The Brown-McCoy radical in categories

by

A. Suliński (Warszawa)

Contents

Introduction. The purpose of this paper is to present an exposition of the theory of the Brown-McCoy radical in categories. This exposition is based on [5] and [9]. In § 1 we shortly recall the fundamental concepts which we shall need later. For details, see [5] and [9]. In § 2 we introduce the subdirect embedding of an object into a direct product of objects, and in § 3 we present an exposition of the general theory of radicals (especially upper radicals) based on [4]. In § 4 we construct the Brown-McCoy radical as the upper radical defined by some class of simple objects. For the radical thus defined we prove facts analogous to those which are known in the category of rings. Finally in § 5 we consider the subdirect products of semi-simple objects.

§ 1. Fundamental concepts. Let $\mathcal{K}$ be a class of elements. The elements of $\mathcal{K}$ will be called *objects* and will be denoted by small Latin letters. Let us assume that to each ordered pair (a, b) of objects of $\mathcal{K}$ corresponds a set $H(a, b)$. The elements of the set $H(a, b)$ will be called *maps* of the object a into the object b and will be denoted by small Greek letters. For convenience, instead of $\alpha \in H(a, b)$ we shall write $\alpha\colon a \to b$. The class $\mathcal{K}$ is said to be a *category* if the following conditions are satisfied

(C_1) *If* $\alpha\colon a \to b$ *and* $\beta\colon b \to c$, *then there is a uniquely defined map* $\alpha\beta\colon a \to c$ *which is called the product of the maps* α *and* β;

(C_2) *If* $\alpha\colon a \to b$, $\beta\colon b \to c$, $\gamma\colon c \to d$ *then*

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)\,;$$

(C_3) *For each object a of $\mathcal{K}$ there is such a map ε_a: $a\to a$ that for any α: $b\to a$ and β: $a\to c$ we have $\alpha\varepsilon_a=\alpha$, $\varepsilon_a\beta=\beta$.*

The map ε_a will be called the *identity map* of the object a. It is easy to see that ε_a is uniquely defined for a given object a.

In this paper we shall consider categories satisfying some additional requirements, which will be introduced successively after the notions necessary to formulate them have been explained.

Two objects a and b are said to be *equivalent* if there are such maps ξ: $a\to b$ and ξ^{-1}: $b\to a$ that $\xi\xi^{-1}=\varepsilon_a$ and $\xi^{-1}\xi=\varepsilon_b$. This fact will be denoted by $a\sim b\,(\xi,\xi^{-1})$.

A map μ: $a\to b$ is said to be a *monomorphism* if for any maps α: $c\to a$, β: $c\to a$ from $\alpha\mu=\beta\mu$ it follows that $\alpha=\beta$. If μ_1: $a\to b$, μ_2: $b\to c$ are monomorphisms, then $\mu_1\mu_2$: $a\to c$ is also a monomorphism. Moreover, if μ is a monomorphism and $\mu=\sigma\delta$, then σ is a monomorphism.

Let a be an object of $\mathcal{K}$. Let us consider all pairs of the form (b,μ), where μ: $b\to a$ is a monomorphism. We shall say that $(b,\mu)\leqslant(b',\mu')$ if there is such a ϱ: $b\to b'$ that $\varrho\mu'=\mu$. Two pairs (b,μ) and (b',μ') are said to be *equivalent* if $(b,\mu)\leqslant(b',\mu')$ and $(b',\mu')\leqslant(b,\mu)$. The equivalence classes of the relation thus defined will be called the *subobjects* of the object a. For convenience the equivalence class determined by the pair (b,μ) will also be denoted by (b,μ). We shall say that *a subobject (b_1,μ_1) is contained in a subobject (b_2,μ_2)* if for the pairs (b_1,μ_1), (b_2,μ_2) we have $(b_1,\mu_1)\leqslant(b_2,\mu_2)$.

An object 0 of $\mathcal{K}$ is said to be a *zero object* if for any object a of $\mathcal{K}$ each of the sets $H(a,0)$ and $H(0,a)$ contains only one map. Two zero objects of $\mathcal{K}$ are always equivalent.

We assume that

(C_4) *$\mathcal{K}$ possesses zero objects.*

We shall say that $\mathcal{K}$ is a *category with zero maps* if for every ordered pair ob objects a, b there is such a map ω_{ab}: $a\to b$ that for any α: $c\to a$, β: $b\to d$ we have $\alpha\omega_{ab}=\omega_{cb}$ and $\omega_{ab}\beta=\omega_{ad}$. If a category $\mathcal{K}$ possesses zero objects, then $\mathcal{K}$ is a category with zero maps. Moreover, an object a of $\mathcal{K}$ is a zero object if and only if $\varepsilon_a=\omega_{aa}$. If there is no doubt between which objects the zero map operates, then that zero map will be shortly denoted by ω.

A subobject (k,μ) of an object a of $\mathcal{K}$ is said to be a *kernel* of the map α: $a\to b$ if: (1) $\mu\alpha=\omega$; (2) for each δ: $c\to a$ satisfying the condition $\delta\alpha=\omega$ there is such δ': $c\to k$ that $\delta'\mu=\delta$. If (k,μ) is a kernel of α: $a\to b$, then the subobject (k,μ) of a will be called the *ideal* of a. Then we shall write $(k,\mu)=\ker\alpha$.

