Singular homology of n-cell-like continua*

by

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1. Introduction. A *compactum* is a compact, metrizable space. A *continuum* is a connected compactum. If a is an open cover of a compactum X, a map f of X onto a compactum Y is called an a-*map* provided that for each y in Y, $f^{-1}(y)$ is contained in some member of a. Let $\mathcal{C}$ be a class of compacta. Following Mardešić and Segal [6], we say a compactum X is $\mathcal{C}$-*like* if for every open cover a of X, there exists an a-map of X onto a member of $\mathcal{C}$. If all the members are continua, it follows that X is a continuum. By [6], a continuum is n-cell-like if and only if it is the limit of an inverse sequence of n-cells with bonding maps onto. Also (for example, by [7]) a continuum X is n-cell-like if and only if every open cover of X can be refined by an open cover whose nerve is an n-cell. Thus 1-cell-like, or arc-like, continua are the *snake-like*, or *chainable* continua studied by R. H. Bing [3] and others.

What homology properties do n-cell-like continua have? Of course they are acyclic in Čech homology. But we shall see that the singular homology can be quite complicated. Let $H_q(X; G)$ denote the q-dimensional singular homology group of X, with coefficients in a group G, *reduced* in dimension 0.

Consider the following two questions for each $n \geqslant 1$.

Q_n: *If X is an arbitrary n-cell-like continuum and G is any coefficient group, is $H_q(X; G) = 0$ for all $q > n$?*

Q'_n: *If X is an arbitrary n-cell-like continuum and G is any coefficient group, is $H_q(X; G) = 0$ for all $q \geqslant n$?*

The main theorem of the paper gives a strong negative answer to Q_n (hence also to Q'_n) for $n \geqslant 3$. Let Q denote the group of rational numbers.

THEOREM 1. *For each $n \geqslant 3$, there exists an n-cell-like continuum X such that for all $q \geqslant 0$ the group $H_q(X; Q)$ is uncountable.*

In view of the fact that X can be given as an inverse limit of n-cells, theorem 1 shows how extremely discontinuous singular homology can be. The result is based on a theorem due to M. G. Barratt and John Milnor [1].

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In contrast to Theorem 1, the following theorem (whose proof is short) answers Q_1' (hence also Q_1) affirmatively.

THEOREM 2. *If X is an arc-like continuum, then $H_q(X; G) = 0$ for all $q \geqslant 1$.*

The questions Q_2 and Q_2' remain unanswered by this paper.

CONVENTION. Throughout Sections 2 and 3, we will assume a fixed but arbitrary coefficient group G for singular homology groups, which will be suppressed from the notation.

2. Methods for costructing n-cell-like continua. If X is a space, let $\mathrm{Cov}(X)$ denote the collection of all open covers of X. If D is an n-cell, let $\mathrm{Bd}\,D$ denote the boundary $(n-1)$-sphere of D.

DEFINITION. A closed subset A of an n-cell-like continuum X is called *extremal* (in X) if for each $\alpha \in \mathrm{Cov}(X)$ there exists an α-map f of X onto an n-cell D which carries A homeomorphically into $\mathrm{Bd}\,D$.

An important special case is when A is an extremal $(n-1)$-sphere. Of course then f must map A homeomorphically *onto* $\mathrm{Bd}\,D$.

The union of two extremal sets does not have to be extremal. For instance, in the (arc-like) "$\sin x^{-1}$-continuum"

$$\{(x, \sin x^{-1}) \colon 0 < x \leqslant 1\} \cup \{(0, y) \colon -1 \leqslant y \leqslant 1\}, \tag{2.1}$$

one may consider the extremal 0-spheres $\{(0, 1), (1, \sin 1)\}$ and $\{(0, -1), (1, \sin 1)\}$.

It is not hard to show (using, for example, the methods of proof in [7]) that in the special case $n = 1$, the notions *extremal point* and *extremal 0-sphere* are equivalent to the notions *end point* and *(pair of) opposite end points*, respectively, investigated by R. H. Bing in [3].

