

For example, let K be the polyhedron constructed by E. G. Begle (see [1], p. 386) in the following manner: Let P be a Poincaré sphere, i.e. a 3-dimensional polyhedron with the homology group of a 3-sphere and with a non-vanishing fundamental group (see [5], p. 245); K is the polyhedron obtained by removing an open 3-simplex from P.

E. G. Begle shows in [1] that K satisfies conditions (I) and (II).

Now, let A be the cone CK over K and let a be the vertex of that cone. Naturally, by (I) and by Lemma 1, a is stable in A.

The interval $\langle -1, 1 \rangle$ is a cone with the vertex 0 over its two-point-subset $D = \{-1\} \cup \{1\}$, whence by Lemma 2 the product $A \times \{-1, 1\rangle = CK \times CD$ is (topologically) a cone with the vertex $(a, 0)$ over the join $K \wedge D$, i.e. over the suspension SK of K. Therefore, by (II) and by Lemma 1, the point $(a, 0)$ is labile in the product $A \times \langle -1, 1 \rangle$.

COROLLARY 4. *The stability of points is not invariant under Cartesian multiplication.*

In fact: the points a and 0 are both stable in A and in $\langle -1, 1 \rangle$ resp.

References

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A structure theory for a class of lattice ordered semirings *

by

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Introduction. A *semiring* S is a set of elements which is closed under two binary associative commutative operations $(+)$ and $(\cdot)$ such that $a(b+c) = ab+ac$ for all $a, b, c \in S$, containing elements 0 and 1 such that $s+0 = s$ and $s\cdot 1 = s$ for all $s \in S$. A semiring S will be called *positive* if $1+s$ has a multiplicative inverse for all $s \in S$.

The theory of semirings is relatively new. Bourne in [1] and Bourne and Zassenhaus in [2] have presented some partial results generalizing the Jacobson structure theory for semirings. In 1955, Słowikowski and Zawadowski in [7] studied the structure space of commutative positive semirings. Although they did not attempt to present an algebraic structure theory, their work seemed to indicate that a structure theory was possible for this class of semirings.

If S is a semiring, let $T(S) = \{x \in S\colon x+x = x\}$ and $K(S) = \{x \in S\colon x+a = x+b \text{ implies } a = b\}$. These elements will be called respectively the *a-idempotent* and *a-cancellable* elements. In Section 1 we prove that every positive semiring is a-idempotent or contains a copy of the non-negative rational numbers.

On every semiring S there is a natural quasi-order defined by letting $a \leqslant b$ if $a+x = b$ is solvable in S. A semiring S will be called an *l-semiring* if S is lattice ordered under the natural quasi-order and $a+(b \vee c) = (a+b) \vee (a+c)$ and $a+(b \wedge c) = (a+b) \wedge (a+c)$ for all $a, b, c \in S$. A semiring S will be called *archimedian* if $nx \leqslant a$ for $n = 1, 2, \ldots$ implies $x \in T$. In Section 3, we show that in a positive archimedian l-semiring S, then $K(S) = \{x \in S\colon x \wedge T = \{0\}\}$ and that if $\{1 \wedge k\colon k \in K\}$ has a supremum in K, then $S = K + \{x \in S\colon x \wedge K = \{0\}\}$. In Section 4, we show that both T and K are an intersection of prime l-ideals but that even under very strong hypothesis, the same is not true for $T+K$.

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