We assume that

(C_5) *Every map has a kernel.*

A map τ: $a\to b$ with the kernel (k,μ) is said to be an ***epimorphism*** ([1]) if for each γ: $a\to c$ satisfying the condition $\mu\gamma=\omega$ there is a unique γ': $b\to c$ such that $\tau\gamma'=\gamma$. If τ: $a\to b$ is an epimorphism, then from $\tau\alpha=\tau\beta$ it follows that $\alpha=\beta$ for any α: $b\to c$, β: $b\to c$. If λ, ϱ are epimorphisms and $\lambda=\varrho\sigma$ then σ is also an epimorphism. Every equivalence map ξ: $a\to b$—i.e. such a map that $a\sim b\,(\xi,\xi^{-1})$—is simultaneously a monomorphism and an epimorphism. If a map ξ: $a\to b$ is simultaneously a monomorphism and an epimorphism, then $a\sim b\,(\xi,\xi^{-1})$. The subobject (a,ε_a) of a is an ideal of a. A map α: $a\to b$ is a zero map if and only if $\ker\alpha=(a,\varepsilon_a)$. If for a given object a there is an object b such that ω_{ab}: $a\to b$ is a monomorphism (or ω_{ba}: $b\to a$ an epimorphism), then a is zero object.

If a map α: $a\to b$ can be represented in the form $\alpha=\tau\nu$, where τ: $a\to l$ is an epimorphism and ν: $l\to b$ is a monomorphism, then the triplet (τ,l,ν) will be called the *image* of the map α. Then of course $\ker\alpha=\ker\tau$.

We assume that

(C_6) *Every map has an image.*

A map μ: $a\to b$ is a monomorphism if and only if $\ker\mu=(0,\omega)$. If (τ,l,ν) is the image of a map α: $a\to b$, then α is an epimorphism if and only if $(l,\nu)=(b,\varepsilon_b)$.

Let (k,μ) be a subobject of an object a and let α: $a\to b$ be an epimorphism. If (τ,l,ν) is the image of the map $\mu\alpha$: $k\to b$, then the subobject (l,ν) of b will be called the *image of (k,μ) by the epimorphism α*.

(C_7) *An image of an ideal by an epimorphism is always an ideal.*

Let a_i, $i\in I$, be a family of objects of the category $\mathcal{K}$. An object g of $\mathcal{K}$ is said to be a *direct product* of the objects a_i, $i\in I$, if there are such maps π_i: $g\to a_i$, $i\in I$, (called the *projections* of g onto a_i) that for each object h of $\mathcal{K}$ and for any system of maps α_i: $h\to a_i$, $i\in I$, there is a unique map γ: $h\to g$ such that $\gamma\pi_i=\alpha_i$ for all $i\in I$. This direct product g will be denoted by $g=\prod_{i\in I}a_i\,(\pi_i)$. The projections π_i are epimorphisms. Moreover, for each $i\in I$ there is a uniquely defined monomorphism ϱ_i: $a_i\to g$ such that $\varrho_i\pi_i=\varepsilon_{a_i}$ and $\varrho_i\pi_j=\omega$ for $i\neq j$. Therefore the direct product g will sometimes be denoted by $g=\prod_{i\in I}a_i\,(\pi_i;\ \varrho_i)$ or,

([1]) A more suitable term would be a normal epimorphism as used in [5], where epimorphism means a notion dual to that of monomorphism. Every normal epimorphism is of course an epimorphism. The converse statement is not true (for a corresponding example, see [5]), although in the category of groups or of rings every epimorphism is normal. In this paper we shall consider only normal epimorphisms, which for convenience will shortly be called *epimorphisms*.

in the case of two summands, by $g = a_1 \times a_2 (\pi_1, \pi_2; \varrho_1, \varrho_2)$. In this case, $(a_1, \varrho_1) = \ker \pi_2$ and $(a_2, \varrho_2) = \ker \pi_1$. Moreover, if $l \sim g\,(\xi, \xi^{-1})$ then $l = a_1 \times a_2 (\xi\pi_1, \xi\pi_2)$.

It is easy to formulate a definition dual to that of the direct product. This leads to the concept of free product.

We assume that

(C_8) *Every family of objects has a direct product and a free product.*

(C_9) *The class of all subobjects of any object is a set.*

Under these assumptions every family (k_i, μ_i), $i \in I$, of subobjects of an object a of $\mathcal{K}$ has a union (k, μ), i.e. $(k, \mu) \geqslant (k_i, \mu_i)$ for $i \in I$ and each subobject (l, δ) of a containing all (k_i, μ_i) is contained in (k, μ). This means that the set L_a of all subobject of a is a complete lattice.

We assume that

(C_{10}) *For each object a of $\mathcal{K}$ the set of all ideals of a is a complete sublattice of the lattice L_a.*

Let α: $a \to b$ be an epimorphism and let (m, χ) be an ideal of b. If (p, ϱ) is an ideal of a such that $(p, \varrho) \geqslant (k, \mu) = \ker \alpha$, and (m, χ) is the image of (p, ϱ) by α, then the ideal (p, ϱ) will be called the *complete counterimage* of (m, χ) by α. We shall need the following facts proved in [9].