LEMMA 2.1. *Let X be an n-cell-like continuum with extremal $(n-1)$-sphere A, and let D be any n-cell. Then for each homeomorphism φ of A onto $\mathrm{Bd}\,D$ and for each $\alpha \in \mathrm{Cov}(X)$, there exists an α-map f of X onto D such that $f|A = \varphi$.*

Proof. Choose an α-map g of X onto an n-cell D' which carries A homeomorphically onto $\mathrm{Bd}\,D'$. Now $\varphi(g|A)^{-1}$ is a homeomorphism of $\mathrm{Bd}\,D'$ onto $\mathrm{Bd}\,D$; extend it to a homeomorphism h of D' onto D. Then $f = hg$ is the required α-map.

Next we define an operation on spaces which will be useful in constructing new n-cell-like continua. A *doubly based space* is a triple (X, x, x') where X is a space, and x and x' are points of X. If (Y, y, y') is another doubly based space we define a new doubly based space

$$(Z, z, z') = (X, x, x') \oplus (Y, y, y'), \tag{2.2}$$

called the *arc join* of (X, x, x') and (Y, y, y'), as follows. Form the disjoint union $X + [0,1] + Y$, and let Z be the quotient space formed by identifying x' with 0 and y with 1. We consider X, Y, and $[0,1]$ as subsets of Z. Then we let the base points of Z be $z = x$ and $z' = y'$. It should be noted that the operation of "arc join" is associative (up to homeomorphism), but not commutative. The next lemma follows from the use of a (reduced) Mayer-Vietoris sequence ([4], p. 39) and two deformation retractions.

LEMMA 2.2. *In equation (2.2) we have $H_q(Z) \approx H_q(X) \oplus H_q(Y)$ for all integers q.*

We call a doubly based space (X, x, x') and *n-cell-like continuum with extremal pair* if X is n-cell-like, $x \neq x'$, and $\{x, x'\}$ is extremal in X.

LEMMA 2.3. *In equation (2.2), if (X, x, x') and (Y, y, y') are n-cell-like continua with extremal pairs, then so is (Z, z, z').*

Before beginning the proof, let us establish the following notation. Let R^n denote Euclidean n-space, with the norm $\|r\| = \big(\sum_{i=1}^{n} r_i^2\big)^{1/2}$. If $r_0 \in R^n$ and $\delta > 0$, let $N_\delta(r_0) = \{r \in R^n \colon \|r - r_0\| \leqslant \delta\}$. Let $D^n = N_1(0)$ and let $S^{n-1} = \mathrm{Bd}\,D^n$. Let $I^n = [0,1]^n$.

Proof. Choose a metric on Z. Let $\varepsilon > 0$ be given. Choose a partition $0 = t_0 < t_1 < \ldots < t_k = 1$ of $[0,1] \subset Z$ such that $\mathrm{diam}\,[t_{i-1}, t_i] < \varepsilon/2$ for all $i = 1, \ldots, k$. We shall produce an ε-map F of Z onto the n-cell $[0, k+2] \times I^{n-1}$.

Choose an $(\varepsilon/4)$-map f of X onto D^n such that $f(x)$ and $f(x')$ are distinct points of S^{n-1}. It is easy to see by compactness that there exists a $\delta_0 > 0$ such that if $E \subset D^n$ and $\mathrm{diam}(E) < \delta_0$, then $\mathrm{diam} f^{-1}(E) < \varepsilon/2$. Hence we may find a number $\delta > 0$ such that $\mathrm{diam} f^{-1}\big(N_\delta(f(x')) \cap D^n\big) < \varepsilon/2$ and $f(x) \notin N_\delta(f(x'))$. Now choose a homeomorphism h of D^n onto $[0,1] \times I^{n-1}$ such that $h\big(N_\delta(f(x')) \cap S^{n-1}\big) = \{1\} \times I^{n-1}$. Let $\varphi = hf$. Obviously φ is an $(\varepsilon/4)$-map of X onto $[0,1] \times I^{n-1}$ such that $\varphi(x') \in \{1\} \times I^{n-1}$, $\varphi(x) \in \mathrm{Bd}([0,1] \times I^{n-1}) - \{1\} \times I^{n-1}$, and $\mathrm{diam}\,\varphi^{-1}(\{1\} \times I^{n-1}) < \varepsilon/2$. Similarly, we obtain an $(\varepsilon/4)$-map ψ of Y onto $[k+1, k+2] \times I^{n-1}$ such that

