Expansive automorphisms of finite-dimensional vector spaces

by

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1. Introduction. We call an automorphism u of a topological vector space E *expansive* if there exists a neighborhood U of 0 in E such that $x \in E$ and $x \neq 0$ implies $xu^i \notin U$ for some integer i. Such an automorphism is expansive in our sense if and only if it is expansive in the sense of Bryant [1] with respect to the uniform structure which E possesses as an abelian topological group.

For example, let E be a finite-dimensional euclidean space, let λ be a nonzero real number, and let u be the homothety $x \to \lambda x$ of E. Then u is expansive iff $|\lambda| \neq 1$. In this paper we establish the following stronger result.

THEOREM 1. *Let E be a separated finite-dimensional real or complex topological vector space and let u be an automorphism of E. Then u is expansive iff $|\lambda| \neq 1$ for each characteristic root λ of u.*

We shall deduce Theorem 1 as a special case of our main result (Theorem 2) in which the scalars are no longer restricted to real or complex numbers.

2. Definitions and statement of main result. All fields considered below are assumed to be commutative.

Let E be a finite-dimensional vector space over a field K and let u be an automorphism of E. If L is an extension field of K, we say that u *has its characteristic roots in L* provided that the characteristic polynomial of u can be written as a product of linear factors over K.

Let K be a field provided with a topology. A subset S of K is said to be *bounded in K* if for every neighborhood U of 0 in K there exists a neighborhood V of 0 in K such that $VS \subset U$, and S is said to be *unbounded in K* if it is not bounded in K.

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Let E be a vector space over a topological field K. By E^* we denote the set of all K-linear maps of E into K. If $x \in E$ and $y^* \in E^*$, we denote the value of y^* at x by $\langle x, y^*\rangle$. By the *K-linear topology of E* is meant the least topology of E with respect to which each element of E^* is continuous. Then E provided with its K-linear topology is a topological vector space over K, and any algebraic automorphism of E is homeomorphic for this topology.

In the absence of any other topology given a priori on a vector space E over a topological field K, we always provide E with its K-linear topology. In particular, if L is an extension field of a topological field K, then we consider L in the usual way as a vector space over K and accordingly provide L with its K-linear topology.

If K is a field and if $\lambda \in K$ with $\lambda \neq 0$, then $Z(\lambda)$ denotes the set of all integral powers of λ.

THEOREM 2. *Let K be a nondiscrete topological field, let E be a nonzero finite-dimensional vector space over K, and let u be an automorphism of E. Then the following statements are equivalement:*

(1) *The automorphism u is expansive.*

(2) *There exists an extension field L of K such that L is of finite degree over K, u has its characteristic roots in L, and for each characteristic root λ of u in L the set $Z(\lambda)$ is unbounded in L.*

(3) *If L is any extension field of K and if λ is a characteristic root of u in L, then the set $Z(\lambda)$ is unbounded in L.*

3. Proof of main result. Our first lemma requires no assumption regarding the dimension of E.

LEMMA 1. *Let E be a topological vector space over a nondiscrete topological field, let u be an expansive automorphism of E, and let λ be a proper value of u. Then $Z(\lambda)$ is unbounded in K.*

Proof. There exists a neighborhood U of 0 in E such that $x \in U$ and $x \neq 0$ implies $xu^i \notin U$ for some integer $i \neq 0$. Let $y \in E$ with $y \neq 0$ and $yu = \lambda y$, and choose a neighborhood V of 0 in K with $Vy \subset U$.

Suppose that $Z(\lambda)$ is bounded in K. There exists a neighborhood W of 0 in K such that $\lambda^i W \subset V$ for each integer $i \neq 0$. Choose $\mu \in W \cap V$ with $\mu \neq 0$, and let $x = \mu y$. Then $x \in U$ and $x \neq 0$, but $xu^i \in U$ for each nonzero integer i.

For the remainder of this section let K, E, and u be as in the hypotheses of Theorem 2.

