Approximation of maps of inverse limit spaces by induced maps

by

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1. Introduction. We use the notation and terminology of [2] for inverse limit systems. In particular, if (X, f) is an inverse limit system over a (directed) index set Λ, then we have the bonding maps $f_\alpha^\beta\colon X_\beta \to X_\alpha$ ($\alpha \leqslant \beta$ in Λ) and the projection maps $f_\alpha\colon X_\infty \to X_\alpha$. Recall that a *map* φ from the inverse limit system (X, f) (indexed over Λ) to the inverse limit system (Y, g) (indexed over M) consists of an order preserving map $\varphi\colon M \to \Lambda$ and a system of maps $\varphi_m\colon X_{\varphi(m)} \to Y_m$ (all $m \in M$) such that if $m \leqslant n$ in M, then $\varphi_m f_{\varphi(m)}^{\varphi(n)} = g_m^n \varphi_n$. Thus φ induces a map $\varphi_\infty\colon X_\infty \to Y_\infty$ defined by the relation $g_m \varphi_\infty = \varphi_m f_{\varphi(m)}$ (all $m \in M$). A map $X_\infty \to Y_\infty$ is called an *induced map* if it is of the form φ_∞, for some map $\varphi\colon (X, f) \to$ $\to (Y, g)$. The following two questions are natural.

QUESTIONS. (1) *Under what conditions on the systems* (X, f) *and* (Y, g) *can every map* $F\colon X_\infty \to Y_\infty$ *be approximated arbitrarily closely by induced maps (for instance, when the space of maps* $X_\infty \to Y_\infty$ *is given the compact-open topology)?*

(2) *Under what conditions is every* F *homotopic to an induced map?*

Question (1) is related to a question asked by J. Mioduszewski ([6], p. 40). Partial answers to these questions are given in Theorems 1 and 2 below.

2. Terminology and statements of theorems. By a *polyhedron* we mean a finitely triangulable space.

DEFINITION 1. A *solenoidal sequence* (Y, g) *of polyhedra* is an inverse limit *sequence* (the index set M is the positive integers), each Y_m being a polyhedron, so that each bonding map $g_1^m\colon Y_m \to Y_1$ is a regular covering map.

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For basic facts on covering maps, see [4], Chapter 6. The (characterizing) property of a *regular* covering map $p\colon \tilde{Z}\to Z$ which we shall use is that whenever $z, z'\in \tilde{Z}$ and $p(z) = p(z')$, there exists a covering transformation $h\colon \tilde{Z}\to\tilde{Z}$ (a homeomorphism with $ph = p$) such that $h(z) = z'$ (see [4], p. 260.) It is an immediate consequence in Definition 1 that each bonding map $g_m^n\colon Y_n\to Y_m$ is also a regular covering map. Solenoidal sequences have been studied in [5].

If ψ and ψ' are maps of a space A into a metric space (B, d), we let $d(\psi,\psi') = \sup\{d(\psi(a),\psi'(a))\colon a\in A\}$. We find it convenient to write homotopies in the form $h^t\colon A\to B$, meaning of course that t varies over the unit inverval I and the function $H\colon A\times I\to B$ defined by $H(a,t) = h^t(a)$ is continuous. We call h^t an *ε-homotopy* if $d(h^s, h^t)<\varepsilon$ whenever $s, t\in I$.

THEOREM 1. *Let (X, f) be an inverse limit system of compact, connected, Hausdorff spaces with all bonding maps onto. Let (Y, g) be a solenoidal sequence of polyhedra. Then for any map $F\colon X_\infty\to Y_\infty$, any metric d_∞ on Y_∞, and any $\varepsilon>0$, there exist an induced map $\varphi_\infty\colon X_\infty\to Y_\infty$ and an ε-homotopy $h_\infty^t\colon X_\infty\to Y_\infty$ from φ_∞ to F. In particular, $d_\infty(\varphi_\infty, F)<\varepsilon$.*

Note that under these circumstances, the topology on the space of maps $X_\infty\to Y_\infty$ defined by the metric d_∞ is equal to the compact-open topology.

According to K. Borsuk [1], an *r-map* $\varphi\colon A\to B$ is a map for which there exists a right inverse $\psi\colon B\to A\,(\varphi\psi = 1)$.