$$\psi(y) \in \{k+1\} \times I^{n-1}, \qquad \psi(y') \in \mathrm{Bd}([k+1, k+2] \times I^{n-1}) - \{k+1\} \times I^{n-1},$$

and

$$\mathrm{diam}\,\psi^{-1}(\{k+1\} \times I^{n-1}) < \varepsilon/2\,.$$

Applying the Hahn-Mazurkiewicz theorem to each interval $[t_{i-1}, t_i]$ $(1 \leqslant i \leqslant k)$, we obtain a map τ of $[0,1]$ onto $[1, k+1] \times I^{n-1}$ such that $\tau(0) = \varphi(x')$, $\tau(1) = \psi(y)$, and for each i $(1 \leqslant i \leqslant k)$

$$\tau([t_{i-1}, t_i]) = [i, i+1] \times I^{n-1}\,.$$

Now we define the map F on Z to be $F = \varphi \cup \tau \cup \psi$. This map is obviously continuous and maps Z onto $[0, k+2] \times I^{n-1}$. Clearly also

$F(z) = F(x) = \varphi(x)$ and $F(z') = F(y') = \psi(y')$ are distinct points of $\mathrm{Bd}([0, k+2] \times I^{n-1})$. Let us check then that F is an ε-map. Suppose $v = (s, r) \in [0, k+2] \times I^{n-1}$.

Case I: $0 \leqslant s < 1$. Then $F^{-1}(v) = \varphi^{-1}(v)$, which has diameter $< \varepsilon/4$.

Case II: $s = 1$. Then

$$F^{-1}(v) = \varphi^{-1}(v) \cup \tau^{-1}(v) \subset \varphi^{-1}(\{1\} \times I^{n-1}) \cup [0, t_1] .$$

The two sets in the latter union intersect (in x') and each have diameter $< \varepsilon/2$. Hence the union has diameter $< \varepsilon$.

Case III: $1 < s < k+1$. Then $F^{-1}(v) = \tau^{-1}(v)$. From the choice of τ, $\tau^{-1}(v)$ is certainly contained in the union of two adjacent intervals of the partition $(t_0, \ldots, t_k)$, hence has diameter $< \varepsilon$.

Case IV: $s = k+1$. Similar to Case II.

Case V: $k+1 < s \leqslant k+2$. Similar to Case I. This completes the proof.

LEMMA 2.4. *If X and Y are respectively m and n-cell-like continua with extremal subsets A and B, then $X \times Y$ is an $(m+n)$-cell-like continuum having $A \times B$ as an extremal subset.*

Proof. Straightforward.

We find it convenient to write the *suspension* $S(X)$ of a space X as the quotient space formed from $X \times [-1, 1]$ by identifying $X \times \{1\}$ to a point and identifying $X \times \{-1\}$ to a point. Let $\nu = \nu_X \colon X \times [-1, 1] \to S(X)$ be the quotient map. We shall often write $[x, s]$ for $\nu(x, s)$. If A is a subset of X, we think of $S(A)$ as a subset of $S(X)$. If $f \colon X \to Y$ is a map, the *suspension* $S(f) \colon S(X) \to S(Y)$ is defined by $S(f)[x, s] = [f(x), s]$. The *n-fold suspension* $S^n(X)$ is defined recursively by

$$S^0(X) = X , \quad S^n(X) = S\big(S^{n-1}(X)\big) \ (n > 0) .$$

LEMMA 2.5. *If X is an n-cell-like continuum having A as an extremal subset, then $S(X)$ is an $(n+1)$-cell-like continuum having $S(A)$ as an extremal subset.*

Proof. (In [8] the obvious generalization of part of the result to $\mathcal{C}$-like continua is given.) Let α be an open cover of $S(X)$. Clearly there exists an open cover β of X such that for each U in β and each s in $[-1, 1]$, there is a V in α with $\nu_X(U \times \{s\}) \subset V$. Choose a β-map f of X onto an n-cell D which takes A homeomorphically into $\mathrm{Bd}D$. Now $S(f)$ maps $S(X)$ onto $S(D)$ and takes $S(A)$ homeomorphically into $S(\mathrm{Bd}D)$. However, $S(D)$ is an $(n+1)$-cell with boundary $S(\mathrm{Bd}D)$. Clearly, for each point $[y, s]$ of $S(D)$, $\big(S(f)\big)^{-1}[y, s] = \nu_X\big(f^{-1}(y) \times \{s\}\big)$. By choice of β, then, $S(f)$ is an α-map. This completes the proof.