LEMMA 2. *Let u have its characteristic roots in K, and let $Z(\lambda)$ be unbounded in K for each proper value λ of u. Then u is expansive.*

Proof. Let E have dimension n. With respect to a suitable base $(e_i \mid i = 1, \ldots, n)$ of E the matrix of u is a triangular matrix $A = (\lambda_{ij})$ whose diagonal entries $\lambda_{11}, \ldots, \lambda_{nn}$ are just the proper values of u. For each $1 \leqslant i \leqslant n$ there exists a neighborhood V_i of 0 in K such that $W_0 Z(\lambda_{ii}) \not\subset V_i$ for each neighborhood W_0 of 0 in K. Let $V = \bigcap_i V_i$, let W be a neighborhood of 0 in K with $W^2 \subset V$, let $(e_i^* \mid i = 1, \ldots, n)$ be the base of E^* dual to $(e_i \mid i = 1, \ldots, n)$, and let

$$U = \{x \mid x \in E \ \&\ \langle x, e_i^*\rangle \in W \ \ (i = 1, \ldots, n)\},$$

whence U is a neighborhood of 0 in E.

Let $x \in U$ with $x \neq 0$. We show that $xu^k \notin U$ for some integer $k \neq 0$. Choose $1 \leqslant m \leqslant n$ such that $\langle x, e_m^*\rangle \neq 0$ and $\langle x, e_i^*\rangle = 0$ for $1 \leqslant i < m$. Let $W_0 = W\langle x, e_m^*\rangle$, whence W_0 is a neighborhood of 0 in K. There exists an integer $k \neq 0$ with $W_0 \lambda_{mm}^k \not\subset V$. Since A^k is a triangular matrix having $\lambda_{11}^k, \ldots, \lambda_{nn}^k$ on its principal diagonal, direct computation yields $\langle xu^k, e_m^*\rangle = \langle x, e_m^*\rangle \lambda_{mm}^k$. Hence $xu^k \notin U$, for otherwise $\langle x, e_m^*\rangle \lambda_{mm}^k \in W$, and so $W_0 \lambda_{mm}^k \subset V$.

Given an extension field L of K, we denote by E_L the vector space $L \otimes_K E$ over L; if L is a topological field, then E_L is of course provided with its L-linear topology.

LEMMA 3. *Let L be a topological field containing K as a topological subfield, and let f be the cannonical injection $x \to 1 \otimes x$ of E into E_L. Then f is homeomorphic.*

Proof. Let g be the canonical map of $(E^*)_L$ into $(E_L)^*$, whence $\mu \in L$, $y \in E$, $\lambda \in L$, and $x^* \in E^*$ implies

$$\langle \mu \otimes y, (\lambda \otimes x^*)g\rangle = \lambda\mu \langle y, x^*\rangle .$$

Since E is finite-dimensional, g is bijective.

We show that f is continuous. Let $y^* \in (E_L)^*$. It is enough to show that $fy^*\colon E \to L$ is continuous. We may write $y^* = \big(\sum_j \lambda_j \otimes x_j^*\big)g$ for some $\lambda_1, \ldots, \lambda_p \in L$ and $x_1^*, \ldots, x_p^* \in E^*$. Then $x \in E$ implies $\langle xf, y^*\rangle = \sum_j \lambda_j \langle x, x_j^*\rangle$, and the continuity of fy^* follows at once.

We show that f is an open map of E onto Ef. Let $x^* \in E^*$ and let U be a neighborhood of 0 in K, whence

$$\{x \mid x \in E \ \&\ \langle x, x^*\rangle \in U\}$$

is a typical neighborhood of 0 in E. Let V be a neighborhood of 0 in L such that $V \cap K = U$, and let $y^* = (1 \otimes x^*)g$. If $x \in E$ and $\langle xf, y^*\rangle \in V$, then $\langle x, x^*\rangle = \langle xf, y^*\rangle \in V \cap K = U$.