DEFINITION 2. An inverse limit sequence (Y, g) is called *retractive* if each bonding map $g_m^{m+1}\colon Y_{m+1}\to Y_m$ is an r-map.

THEOREM 2. *Let (X, f) be an inverse limit system of compact Hausdorff spaces, and let (Y, g) be a retractive inverse limit sequence of polyhedra. Then every map $F\colon X_\infty\to Y_\infty$ can be approximated arbitrarily closely by induced maps.*

3. Preliminaries. The following lemma is well known. See for instance [4], pp. 262-264.

LEMMA 1. *Suppose that $p\colon\tilde{Y}\to Y$ is a covering map, where Y is a polyhedron with a given triangulation. Then $\tilde{Y}$ can be triangulated so that p is simplicial. With this done, then for each vertex v of Y, $p^{-1}(\overline{\mathrm{star}}\,(v))$ is the disjoint union of the closed stars of the vertices of $\tilde{Y}$ lying over v, each of which is mapped isomorphically onto $\overline{\mathrm{star}}\,(v)$ by p.*

DEFINITION 3. If Y is a polyhedron with a given triangulation, $x\in Y$, and v is a vertex of Y, let $x(v)$ be the barycentric coordinate of x with respect to v. Define the *barycentric metric* d on Y by

$$d(x, y) = \sum\{|x(v)-y(v)|\colon v \text{ a vertex of } Y\}.$$

Note that if σ and τ are disjoint closed simplexes and $x\in\sigma$, $y\in\tau$, then $d(x, y) = 2$.

It will be assumed throughout this paper that triangulated polyhedra are given the barycentric metric.

The following lemma is straightforward to verify.

LEMMA 2. *If $p\colon Y\to Z$ is a simplicial map, then $d(p(x), p(y))\leqslant d(x, y)$ whenever $x, y\in Y$.*

In particular, every isomorphism $Y\to Z$ is an isometry. Thus in Lemma 1, one may add to the conclusion that for each vertex v in Y and each $\tilde{v}$ in $p^{-1}(v)$, p maps the closed star of $\tilde{v}$ *isometrically* onto the closed star of v.

4. Lemmas on solenoidal sequences. Throughout this section, let (Y, g) be a solenoidal sequence of polyhedra. Choose a triangulation of Y_1; and by Lemma 1, triangulate all Y_n so that all bonding maps $g_m^n\colon Y_n\to Y_m$ $(m\leqslant n)$ are simplicial covering maps.

Choice of η'. Choose a positive number η' such that every subset of Y_1 of diameter $\leqslant\eta'$ is contained in some open star in Y_1 (see for example [2], p. 65).

LEMMA 3. *Suppose that $m<n$, $0<\varepsilon\leqslant\eta'$, A is a space, and $h_m^t\colon A\to Y_m$ and $h_n^t\colon A\to Y_n$ are homotopies such that $g_m^n h_n^t = h_m^t$, where h_m^t is an ε-homotopy. Then,* (i) *h_n^t is an ε-homotopy; and* (ii) *if $\psi\colon A\to Y_n$, $d(h_n^0,\psi)<2$, and $g_m^n\psi = h_m^1$, then $\psi = h_n^1$.*