If β is a cardinal number, let G^β denote the direct sum of β copies of the coefficient group G ($G^0 = 0$).

LEMMA 2.6. *Let $0 \leqslant m < n$ and let β be a cardinal number which is finite, $\aleph_0$, or c (the cardinal number of the continuum). Then there exists an n-cell-like continuum (Y, y, y') with extremal pair such that $H_m(Y) \approx G^\beta$ and $H_q(Y) = 0$ for $q \neq m$.*

Proof. First we produce an arc-like continuum (X, x, x') with extremal pair such that $H_0(X) \approx G^\beta$ and $H_q(X) = 0$ for $q > 0$ (the latter is of course true for any arc-like continuum, by Theorem 2). In case $\beta = 0$, we take $X = [0, 1]$. In case $0 < \beta < \aleph_0$, we take X to be an arc join of β copies of the $\sin x^{-1}$-continuum. Similarly, if $\beta = \aleph_0$, we can construct a "countable arc join" (in an obvious sense) of copies of the $\sin x^{-1}$-continuum. In case of $\beta = c$, we can take X to be the pseudo-arc ([5], [9], [2]). Since the pseudo-arc is hereditarily indecomposable, its arc-components are simply its points. Hence X has the required homology. Furthermore, by [2], proof of Theorem 1, X possesses extremal pairs (any two points in different composants).

Now we let $Y = S^m(X) \times I^{n-m-1}$. By Lemmas 2.4 and 2.5, Y is an n-cell-like continuum possessing an extremal pair. By the iterated suspension isomorphism, Y has the required homology.

3. A generalization of the concept of deformation retract.

A *deformation* f_t of a space X is a homotopy $f_t \colon X \to X$ ($0 \leqslant t \leqslant 1$) such that $f_0 = 1_X$ (the identity on X).

DEFINITION. Let $\mathcal{C}$ be a class of topological spaces. We say that a subset A of a space X is a *$\mathcal{C}$-deformation retract* of X provided that for every subset K of X with $K \in \mathcal{C}$, there exists a deformation f_t of X such that $f_1(K) \subset A$ and $f_1(x) = x$ for all x in A. If we can always choose f_t so that $f_t(x) = x$ for all t and all x in A, then A is a *strong* $\mathcal{C}$-deformation retract of X.

Note that in case $\mathcal{C}$ is the class of all spaces, the notion of *$\mathcal{C}$-deformation retract* coincides with the usual notion of *deformation retract*.

Let $\mathcal{LC}$ denote the class of locally connected compacta.

LEMMA 3.1. *Let X be a Hausdorff space and let A be an $\mathcal{LC}$-deformation retract of X. Then the inclusion $i \colon A \to X$ induces isomorphisms on singular homology groups.*

Proof. Let $C_*(X) = \big(C_q(X)\big)_q$ be the singular chain complex of X. For each $c \in C_q(X)$, let $|c|$ denote the carrier of c — the union of the images of the singular simplexes appearing in c. Since X is Hausdorff, $|c| \in \mathcal{LC}$.

To show that $i_* \colon H_q(A) \to H_q(X)$ is onto, suppose that z is a singular q-cycle in X. Choose $f_t \colon X \to X$ as in the above definition for $K = |z| \subset X$.

By [4], p. 195, there exists a chain homotopy D between the chain maps induced by $f_0 = 1$ and f_1. Thus $z - f_1 z = D_q z$, so that z is homologous to the cycle $f_1 z$ in A.