THEOREM 1.1. *Let α: $a \to b$ be an epimorphism and let (m, χ) be an ideal of b. Then there exists a unique (up to an equivalence) complete counterimage of (m, χ) by α.*

THEOREM 1.2. *Let (k_1, μ_1), (k_2, μ_2) be two ideals of an object a, let α_1: $a \to b$ be such an epimorphism that $(k_1, \mu_1) = \ker \alpha_1$, and let (m, χ) be the image of (k_2, μ_2) by α_1. Then the union $(k_1, \mu_1) \cup (k_2, \mu_2)$ is the complete counterimage of (m, χ) by α_1.*

THEOREM 1.3. *If α: $a \to b$, β: $b \to c$ are epimorphisms, then $\alpha\beta$: $a \to c$ is also an epimorphism.*

THEOREM 1.4. *Let α: $a \to b$ be an epimorphism and let (m_i, χ_i), $i = 1, 2$, be two ideals of b. If $(m_1, \chi_1) \leqslant (m_2, \chi_2)$ then $(p_1, \varrho_1) \leqslant (p_2, \varrho_2)$, where (p_i, ϱ_i), $i = 1, 2$, is the complete counterimage of (m_i, χ_i) by α.*

§ 2. Direct products and subdirect embeddings.

PROPOSITION 2.1. *Let (k_i, μ_i), $i \in I$, be a family of ideals of an object a. Consider the direct product $g = \prod_{i \in I} a_i (\pi_i)$, where α_i: $a \to a_i$ are epimorphisms such that $(k_i, \mu_i) = \ker \alpha_i$. If γ: $a \to g$ is a map such that $\gamma\pi_i = \alpha_i$ for $i \in I$, then the ideal $(k, \mu) = \ker \gamma$ is the intersection of all (k_i, μ_i), $i \in I$, i.e. (k, μ) is contained in all (k_i, μ_i) and for any subobject (l, δ) of a contained in all (k_i, μ_i) we have $(l, \delta) \leqslant (k, \mu)$.*

Proof. For each $i \in I$ there is such a $\bar{\mu}_i$: $k \to k_i$ that $\bar{\mu}_i \mu_i = \mu$ since $\mu\alpha_i = \mu\gamma\pi_i = \omega\pi_i = \omega$ and $(k_i, \mu_i) = \ker \alpha_i$. Therefore (k, μ) is contained in all (k_i, μ_i).

Suppose that (l, δ) is contained in all (k_i, μ_i). Then for each $i \in I$ there is such δ_i: $l \to k_i$ that $\delta_i \mu_i = \delta$. Hence $(\delta\gamma)\pi_i = \delta(\gamma\pi_i) = \delta\alpha_i = \delta_i \mu_i \alpha_i = \delta\omega = \omega$. But on the other hand $\omega\pi_i = \omega$. Therefore by the uniqueness of $\delta\gamma$: $l \to g$ we get $\delta\gamma = \omega$. Hence there is such a $\bar{\mu}: l \to k$ that $\bar{\mu}\mu = \delta$ since $(k, \mu) = \ker \gamma$. Therefore $(l, \delta) \leqslant (k, \mu)$.

DEFINITION 2.2. An object a is said to be *subdirectly embedded* into the direct product $g = \prod_{i \in I} a_i (\pi_i)$ if there is such a monomorphism γ: $a \to g$ that all maps $\alpha_i = \gamma\pi_i$: $a \to a_i$, $i \in I$, are epimorphisms.

THEOREM 2.3. *An object a can be subdirectly embedded into the direct product $g = \prod_{i \in I} a_i (\pi_i)$ if and only if there is such a family of ideals (k_i, μ_i), $i \in I$, of a that $(k_i, \mu_i) = \ker \alpha_i$, where α_i: $a \to a_i$ are epimorphisms, and $(k, \mu) = (0, \omega)$, where (k, μ) is the intersection of all (k_i, μ_i), $i \in I$.*

Proof. Let a be subdirectly embedded into g by a monomorphism γ: $a \to g$. If $(k_i, \mu_i) = \ker \gamma\pi_i$ and (k, μ) is the intersection of all (k_i, μ_i), $i \in I$, then, by Proposition 2.1, $(k, \mu) = \ker \gamma$. Therefore $(k, \mu) = (0, \omega)$ since γ is a monomorphism. Conversely, let (k_i, μ_i) be such a family of ideals of a that $(k_i, \mu_i) = \ker \alpha_i$, where α_i: $a \to a_i$ are epimorphisms and the intersection $(k, \mu) = (0, \omega)$. Then there is such a γ: $a \to g$ that $\gamma\pi_i = \alpha_i$ for $i \in I$. Applying Proposition 2.1 again, we get $\ker \gamma = (k, \mu) = (0, \omega)$. Therefore γ is a monomorphism.

An object is said to be *subdirectly irreducible* if the intersection of all its non-zero ideals is a non-zero ideal. The author does not know whether every object of a category satisfying (C_1)-(C_{10}) can be subdirectly embedded into a direct product of subdirectly irreducible objects. Such a theorem for universal algebras was proved in [1].