Proof. By Lemma 2, the homotopy $h_1^t = g_1^m h_m^t\colon A\to Y_1$ is also an ε-homotopy. And $g_1^n h_n^t = g_1^m g_m^n h_n^t = h_1^t$. Hence we may assume that $m = 1$. Let a be any point of A, and let paths $\gamma_j\colon I\to Y_j$ $(j = 1, n)$ be defined by $\gamma_j(t) = h_j^t(a)$. Thus $g_1^n\gamma_n = \gamma_1$ and $\operatorname{diam}\gamma_1(I)<\varepsilon$. For part (i), it suffices to show that $\operatorname{diam}\gamma_n(I)<\varepsilon$. Now since $\operatorname{diam}\gamma_1(I)<\eta'$, there exists a vertex v of Y_1 such that $\gamma_1(I)\subset\mathrm{star}(v)$. Let $(g_1^n)^{-1}(v) = \{v_1,\dots,v_r\}$. By Lemma 1, $(g_1^n)^{-1}(\overline{\mathrm{star}}\,(v))$ is the disjoint union of the sets $\overline{\mathrm{star}}\,(v_j)$ $(j = 1,\dots,r)$, each of which is mapped isometrically onto $\overline{\mathrm{star}}\,(v)$ by g_1^n. Now $\gamma_n(I)$, being connected, is contained in some $\overline{\mathrm{star}}\,(v_j)$. Thus $\operatorname{diam}\gamma_n(I) = \operatorname{diam}\gamma_1(I)<\varepsilon$. This completes part (i). Suppose that ψ is given as in part (ii). Since $g_1^n\psi(a) = \gamma_1(1)\in\overline{\mathrm{star}}\,(v)$, $\psi(a)$ must be in some $\overline{\mathrm{star}}\,(v_k)$. However, $d(\gamma_n(0),\psi(a))<2$. Thus $j = k$. Since g_1^n is 1-1 on $\overline{\mathrm{star}}\,(v_j)$ and $g_1^n\psi(a) = g_1^n\gamma_n(1)$, we see that $\psi(a) = \gamma_n(1) = h_n^1(a)$. This completes the proof.

LEMMA 4. *There exists a positive number η'' such that for any space A and any maps $\varphi,\psi\colon A\to Y_1$ with $d(\varphi,\psi)<\eta''$, there is an η'-homotopy from φ to ψ.*

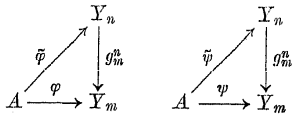
This result is well known. It can be seen by imbedding Y_1 in a Euclidean space and taking an open set that retracts onto Y_1.

Choice of η. Let η'' be chosen as in the preceding lemma, and let $\eta = \min(\eta', \eta'', 1)$.

LEMMA 5. *For any space A, any $n \geqslant 1$, and any maps $\varphi, \psi\colon A \to Y_n$ such that $d(\varphi, \psi) < \eta$, we get $\varphi \simeq \psi$.*

Proof. By Lemma 2, $d(g_1^n\varphi, g_1^n\psi) < \eta \leqslant \eta''$. Then by Lemma 4 there exists an η'-homotopy $h_1^t\colon A \to Y_1$ such that $h_1^0 = g_1^n\varphi$ and $h_1^1 = g_1^n\psi$. Since the covering map g_1^n has the covering homotopy property, there exists a homotopy $h_n^t\colon A \to Y_n$ such that $g_1^n h_n^t = h_1^t$ and $h_n^0 = \varphi$. Then $d(h_n^0, \psi) < \eta < 2$. Hence by part (ii) of Lemma 3, $\psi = h_n^1$. This completes the proof.

LEMMA 6. *Suppose that $0 < \varepsilon \leqslant \eta$, $m < n$, and there are given commutative diagrams*



where A is a connected space, and there is given an ε-homotopy $h_m^t\colon \varphi \simeq \psi$. Then there exist a covering transformation $\varrho\colon Y_n \to Y_n$ and an ε-homotopy $h_n^t\colon \varrho\widetilde{\varphi} \simeq \widetilde{\psi}$ such that $g_m^n h_n^t = h_m^t$.

Proof. By the covering homotopy property, choose a homotopy $h_n^t\colon A \to Y_n$ such that $g_m^n h_n^t = h_m^t$ and $h_n^1 = \widetilde{\psi}$. By Lemma 3, h_n^t is an ε-homotopy. Choose a point a_0 in A. Since $g_m^n\widetilde{\varphi}(a_0) = \varphi(a_0) = h_m^0(a_0) = g_m^n h_n^0(a_0)$, and since g_m^n is regular, there exists a covering transformation $\varrho\colon Y_n \to Y_n$ such that $\varrho\widetilde{\varphi}(a_0) = h_n^0(a_0)$. Since A is connected, it is easy to see from the usual open-closed argument that $\varrho\widetilde{\varphi}(a) = h_n^0(a)$ for all a in A. This completes the proof.