Suppose that z is a q-cycle in A that bounds a $(q+1)$-chain c in X. Choose f_t as in the definition for $K = |c|$. Then $z = f_1 z = f_1 \partial c = \partial f_1 c$, so that z bounds in A. Thus $\ker i_* = 0$.

4. Iterated suspensions of a modified $\sin x^{-1}$-continuum. Let C denote the union of the arc $\{(x, -1): -1 \leqslant x \leqslant 0\}$ and the $\sin x^{-1}$-continuum (2.1). Define a 1-1 correspondence $\varphi: [-2, 1] \to C$ as follows:

$$\varphi(x) = \begin{cases} (x+1, -1) & \text{for} \quad -2 \leqslant x \leqslant -1, \\ (0, 2x+1) & \text{for} \quad -1 \leqslant x \leqslant 0, \\ (x, \sin x^{-1}) & \text{for} \quad 0 < x \leqslant 1. \end{cases}$$

Throughout this section and Section 6, let B denote the interval $[-2, 1]$ endowed with the topology making φ a homeomorphism. Thus B is simply another version of C with a simpler notation which we find convenient. In B consider the following subspaces, each of which clearly has the ordinary topology as a subset of the real line:

$$B^0 = \{-2, 1\}, \quad B' = [-2, 0] \cup \{1\}.$$

LEMMA 4.1. *For each $n \geqslant 0$, $S^n(B)$ is an $(n+1)$-cell-like continuum having $S^n(B^0)$ as an extremal n-sphere.*

Proof. This follows from Lemma 2.5, induction, and the fact that B is arc-like with extremal pair $\{-2, 1\} = B^0$.

LEMMA 4.2. *For each $n \geqslant 0$, the n-sphere $S^n(B^0)$ is a strong deformation retract of $S^n(B')$.*

Proof. Obviously B^0 is a strong deformation retract of B'. Then the result follows by induction and the general fact that if A is a strong deformation retract of X, then $S(A)$ is a strong deformation retract of $S(X)$.

LEMMA 4.3. *For each $n \geqslant 0$, $S^n(B')$ is a strong $\mathfrak{LC}$-deformation retract of $S^n(B)$.*

Remark. $S^n(B')$ is of course not a deformation retract of $S^n(B)$ since $S^n(B')$ has the Čech homology of an n-sphere, whereas $S^n(B)$ is acyclic in Čech homology.

Before beginning the proof let us set the following notation. Let $J^n = [-1, 1]^n$, $J^{on} = (-1, 1)^n$ (usual topology). For any space X, $S^n(X)$ can be viewed as a quotient space of $X \times J^n$. We define the quotient map

$$\nu = \nu_n: X \times J^n \to S^n(X)$$

recursively: ν_1 has already been defined. If ν_n is defined, let ν_{n+1} be given by

$$\nu_{n+1}\big(x, (s_1, \ldots, s_{n+1})\big) = \nu_1\big(\nu_n(x, (s_1, \ldots, s_n)), s_{n+1}\big).$$

Again we find it convenient to write $[x, s] = \nu(x, s)$ for $(x, s) \in X \times J^n$. Note that ν maps $X \times J^{on}$ homeomorphically into $S^n(X)$. If $s \in J^n$, let $|s| = \max\{|s_1|, \ldots, |s_n|\}$.

Proof of Lemma 4.3. For each ε in $(0, 1]$ we wish to define a certain homotopy $f_t^\varepsilon: B \to B$ $(0 \leqslant t \leqslant 1)$. Let $\varepsilon_t = (1-t)\varepsilon + t$. Then f_t^ε is the identity on $[-2, \varepsilon/2]$; f_t^ε maps $[\varepsilon/2, \varepsilon]$ linearly onto $[\varepsilon/2, \varepsilon_t]$; and f_t^ε maps $[\varepsilon, 1]$ linearly onto $[\varepsilon_t, 1]$. Note that $f_t^\varepsilon(x)$, x in B, is continuous simultaneously in x, t, and ε. For each ε,

$$f_0^\varepsilon = 1_B, \tag{4.1}$$

$$f_t^\varepsilon(x) = x \quad \text{for all } x \text{ in } B' \text{ and all } t, \tag{4.2}$$

$$f_1^\varepsilon \text{ maps } [-2, 0] \cup [\varepsilon, 1] \text{ onto } B'. \tag{4.3}$$