PROPOSITION 2.4 (Šulgeifer [10]). *Let (b_1, ν_1), (b_2, ν_2) be two ideals of an object a and let $(k, \mu) = (b_1, \nu_1) \cap (b_2, \nu_2)$. If μ_1: $k \to b_1$, μ_2: $k \to b_2$ are such monomorphisms that $\mu_1 \nu_1 = \mu_2 \nu_2 = \mu$, then $(k, \mu_2) = \ker \nu_2 \alpha_1$, where α_1: $a \to a_1$ is the epimorphism with the kernel (b_1, ν_1).*

Proof. We have $\mu_2 \nu_2 \alpha_1 = \mu_1 \nu_1 \alpha_1 = \mu_1 \omega = \omega$. Let φ: $f \to b_2$ be such a map that $\varphi\nu_2\alpha_1 = \omega$. Let us denote by α_2: $a \to a_2$ the epimorphism with the kernel (b_2, ν_2) and let us consider the direct product $g = a_1 \times a_2 (\pi_1, \pi_2)$. Then there is such a γ: $a \to g$ that $\gamma\pi_1 = \alpha_1$, $\gamma\pi_2 = \alpha_2$. We have $\varphi\nu_2\gamma\pi_1 = \varphi\nu_2\alpha_1 = \omega$ and $\varphi\nu_2\gamma\pi_2 = \varphi\nu_2\alpha_2 = \varphi\omega = \omega$. But, on the other hand, $\omega_{fg}\pi_1 = \omega$ and $\omega_{fg}\pi_2 = \omega$. Hence by the uniqueness of $\varphi\nu_2\gamma$: $f \to g$ we have $\varphi\nu_2\gamma = \omega$. Then there is such a φ': $f \to k$ that $\varphi'\mu = \varphi\nu_2$ since, by Proposition 2.1, $(k, \mu) = \ker \gamma$. Therefore $\varphi\nu_2 = \varphi'\mu = \varphi'\mu_2\nu_2$, whence $\varphi = \varphi'\mu_2$ since ν_2 is a monomorphism.

THEOREM 2.5. *Let* (k_1, μ_1), (k_2, μ_2) *be such ideals of* a *that* $(k_1, \mu_1) \cap (k_2, \mu_2) = (0, \omega)$ *and* $(k_1, \mu_1) \cup (k_2, \mu_2) = (a, \varepsilon_a)$. *If* $a_i\colon a \to a_i$, $i = 1, 2$, *is such an epimorphism that* $(k_i, \mu_i) = \ker a_i$, *then* $a_1 \sim k_2 (\xi, \xi^{-1})$, $a_2 \sim k_1 (\eta, \eta^{-1})$ *and the object* a *is a direct product of* a_1 *and* a_2.

Proof. If we take $g = a_1 \times a_2 (\pi_1, \pi_2; \varrho_1, \varrho_2)$, then there exists a unique $\gamma\colon a \to g$ such that $\gamma\pi_i = a_i$ for $i = 1, 2$. By Proposition 2.1, $\ker\gamma = (k_1, \mu_1) \cap (k_2, \mu_2) = (0, \omega)$. Therefore γ is a monomorphism.

The map $\mu_2 a_1\colon k_2 \to a_1$ is a monomorphism since by Proposition 2.4 $\ker \mu_2 a_1 = (0, \omega)$. We shall show that $\mu_2 a_1$ is also an epimorphism. Indeed by (C_6) we have $\mu_2 a_1 = \tau\nu$, where $\tau\colon k_2 \to l$ is an epimorphism and $\nu\colon l \to a_1$, a monomorphism. By (C_7), (l, ν) is an ideal of a_1. But, on the other hand, by Theorem 1.2, the ideal $(k_1, \mu_1) \cup (k_2, \mu_2) = (a, \varepsilon_a)$ is a complete counterimage of (l, ν) by the epimorphism a_1. Then $(l, \nu) = (a_1, \varepsilon_a)$, i.e. $\mu_2 a_1$ is an epimorphism. Therefore we have proved that $k_2 \sim a_1(\xi, \xi^{-1})$, where $\xi = \mu_2 a_1$. In an analogous way one can prove that $k_1 \sim a_2(\eta, \eta^{-1})$, where $\eta = \mu_1 a_2$.

Let us consider the following maps: $\omega\colon k_2 \to a_2$, $\xi\colon k_2 \to a_1$. There is a unique map $\delta\colon k_2 \to g$ such that $\delta\pi_2 = \omega$ and $\delta\pi_1 = \xi$. But, on the other hand, $(\mu_2\gamma)\pi_2 = \mu_2 a_2 = \omega$ and $(\mu_2\gamma)\pi_1 = \mu_2 a_1 = \xi$. Moreover, $(\xi\varrho_1)\pi_2 = \xi\omega = \omega$ and $(\xi\varrho_1)\pi_1 = \xi\varepsilon = \xi$. Therefore by the uniqueness of $\delta\colon k_2 \to g$ we have $\delta = \mu_2\gamma = \xi\varrho_1$, i.e. $(k_2, \mu_2\gamma) \leqslant (a_1, \varrho_1)$. Moreover, $\xi^{-1}(\mu_2\gamma) = \xi^{-1}\xi\varrho_1 = \varrho_1$, whence $(k_1, \mu_2\gamma) \geqslant (a_1, \varrho_1)$. Therefore the subobjects $(k_2, \mu_2\gamma)$ and (a_1, ϱ_1) are equivalent, hence $(k_2, \mu_2\gamma)$ is an ideal of g since (a_1, ϱ_1) is an ideal of g. In an analogous way one can prove that $(k_1, \mu_1\gamma)$ is an ideal of g equivalent to (a_2, ϱ_2). But, on the other hand, $\varrho_1\pi_1 = \varepsilon_{a_1}$, whence (a_1, ε_{a_1}) is the image of the ideal (a_1, ϱ_1) by the epimorphism $\pi_1\colon g \to a_1$. Applying Theorem 1.2 and taking into account that $(a_2, \varrho_2) = \ker\pi_1$, we have $(a_1, \varrho_1) \cup (a_2, \varrho_2) = (g, \varepsilon_g)$. Moreover,