Let L be an extension field of K which is of finite degree over K. Then L provided with its K-linear topology is a topological field containing K as a topological subfield. This fact is proved by Hinrichs [2] for L a simple algebraic extension of K, and our assertion follows easily from Hinrichs' result.

Proof of Theorem 2. Assume (1). We show (3). Let L be an extension of K and let λ be a characteristic root of u in L. Let F be the subfield $K(\lambda)$ of L, and let u_F be the automorphism of E_F induced by u, whence $\mu \in F$ and $x \in E$ implies $(\mu \otimes x)u_F = \mu \otimes xu$. Then F is of finite degree over K and λ is a proper value of u_F. Now the K-linear topology of L induces on F the K-linear topology of F. In order to show that $Z(\lambda)$ is unbounded in L it is therefore enough to show that $Z(\lambda)$ is unbounded in F. In view of Lemma 1 it is enough to show that u_F is an expansive automorphism of E_F.

We show that u_F is expansive. By (1) there exists a neighborhood N of 0 in E such that $x \in E$ and $x \neq 0$ implies $xu^r \notin N$ for some integer r. We may assume that N has the form

$$N = \{x|\, x \in E \;\&\; \langle x, x_j^*\rangle \in U \;\; (j = 1, \ldots, m)\}$$

where $x_1^*, \ldots, x_m^* \in E^*$ and U is a neighborhood of 0 in K. For $j = 1, \ldots, m$, let y_j^* be that element of $(E_F)^*$ such that $\mu \in F$ and $x \in E$ implies $\langle \mu \otimes x, y_j^*\rangle = \mu\langle x, x_j^*\rangle$. Let $(\mu_i|\, i = 1, \ldots, n)$ be a base of F over K, let $(\mu_i^*|\, i = 1, \ldots, n)$ be the dual base of F^*, let

$$W = \{\mu|\, \mu \in F \;\&\; \langle \mu, \mu_i^*\rangle \in U \;\; (i = 1, \ldots, n)\},$$

and let

$$V = \{z|\, z \in E_F \;\&\; \langle z, y_j^*\rangle \in W \;\; (j = 1, \ldots, m)\}.$$

Let $z \in E_F$ with $z \neq 0$. Suppose that $zu_F^r \in V$ for every integer r. Choose $\eta_1, \ldots, \eta_p \in F$ and $y_1, \ldots, y_p \in E$ with $z = \sum_k \eta_k \otimes y_k$. Defining

$$x_i = \sum_k \langle \eta_k, \mu_i^*\rangle y_k \qquad (i = 1, \ldots, n),$$

we have $z = \sum_i \mu_i \otimes x_i$. There exists $1 \leqslant t \leqslant n$ such that $x_t \neq 0$. Then for each integer r we have

$$\sum_i \mu_i \otimes x_i u^r = zu_F^r \in V,$$

$$\sum_i \langle x_i u^r, x_j^*\rangle \mu_i \in W \qquad (j = 1, \ldots, m),$$

$$\langle x_t u^r, x_j^*\rangle \in U \qquad (j = 1, \ldots, m),$$

and $x_t u^r \in N$, which is impossible. Hence u_F is expansive.

By taking for L a field of roots of the characteristic polynomial of u, (2) follows at once from (3).

Assume (2). We show (1). The automorphism u_L of E_L induced by u has its characteristic roots in L. Then u_L is expansive by Lemma 2, and consequently u is expansive by Lemma 3.

4. Specialization to valued fields. Recall that a *valued field* is a field K provided with a map $x \to |x|$ of K into the nonnegative reals such that $|x| = 0$ iff $x = 0$, $|x+y| \leqslant |x|+|y|$, and $|xy| = |x|\cdot|y|$ $(x, y \in K)$. If K is a valued field, we provide K with the metric $(x, y) \to |x-y|$ under which K is a topological field.

Remark. A subset S of a nondiscrete valued field K is bounded in K iff there is a real number $c > 0$ such that $|x| < c$ for all $x \in S$.