Now suppose that K is a locally connected compactum in $S^n(B)$. Using the quotient map $\nu: B \times J^n \to S^n(B)$, we get the compact subset $\nu^{-1}(K)$ of $B \times J^n$. We claim that for each number r in $[0, 1)$, there exists an ε in $(0, 1)$ such that

$$\nu^{-1}(K) \cap \{(x, s) \in B \times J^n: 0 < x < \varepsilon, \ |s| \leqslant r\} = \emptyset.$$

Suppose that no such ε can be found. Then, by the compactness of $\nu^{-1}(K)$ and the definition of the topology of B, there exists a sequence of points (x_i, s^i) $(i = 0, 1, 2, \ldots)$ of $\nu^{-1}(K)$ such that $x_0 \in [-1, 0]$, $0 < x_i < 1/i$ for $i \geqslant 1$, $|s^i| \leqslant r$, and $\big((x_1, s^1), (x_2, s^2), \ldots\big)$ converges to (x_0, s^0) in $B \times J^n$. It is clear then that every (sufficiently small) neighborhood of (x_0, s^0) in $\nu^{-1}(K)$ is disconnected. However, since ν is a homeomorphism on $B \times J^{on}$, $\nu^{-1}(K)$ is locally connected at (x_0, s^0).

Thus we obtain a sequence $1 > \varepsilon_1 > \varepsilon_2 > \ldots > 0$ such that for all $m \geqslant 1$,

$$\nu^{-1}(K) \cap \{(x, s) \in B \times J^n: 0 < x < \varepsilon_m, \ |s| \leqslant 1 - m^{-1}\} = \emptyset. \tag{4.4}$$

Clearly there exists a map $\psi: J^{on} \to (0, 1)$ such that

$$\psi(s) \leqslant \varepsilon_m \quad \text{whenever} \quad |s| \leqslant 1 - m^{-1}. \tag{4.5}$$

Now define the homotopy $g_t: S^n(B) \to S^n(B)$ by

$$g_t[x, s] = \begin{cases} [f_t^{\psi(s)}(x), s] & \text{if} \quad |s| < 1; \\ [x, s] & \text{if} \quad |s| = 1. \end{cases} \tag{4.6}$$

It is clear that g_t is continuous. By (4.1), g_0 is the identity on $S^n(B)$. By (4.2), $g_t[x, s] = [x, s]$ whenever $[x, s] \in S^n(B')$. Finally, we claim that g_1 maps K into $S^n(B')$. For suppose that $[x, s] \in K$. If $|s| = 1$, then $g_1[x, s] = [x, s]$ is in the n-fold suspension of *any* subset of B. If $|s| < 1$, then for some $m \geqslant 1$, $|s| \leqslant 1 - m^{-1}$. Hence by (4.4), $x \in [-2, 0] \cup \cup [\varepsilon_m, 1]$. Then by (4.5), $x \in [-2, 0] \cup [\psi(s), 1]$. Thus by (4.3) and (4.6), $g_1[x, s] \in S^n(B')$.

We shall actually need a lemma a little stronger than Lemma 4.3. It is clear from the second paragraph of the proof that the same proof actually gives the following result:

LEMMA 4.4. *Let K be a compact subset of $S^n(B)$ which is locally connected at each point it has in common with the subset $\nu([-1, 0] \times J^{\infty n})$ of $S^n(B)$. Then there exists a deformation g_t of $S^n(B)$ leaving $S^n(B')$ pointwise fixed such that $g_1(K) \subset S^n(B')$.*

5. The example of Barratt and Milnor. In R^{n+1} $(n \geqslant 0)$, for each $i \geqslant 1$, let S_i^n denote the sphere with the center $(2^{-i}, 0, \ldots, 0)$ and radius 2^{-i}; and let $M^n = \bigcup_{i=1}^{\infty} S_i^n$.