$$(a, \gamma) \geqslant (k_1, \mu_1\gamma) = (a_2, \varrho_2); \qquad (a, \gamma) \geqslant (k_2, \mu_2\gamma) = (a_1, \varrho_1).$$

By (C_{10}) the subobject (a, γ) of g contains the ideal $(a_1, \varrho_1) \cup (a_2, \varrho_2) = (g, \varepsilon_g)$. Therefore $a \sim g(\gamma, \gamma^{-1})$, i.e. $a = a_1 \times a_2 (\gamma\pi_1, \gamma\pi_2)$.

§ 3. General theory of radicals. Let S be a property of objects of $\mathfrak{K}$. An object a possessing the property S will be called an *S-object*. We assume that if a is an S-object and $a \sim b (\xi, \xi^{-1})$, then b is also an S-object. An ideal (l, δ) of an object a will be called an *S-ideal* if l is an S-object. If there exists an S-ideal (a_S, σ_S) of an object a which contains all S-ideals of a, then (a_S, σ_S) will be called the *S-radical* of a. An object containing no non-zero S-ideals will be called *S-semisimple*.

We shall call S a *radical property* if the following conditions hold:

(a) *If* a *is an* S*-object and* $a\colon a \to b$ *is an epimorphism, then* b *is also an* S*-object*;

(b) *For each object* a *of* $\mathfrak{K}$ *there is an* S*-radical* (a_S, σ_S) *of* a;

(c) *If* $a\colon a \to b$ *is such an epimorphism that* $(a_S, \sigma_S) = \ker a$, *then the object* b *is* S*-semisimple.*

If S is a radical property, then S-objects will be called *S-radical objects*.

Let S and S' be two radical properties. We shall say that $S \leqslant S'$ if each S-radical object is also S'-radical.

Let $\mathcal{M}$ be a class of objects of $\mathfrak{K}$. We shall say that an object a of $\mathfrak{K}$ satisfies condition $(\mathcal{M})$ if for each non-zero ideal (k, μ) of a there is such an epimorphism $\theta\colon k \to k'$, $\theta \neq \omega$, that $k' \in \mathcal{M}$. A class $\mathcal{M}$ will be called *regular* if each of its objects satisfies condition $(\mathcal{M})$. An object a is said to be an *$U_{\mathcal{M}}$-object* if there is no epimorphism $\theta\colon a \to b$, $\theta \neq \omega$, such that $b \in \mathcal{M}$.

THEOREM 3.1. *If* $\mathcal{M}$ *is a regular class, then* $U_{\mathcal{M}}$ *is a radical property.*

Proof. Let a be a $U_{\mathcal{M}}$-object and let $\theta\colon a \to b$, $\theta \neq \omega$, be an epimorphism. If b is not a $U_{\mathcal{M}}$-object, then there is such an epimorphism $\theta'\colon b \to c$, $\theta' \neq \omega$, that $c \in \mathcal{M}$. Then by Theorem 1.3 the map $\theta\theta'\colon a \to c$, $\theta\theta' \neq \omega$ is an epimorphism, which is impossible since a is a $U_{\mathcal{M}}$-object. Thus condition (a) is satisfied.

Now let a be an object of $\mathfrak{K}$. By (C_8) and (C_9) there exists a union (a_U, σ_U) of all $U_{\mathcal{M}}$-ideals of a. If $(a_U, \sigma_U) = (0, \omega)$, then the theorem is proved. Suppose that $(a_U, \sigma_U) \neq (0, \omega)$. We shall show that (a_U, σ_U) is a $U_{\mathcal{M}}$-ideal of a. Indeed, let us suppose that there is such an epimorphism $\theta\colon a_U \to b$, $\theta \neq \omega$, that $b \in \mathcal{M}$. If $(k, \mu) = \ker\theta$, then there is a $U_{\mathcal{M}}$-ideal (l, δ) of a which is not contained in the subobject $(k, \mu\sigma_U)$ of a, since otherwise each $U_{\mathcal{M}}$-ideal of a would be contained in $(k, \mu\sigma_U)$, whence, by (C_{10}), $(k, \mu\sigma_U) = (a_U, \sigma_U)$, which is impossible because of $\theta \neq \omega$. But, on the other hand, the $U_{\mathcal{M}}$-ideal (l, δ) is contained in (a_U, σ_U), i.e. there is such a $\varrho\colon l \to a_U$ that $\varrho\sigma_U = \delta$. Taking into account that $(l, \delta) \cap (a_U, \sigma_U) = (l, \delta)$ and applying Proposition 2.4 we find that (l, ϱ) is a $U_{\mathcal{M}}$-ideal of a_U. If we denote by (l', ϱ') the image of the ideal (l, ϱ) by the epimorphism $\theta\colon a_U \to b$, then $(l', \varrho') \neq (0, \omega)$ since (l, δ) is not contained in $(k, \mu\sigma_U)$. By (C_7) and condition (a), (l', ϱ') is a $U_{\mathcal{M}}$-ideal of b, which is impossible since b belongs to the regular class $\mathcal{M}$. Therefore (a_U, σ_U) is the $U_{\mathcal{M}}$-radical of a and condition (b) is satisfied.