5. Completion of the proof of Theorem 1. Let now (X, f), (Y, g), $F\colon X_\infty \to Y_\infty$, d_∞, and $\varepsilon > 0$ be given as in the statement of Theorem 1. Let Λ be the index set for (X, f). We retain the considerations of the preceding section for (Y, g), in particular the choice of η. Clearly we may assume $\varepsilon < \eta/2$. Since Y_∞ is a compact metric space, if we prove the result for some metric on Y_∞, it is true for any other metric. Hence we may assume that d_∞ given by

$$d_\infty(y, y') = \sum_{n=1}^{\infty} 2^{-n} d\big(g_n(y), g_n(y')\big). \tag{5.1}$$

(Recall that we use the barycentric metric on each Y_n.)

A slightly weaker version of the following lemma was used by J. Mioduszewski [6]. The lemma requires only a slight modification of the proof of Theorem 11.9 in [2], p. 287.

LEMMA 7. *For each positive integer n and each $\delta > 0$ there is a $\lambda \in \Lambda$ such that for each $\beta \geqslant \lambda$ $(\beta \in \Lambda)$, there exist a map $\psi\colon X_\beta \to Y_n$ and a δ-homotopy $h^t\colon X_\infty \to Y_n$ so that $h^0 = \psi f_\beta$ and $h^1 = g_n F$. In particular, $d(\psi f_\beta, g_n F) < \delta$.*

The following lemma is the recursive step in the proof of the Theorem.

LEMMA 8. *Suppose that m and n are positive integers, $m < n$, $a \in \Lambda$, $\varphi_m\colon X_a \to Y_m$, and $h_m^t\colon X_\infty \to Y_m$ is an ε-homotopy from $\varphi_m f_a$ to $g_m F$. Then there exist* (i) *an index β in Λ such that $\beta \geqslant a$,* (ii) *a map $\varphi_n\colon X_\beta \to Y_n$ such that $\varphi_m f_a^\beta = g_m^n \varphi_n$, and* (iii) *an ε-homotopy $h_n^t\colon X_\infty \to Y_n$ from $\varphi_n f_\beta$ to $g_n F$ such that $g_m^n h_n^t = h_m^t$.*

The reader is urged to draw the appropriate mapping diagrams.

Proof. From the fact that Λ is directed, and from Lemma 7, we see that there is an index $\beta \geqslant a$ and a map $\psi\colon X_\beta \to Y_n$ such that $d(\psi f_\beta, g_n F) < \varepsilon$. Hence by Lemma 2, we have

$$d(g_m F, g_m^n \psi f_\beta) = d(g_m^n g_n F, g_m^n \psi f_\beta) < \varepsilon. \tag{5.2}$$

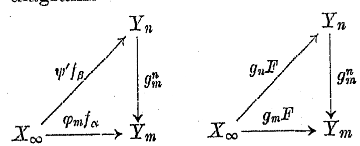
Since there is an ε-homotopy from $\varphi_m f_a$ to $g_m F$, $d(\varphi_m f_a, g_m F) < \varepsilon$. Hence

$$d(\varphi_m f_a^\beta f_\beta, g_m F) < \varepsilon. \tag{5.3}$$

The triangle inequality applied to (5.2) and (5.3) gives

$$d(\varphi_m f_a^\beta f_\beta, g_m^n \psi f_\beta) < 2\varepsilon < \eta. \tag{5.4}$$

By [2], Corollary 3.9, p. 218, f_β is onto. Hence (5.4) gives $d(\varphi_m f_a^\beta, g_m^n\psi) < \eta$. Therefore, by Lemma 5, $g_m^n \psi \simeq \varphi_m f_a^\beta$. Since g_m^n has the covering homotopy property, then there exists a map $\psi'\colon X_\beta \to Y_n$ such that $g_m^n\psi' = \varphi_m f_a^\beta$. By [2], p. 229, X_∞ is connected. Hence we may apply Lemma 6 to the two commutative diagrams



Thus there exist a covering transformation $\varrho\colon Y_n \to Y_n$ and an ε-homotopy $h_n^t\colon X_\infty \to Y_n$ from $\varrho\psi' f_\beta$ to $g_n F$ such that $g_m^n h_n^t = h_m^t$. We let $\varphi_n = \varrho\psi'\colon X_\beta \to Y_n$. Then $\varphi_m f_a^\beta = g_m^n\psi' = g_m^n\varrho\psi' = g_m^n\varphi_n$, and the proof of the lemma is complete.