THEOREM. (Barratt and Milnor [1].) *For $n \geqslant 2$, $H_q(M^n; Q)$ is uncountable when $q \equiv 1 \pmod{n-1}$ and $q > 1$.*

We now describe another continuum N^{n+1} having the same homotopy type as M^n (so that the above theorem also holds for N^{n+1}). In the unit disk D^{n+1} with center at the origin, for each $i \geqslant 1$ let E_i denote the open disk with center $(2^{-i} + 2^{-i-1}, 0, \ldots, 0)$ and radius 2^{-i-2}. Then let

$$N^{n+1} = D^{n+1} - \bigcup_{i=1}^{\infty} E_i .$$

It can be seen that N^{n+1} in fact are deformation retracts onto M^n.

6. Completion of the proof of Theorem 1. We construct an $(n+1)$-cell-like continuum X^{n+1} obtained from N^{n+1} by filling in each hole with a copy of the $(n+1)$-cell-like continuum $S^n(B)$ (see Section 4). Let us describe X^{n+1} more precisely.

Let P denote the positive integers with the discrete topology. For each $i \in P$ choose a homeomorphism φ_i of the n-sphere $S^n(B^0)$ onto the n-sphere $\mathrm{Bd}\, E_i$. Let X^{n+1} be the quotient space formed from the disjoint union $\big(S^n(B) \times P\big) \cup N^{n+1}$ by identifying (x, i) with $\varphi_i(x)$ for each $x \in S^n(B)$. Let

$$\mu\colon \big(S^n(B) \times P\big) \cup N^{n+1} \to X^{n+1}$$

be the quotient map, and for each $i \in P$, let the imbedding

$$\mu_i\colon S^n(B) \to X^{n+1}$$

be defined by $\mu_i(x) = \mu(x, i)$.

LEMMA 6.1. *X^{n+1} is an $(n+1)$-cell-like continuum.*

Proof. Let $\alpha \in \mathrm{Cov}(X^{n+1})$. For each $i \in P$, define $\alpha_i \in \mathrm{Cov}\big(S^n(B)\big)$ to be $\{\mu_i^{-1}(U)\colon U \in \alpha\}$. By Lemma 4.1 and Lemma 2.1, we get an α_i-map f_i of $S^n(B)$ onto the $(n+1)$-cell $\bar{E}_i$ such that

$$f_i | S^n(B^0) = \varphi_i . \tag{6.1}$$

Now define a map f of X^{n+1} onto D^{n+1} as follows: If $y \in N^{n+1}$, let $f\mu(y) = y$; if $(x, i) \in S^n(B) \times \{i\}$, let $f\mu(x, i) = f_i(x)$. The map f is well-defined and continuous, because of condition (6.1). Finally, f is an α-map. For suppose that $y \in D^{n+1}$.

Case I: *y is in no $\bar{E}_i$.* Then $f^{-1}(y) = \mu(y)$ (a single point).

Case II: *$y \in \bar{E}_i$.* (This can happen for at most one i.) Since f_i is an α_i-map, there exists a member U of α such that $f_i^{-1}(y) \subset \mu_i^{-1}(U)$. Then $f^{-1}(y) = \mu_i f_i^{-1}(y) \subset \mu_i \mu_i^{-1}(U) \subset U$. This completes the proof.

Remark. It is clear from the proof that $\mu(\mathrm{Bd}\, D^{n+1})$ is an extremal n-sphere in X^{n+1}.

For the next two lemmas, we introduce the following notation. If A is a subset of B, let

$$N^{n+1}(A) = \mu\Big(\big(S^n(A) \times P\big) \cup N^{n+1}\Big) \subset X^{n+1} .$$

Then clearly

$$N^{n+1} \equiv N^{n+1}(B^0) \subset N^{n+1}(B') \subset N^{n+1}(B) = X^{n+1} .$$

LEMMA 6.2. *$N^{n+1}(B^0)$ is a strong deformation retract of $N^{n+1}(B')$.*

Proof. Lemma 4.2 says that $S^n(B^0)$ is a strong deformation retract of $S^n(B')$. Hence choose the appropriate deformation $f_t\colon S^n(B') \to S^n(B')$. Then we define the required deformation $g_t\colon N^{n+1}(B') \to N^{n+1}(B')$ by $g_t \mu(x, i) = \mu\big(f_t(x), i\big)$ if $x \in S^n(B')$ and $g_t \mu(y) = \mu(y)$ if $y \in N^{n+1}$.