Now let $\theta\colon a \to b$ be such an epimorphism that $(a_U, \sigma_U) = \ker\theta$ and let $(m, \nu) \neq (0, \omega)$ be a $U_{\mathcal{M}}$-ideal of b. By Theorem 1.1 there exists a unique complete counterimage (p, ϱ) of (m, ν) by θ. Then $\varrho\theta = \tau\nu$, where $\tau\colon p \to m$ is an epimorphism. We shall show that (p, ϱ) is a $U_{\mathcal{M}}$-ideal of a. Indeed, let us suppose that there is such an epimorphism $\lambda\colon p \to q$, $\lambda \neq \omega$, that $q \in \mathcal{M}$ and let $(l, \delta) = \ker\lambda$. If $\mu\colon a_U \to p$ is such a monomorphism that $\mu\varrho = \sigma_U$, then taking into account that $(a_U, \sigma_U) \cap$

$\cap (p, \varrho) = (a_U, \sigma_U)$ and applying Proposition 2.4 we obtain $(a_U, \mu) = \ker \varrho\theta$, whence $(a_U, \mu) = \ker\tau$. If $(l, \delta) \geqslant (a_U, \mu)$ then $\bar{\delta}\delta = \mu$ for a certain monomorphism $\bar{\delta}$: $a_U \to l$. Then $\mu\lambda = \bar{\delta}\delta\lambda = \bar{\delta}\omega = \omega$, whence there is such a τ^*: $m \to q$ that $\tau\tau^* = \lambda$ since τ is an epimorphism. Therefore τ^* is an epimorphism since τ and λ are epimorphisms. Moreover, $\tau^* \neq \omega$ since $\lambda \neq \omega$. But this is impossible since m is a $U_{\mathcal{M}}$-object and q belongs to the regular class $\mathcal{M}$. If, however, (a_U, μ) is not contained in (l, δ), then $(a'_U, \mu') \neq (0, \omega)$, where (a'_U, μ') is the image of the $U_{\mathcal{M}}$-ideal (a_U, μ) of p by the epimorphism λ. By (C_7) and (a), (a'_U, μ') is a $U_{\mathcal{M}}$-ideal of q, which is impossible since q belongs to the regular class $\mathcal{M}$. Therefore (p, ϱ) is a $U_{\mathcal{M}}$-ideal of a, whence $(p, \varrho) = (a_U, \sigma_U)$ and $(m, \nu) = (0, \omega)$. Thus condition (c) is also satisfied.

PROPOSITION 3.2. *If $\mathcal{M}$ is a regular class, then $U_{\overline{\mathcal{M}}} = U_{\mathcal{M}}$, where $\overline{\mathcal{M}}$ is the class of all objects of $\mathcal{K}$ satisfying condition ($\mathcal{M}$).*

Proof. We have $\mathcal{M} \subseteq \overline{\mathcal{M}}$ since the class $\mathcal{M}$ is regular. Therefore each $U_{\overline{\mathcal{M}}}$-object is $U_{\mathcal{M}}$-radical, i.e. $U_{\overline{\mathcal{M}}} \leqslant U_{\mathcal{M}}$. Conversely, let us suppose that a is not a $U_{\overline{\mathcal{M}}}$-object. Then there is such an epimorphism θ: $a \to b$, $\theta \neq \omega$, that b belongs to $\overline{\mathcal{M}}$. Hence by the definition of $\overline{\mathcal{M}}$ there is such an epimorphism θ': $b \to c$, $\theta' \neq \omega$ that c belongs to $\mathcal{M}$. By Theorem 1.3, $\theta\theta'$: $a \to c$ is an epimorphism and a cannot be a $U_{\mathcal{M}}$-object. Therefore $U_{\mathcal{M}} \leqslant U_{\overline{\mathcal{M}}}$.

PROPOSITION 3.3. *Let S be a radical property and let $\mathcal{M}$ be the class of all S-semisimple objects of $\mathcal{K}$. Then the class $\mathcal{M}$ is regular and $\overline{\mathcal{M}} = \mathcal{M}$.*

Proof. Let a be an S-semisimple object and let (l, δ) be a non-zero ideal of a. Since (l, δ) cannot be an S-ideal, then by condition (c) there is such an epimorphism θ: $l \to c$, $\theta \neq \omega$, that c is S-semisimple, i.e. c belongs to $\mathcal{M}$. Therefore the class $\mathcal{M}$ is regular and $\mathcal{M} \subseteq \overline{\mathcal{M}}$. Now let us suppose that an object a is not S-semisimple. Then $(a_S, \sigma_S) \neq (0, \omega)$ and by (a) there is no epimorphism θ: $a_S \to b$, $\theta \neq \omega$, where b is S-semisimple. Therefore a does not belong to $\overline{\mathcal{M}}$, i.e. $\overline{\mathcal{M}} \subseteq \mathcal{M}$.