THEOREM 3. *Let K be a complete nondiscrete valued field, let L be an algebraic closure of K, and provide L with its unique absolute value which extends the absolute value on K. Let E be a nonzero separated finite-dimensional topological vector space over K, and let u be an automorphism of E. Then a necessary and sufficient condition for u to be expansive is that $|\lambda| \neq 1$ for each characteristic root λ of u in L.*

Proof. Since E is homeomorphically isomorphic with the product space K^n, where n is the dimension of E, the topology of E is its K-linear topology. Similarly, if F is any subfield of L containing K and of finite degree over K, then the topology of F induced by the absolute value on L is just the K-linear topology of F. Now use Theorem 2 and the preceding Remark.

Theorem 1 is an immediate consequence of Theorem 3.

5. Example. As an application of Theorem 2 in which K is not a valued field, let K be a simple transcendental extension of the field C of complex numbers. Williamson [3] has shown that there exists a metrizable topology $\mathcal{T}$ on K for which K is a topological field containing C (with its usual topology) as a topological subfield. Now $\mathcal{T}$ cannot be induced by an absolute value on K, for otherwise K would be a normed division algebra over C and hence by the Gelfand-Mazur theorem would be isomorphic to C. Provide K with $\mathcal{T}$, let E be the vector space $K \times K$ over K, and provide E with its product toplogy. Then the automorphism u of E such that $(1, 0)u = (2, 1)$ and $(0, 1)u = (0, 1/2)$ is expansive.

6. Remarks. Trivial modifications in our arguments yield necessary and sufficient conditions for u to be *positively expansive*, that is, for the existence of a neighborhood U of 0 in E such that $x \in E$ and $x \neq 0$ implies $xu^i \notin U$ for some *nonnegative* integer i. Statements of these conditions are obtained by redefining $Z(\lambda)$ in Theorem 2 to be the set of all nonnegative powers of λ and by replacing the inequality $|\lambda| \neq 1$ in Theorems 1 and 3 by the inequality $|\lambda| > 1$.

It would be desirable to extend Theorem 1 in two directions: (1) replace the single automorphism u by a group of automorphisms of E; (2) remove the assumption that E has finite dimension and use the weak topology of E.

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Expansive transformation semigroups of endomorphisms

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1. Introduction. A number of examples of expansive homeomorphisms [1] on compact uniform spaces are actually automorphisms of topological groups: the symbolic flows ([4], 12.24), a homeomorphism on the dyadic solenoid constructed by Williams [7], expansive homeomorphisms of tori ([5], [6]). We formulate and study below notions of a semigroup being expansive when it acts on a uniform space and of a semigroup being regionally expansive when it acts with fixed points on a topological space; these two notions coincide in the case of a semigroup of endomorphisms of a topological group.

In section 4 we generalize the example of Williams by showing how to manufacture an expansive automorphism out of a "positively expansive" endomorphism.

In section 5 we apply the general considerations of sections 2 and 3 in order to characterize completely the expansive automorphisms of finite-dimensional toral groups: they are the automorphisms induced by unimodular matrices whose eigenvalues have modulus different from unity. We show that the toral groups are the only compact connected Lie groups which admit expansive groups of automorphisms, and that a compact connected group G is toral if and only if the power map $x \to x^k$ of $G \to G$ is positively expansive.

2. Expansive transformation semigroups. A topological space T provided with an associative binary operation for which T has a bilateral identity 1 is called a *topologized semigroup* and, if T is discrete, a *discrete semigroup*.

A *transformation semigroup* (tsg) is a triple (X, T, π) where X is a topological space, T is a topologized semigroup, and π is a continuous map of $X \times T$ into X such that, if π^t denotes the map $x \to (x, t)\pi$ of $X \to X$ for each $t \in T$, then π^1 is the identity map of X and $\pi^{tr} = \pi^t \pi^r$ for all t,

* These results were obtained while the author was a National Science Foundation Cooperative Graduate Fellow at Wesleyan University and are contained in a doctoral dissertation written under the supervision of Professor W. H. Gottschalk.