LEMMA 6.3. *$N^{n+1}(B')$ is a strong $\mathfrak{LC}$-deformation retract of $N^{n+1}(B) = X^{n+1}$.*

Proof. Let K be a locally connected compactum in X^{n+1}. For each $i \in P$, let K_i be the compact subset $\mu_i^{-1}\big(\mu_i(S^n(B)) \cap K\big)$ of $S^n(B)$. Then K_i satisfies the hypotheses (replacing K) in Lemma 4.4. This is so because the imbedding $\mu_i\colon S^n(B) \to X^{n+1}$ obviously maps $\nu([-1, 0] \times J^{\infty n})$ into the *interior* of $\mu_i(S^n(B)) \subset X^{n+1}$. Thus by Lemma 4.4, choose a deformation g_t^i of $S^n(B)$ leaving $S^n(B')$ pointwise fixed such that $g_1^i(K_i) \subset S^n(B')$. Then we define the required deformation $g_t\colon X^{n+1} \to X^{n+1}$ by $g_t \mu_i(x) = \mu_i(g_t^i(x))$ for each $i \in P$ and each $x \in S^n(B)$, and $g_t \mu(y) = \mu(y)$ for each $y \in N^{n+1}$. Clearly g_t leaves $N^{n+1}(B')$ pointwise fixed and $g_1(K) \subset N^{n+1}(B')$.

COROLLARY 6.4. *The (singular) homology groups of X^{n+1} are isomorphic to those of M^n. In particular, $H_q(X^{n+1}; Q)$ is uncountable for $q \equiv 1 \pmod{n-1}$, $q > 1$.*

Proof. This follows from Lemmas 6.3, 3.1, 6.2, and the fact that $N^{n+1} \equiv N^{n+1}(B^0)$ deformation retracts onto M^n.

In particular, for the 3-cell-like continuum X^3, we have

$$H_q(X^3; Q) \text{ is uncountable for all } q \geqslant 2 . \tag{6.2}$$

Now X^3 possesses an extremal 2-sphere (see the remark following Lemma 6.1). In particular, X^3 possesses an extremal pair $\{x, x'\}$. By Lemma 2.6, choose 3-cell-like continua (Y^3, y, y') and (Z^3, z, z') with extremal pairs such that

$$H_0(Y^3; Q) \text{ and } H_1(Z^3; Q) \text{ are uncountable} . \tag{6.3}$$

Then let W^3 be the continuum formed by taking the arc join

$$(X^3, x, x') \oplus (Y^3, y, y') \oplus (Z^3, z, z') .$$

By Lemma 2.3, W^3 is 3-cell-like. And from Lemma 2.2, (6.2), and (6.3) it follows that $H_q(W^3; Q)$ is uncountable for all $q \geqslant 0$. By Lemma 2.5, for each $n \geqslant 3$, the continuum $W^n = W^3 \times I^{n-3}$ is n-cell-like. And $H_*(W^n; Q) \approx H_*(W^3; Q)$ by a deformation retraction. This completes the proof of Theorem 1.

7. Proof of Theorem 2. The proof depends on the following lemma.

LEMMA 7.1. *Every non-degenerate locally connected subcontinuum of an arc-like continuum is an arc.*

Proof. It is clear (for instance, from the α-map definition) that every non-degenerate subcontinuum of an arc-like continuum is itself arc-like. If X is a locally connected arc-like continuum, then X contains no simple closed curves and no triods (since these are not arc-like), so that X is a dendrite containing no triods. From [10], p. 88, (1.1) (ii), it follows that such a dendrite is an arc. See also [6], p. 163.

A stronger result than Theorem 2 holds; namely, if X is an arc-like continuum, then the homotopy groups $\pi_q(X, x_0)$ ($q > 0$, any base point x_0) are zero. For by Lemma 7.1, if f is a map of a q-sphere into X, the image of f must be an arc or a point. One may also easily argue directly for the homology.

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