THEOREM 3.4. *Let S be a radical property. If an object a can be subdirectly embedded into a direct product of S-semisimple objects, then the object a is also S-semisimple.*

Proof. Let a_i, $i \in I$, be a family of S-semisimple objects and let an object a be subdirectly embedded into the direct product $g = \prod_{i \in I} a_i (\pi_i)$. Then there is such a monomorphism γ: $a \to g$ that $\alpha_i = \gamma\pi_i$: $a \to a_i$, $i \in I$, are epimorphisms. Let (p, δ) be a non-zero ideal of a. Then there exists such an $i_0 \in I$ that $\delta\alpha_{i_0} \neq \omega$, since otherwise (p, δ) would be contained in the intersection (k, μ) of all (k_i, μ_i), where $(k_i, \mu_i) = \ker\alpha_i$. But by Theorem 2.3, $(k, \mu) = (0, \omega)$, whence $(p, \delta) = (0, \omega)$, which is impossible. By (C_6) we have $\delta\alpha_{i_0} = \tau\nu$, where τ: $p \to l$ is an epimorphism and ν: $l \to a_i$

a monomorphism. By (C_7), (l, ν) is a non-zero ideal of the S-semisimple object a_{i_0}. Then by Proposition 3.3 there is such an epimorphism θ: $l \to q$ that q is S-semisimple. By Theorem 1.3 the map $\tau\theta$: $p \to q$ is an epimorphism, whence a belongs to $\overline{\mathcal{M}}$, where $\mathcal{M}$ is the class of all S-semisimple objects. Therefore, applying Proposition 3.3 again, we find that the object a is S-semisimple.

DEFINITION 3.5. If $\mathcal{M}$ is a regular class, then the property $U_{\mathcal{M}}$ will be called the *upper radical property* defined by $\mathcal{M}$.

The term upper radical property can be explained by the following fact.

PROPOSITION 3.6. *Let S be a radical property and let $\mathcal{M}$ be a regular class. If each object from $\mathcal{M}$ is S-semisimple, then $S \leqslant U_{\mathcal{M}}$.*

Proof. Let a be an S-radical object. Then by condition (a) for each epimorphism θ: $a \to b$, $\theta \neq \omega$, the object b is S-radical, i.e. b does not belong to $\mathcal{M}$. Therefore a is $U_{\mathcal{M}}$-radical.

§ 4. Brown-McCoy radical. A non-zero object a will be called *simple* if its only ideals are (o, ω) and (a, ε_a). Let $\mathcal{M}$ be a class of simple objects of $\mathcal{K}$. We assume that if $a \in \mathcal{M}$ and $a \sim b$ (ξ, ξ^{-1}) then $b \in \mathcal{M}$. If a is an object of $\mathcal{K}$ and α: $a \to b$ is such an epimorphism that $b \in \mathcal{M}$, then by Theorem 1.1, $(m, \mu_m) = \ker\alpha$ is a maximal ideal of a. The ideal (m, μ_m) will be called an *$\mathcal{M}$-maximal ideal* of a. By (C_9) the class of all $\mathcal{M}$-maximal ideals of a is a set. This set will be denoted by M_a and will be called the *structure $\mathcal{M}$-space* of a. If $(m, \mu_m) \in M_a$ then the epimorphism with the kernel (m, μ_m) will always be denoted by α_m: $a \to a_m$, $a_m \in \mathcal{M}$. If there is no doubt that we deal with the structure $\mathcal{M}$-space of a, then instead of M_a we shall write M. If $N \subseteq M$, then by (k_N, μ_N) we denote the intersection of all $(m, \mu_m) \in N$, and by $\overline{N}$ we denote the set of all $\mathcal{M}$-maximal ideals of a containing the ideal (k_N, μ_N). Moreover, the direct product $\prod_{(m,\mu_m) \in N} a_m(\pi_m)$ will always be denoted by $g_N(\pi_m)$.

PROPOSITION 4.1. *Let (l, δ) be an ideal of an object a. If $\delta\alpha_m \neq \omega$ for some $(m, \mu_m) \in M_a$, then $\delta\alpha_m$: $d \to a_m$ is an epimorphism.*

Proof. By (C_6), $\delta\alpha_m = \tau\nu$, where τ: $l \to p$ is an epimorphism and ν: $p \to a_m$ a monomorphism. By (C_7), (p, ν) is an ideal of a_m. Therefore, $(p, \nu) = (a_m, \varepsilon)$ since a_m is simple and $\delta\alpha_m \neq \omega$.

DEFINITION 4.2. An ideal (p, δ) of an object a is said to be a *retract* of a if there is such a map π: $a \to p$ that $\delta\pi = \varepsilon_p$.