Now we construct a map $\varphi\colon (X, f) \to (Y, g)$ and a homotopy $h_\infty^t\colon X_\infty \to Y_\infty$ by recursion. First, by Lemma 7, we get an index $a(1)$, a map $\varphi_1\colon X_{\alpha(1)}\to Y_1$, and an ε-homotopy $h_1^t\colon X_\infty\to Y_1$ from $\varphi_1 f_{\alpha(1)}$ to g_1F. Applying Lemma 8 recursively, we get an increasing sequence $\alpha(1)\leqslant\alpha(2)\leqslant\ldots$ of indices from Λ, sequences $\varphi=(\varphi_1,\varphi_2,\ldots)$ and $(h_1^t, h_2^t,\ldots)$ such that for each n, φ_n is a map $X_{\alpha(n)}\to Y_n$, $h_n^t\colon X_\infty\to Y_n$ is an ε-homotopy from $\varphi_n f_{\alpha(n)}$ to g_nF, $\varphi_n f_{\alpha(n)}^{\alpha(n+1)} = g_n^{n+1}\varphi_{n+1}$, and $g_n^{n+1}h_{n+1}^t = h_n^t$. Thus φ is a map $(X,f)\to(Y,g)$ and induces a map $\varphi_\infty\colon X_\infty\to Y_\infty$ by the relation $g_n\varphi_\infty=\varphi_n f_{\alpha(n)}$. Similarly, the homotopies h_n^t define a homotopy $h_\infty^t\colon X_\infty\to Y_\infty$ by the relation $g_n h_\infty^t = h_n^t$. Clearly, $h_\infty^0=\varphi_\infty$ and $h_\infty^1 = F$. Finally, by (5.1), h_∞^t is an ε-homotopy; for if $s,t\in I$, then

$$d_\infty(h_\infty^s, h_\infty^t) = \sum 2^{-n} d(h_n^s, h_n^t) < \sum 2^{-n}\varepsilon = \varepsilon\,.$$

This completes the proof of Theorem 1.

6. Proof of Theorem 2. Let (X,f), (Y,g), and $F\colon X_\infty\to Y_\infty$ be given as in the statement of the Theorem. Again, we may take the metric d_∞ on Y_∞ to be given by (5.1).

Clearly a map $\gamma\colon A\to B$ is an r-map if and only if for every map $\varphi\colon C\to B$ there exists a map $\tilde\varphi\colon C\to A$ such that $\gamma\tilde\varphi=\varphi$.

Choose a positive integer n such that $\sum_{m\geqslant n} 2^{-m} < \varepsilon/2$. By uniform continuity, there exists a $\delta>0$ such that if $d(y,y')<\delta$ in Y_n, then $d(g_m^n y, g_m^n y')<\varepsilon/2$ for all $m\leqslant n$. Now from Lemma 7 (which is also applicable in the present situation) we get an index β and a map $\varphi_n\colon X_\beta\to Y_n$ such that $d(\varphi_n f_\beta, g_nF)<\delta$. For $m\leqslant n$ define $\varphi_m\colon X_\beta\to Y_m$ by $\varphi_m = g_m^n\varphi_n$. Hence $d(\varphi_m f_\beta, g_mF)<\varepsilon/2$ for $m\leqslant n$. Now, using the fact that each g_m^{m+1} is an r-map, choose maps $\varphi_m\colon X_\beta\to Y_m$, $m>n$, such that $g_m^{m+1}\varphi_{m+1}=\varphi_m$ for $m\geqslant n$. Thus $\varphi=(\varphi_1,\varphi_2,\ldots)$ induces a map $\varphi_\infty\colon X_\infty\to Y_\infty$. Recall that diam $Y_m\leqslant 2$. Hence

$$d_\infty(\varphi_\infty, F) = \sum_m 2^{-m} d(\varphi_m f_\beta, g_mF) < \sum_{m\leqslant n} 2^{-m}(\varepsilon/2) + \sum_{m>n} 2^{-m}\cdot 2 < \varepsilon/2+\varepsilon/2 = \varepsilon\,.$$

This completes the proof.

Remark. In case the index set Λ for (X,f) is the positive integers, we can clearly alternately choose the maps φ_m, so that for $m\geqslant n$, $\varphi_m\colon X_{\beta+m-n}\to Y_m$.

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