PROPOSITION 4.3. ([6].) *If $\delta\pi = \varepsilon$ then π is an epimorphism.*

Proof. Applying (C_6) to the map π: $a \to p$ we have $\pi = \tau_1\nu_1$, where τ_1: $a \to p_1$ is an epimorphism and ν_1: $p_1 \to p$, a monomorphism. Applying (C_6) to the map $\delta\tau_1$: $p \to p_1$ we have $\delta\tau_1 = \tau_2\nu_2$, where τ_2: $p \to p_2$ is an epimorphism and ν_2: $p_2 \to p_1$, a monomorphism. Then $\varepsilon_p = \delta\pi = \delta(\tau_1\nu_1)$

$= (\delta\tau_1)\nu_1 = (\tau_2\nu_2)\nu_1$, i.e. $\tau_2\nu_2\colon p \to p_1$ is a left inverse map for the monomorphism ν_1. We shall show that $\tau_2\nu_2$ is also a right inverse map for ν_1. Indeed, $[\nu_1(\tau_2\nu_2)]\nu_1 = \nu_1[(\tau_2\nu_2)\nu_1] = \nu_1\varepsilon = \nu_1$, whence $\nu_1(\tau_2\nu_2) = \varepsilon$ since ν_1 is a monomorphism. Therefore $p_1 \sim p\,(\nu_1, \nu_1^{-1})$, whence $\pi = \tau_1\nu_1$ is an epimorphism.

DEFINITION 4.4. Let $\mathcal{M}$ be a class of simple objects. An ideal (p, σ) of a will be called a *simple $\mathcal{M}$-ideal* of a if $p \in \mathcal{M}$.

DEFINITION 4.5. A class $\mathcal{M}$ of simple objects will be called *modular* if the following conditions hold:

4.5.I. If (p, σ) is a simple $\mathcal{M}$-ideal of a, then (p, σ) is a retract of a, and there is a unique ideal $(m, \mu_m) \in M_a$ such that $(p, \sigma) \cap (m, \mu_m) = (0, \omega)$;

4.5.II. If (l, δ) is an ideal of an object a and (q, σ) is an $\mathcal{M}$-maximal ideal of l, then $(q, \sigma\delta)$ is an ideal of a.

A modular class is always regular since it consists of simple objects. Therefore we can formulate the following definition.

DEFINITION 4.6. The upper radical property $U_{\mathcal{M}}$ defined by a modular class $\mathcal{M}$ will be called the *Brown-McCoy radical property*.

In the category of all alternative rings a class $\mathcal{M}$ of simple rings is modular if $\mathcal{M}$ contains no simple ring without the unity element. Indeed, let $P \in \mathcal{M}$ be an ideal of an alternative ring A. If P^* is the set of all $x \in A$ such that $xP = Px = 0$, then $A = P \oplus P^*$ since P contains the unity element. Therefore P is a retract of A. Now let M be such an $\mathcal{M}$-maximal ideal of A that $P \cap M = 0$. Then $P \cup M = A$ since M is maximal. Hence $A = P \oplus M$. If $x \in M$ then $xP, Px \subseteq P \cap M = 0$, whence $x \in P^*$, i.e. $M \subseteq P^*$. Conversely, if $x \in P^*$ then $x = p + m$, where $p \in P$, $m \in M$. If e is the unity element of P, then $0 = xe = pe + me = p + me$. Hence $p = -me \in P \cap M = 0$ since P and M are ideals of A. Therefore $x = m \in M$, whence $P^* \subseteq M$. Thus we have proved that condition 4.5.I is satisfied.

Now we shall prove 4.5.II. Let L be an ideal of an alternative ring A and let Q be an $\mathcal{M}$-maximal ideal of L. By h we denote the natural homomorphism of L onto L/Q. Take any counterimage $i \in L$ of the unity element from L/Q by the homomorphism h. We set the map $\bar{h}\colon A \to L/Q$ as follows

$$\bar{h}(x) = h(ixi) \quad \text{for any} \quad x \in A\,.$$

It is obvious that $\bar{h}$ thus defined is an additive map. Take any $x, y \in A$. Since $i(xy)i = (ix)(yi)$ (cf. [3], Lemma 3), we have $\bar{h}(xy) = h[i(xy)i] = h[(ix)(yi)] = h(ix)h(yi) = h(ix)h(i)h(i)h(yi) = h(ixi)h(iyi) = \bar{h}(x)\bar{h}(y)$. Therefore the map $\bar{h}$ is a homomorphism. It is a homomorphism onto L/Q since $\bar{h}$ coincides with h on the ideal L. Moreover, $\ker\bar{h} \cap L = Q$, whence Q is an ideal of A and condition 4.5.II is also satisfied.

If $\mathcal{M}$ consists of all simple rings with the unity element, then $U_{\mathcal{M}}$ becomes the well-known Brown-McCoy radical ([2] and [8]). Radicals $U_{\mathcal{M}}$, where $\mathcal{M}$ consists of some simple rings with the unity element were considered in [11].

In the category of Lie rings a class $\mathcal{M}$ of simple complete rings (a Lie ring is said to be *complete* if its centre is zero and each of its derivations is an inner derivation), satisfies condition 4.5.I. Indeed, let $P \in \mathcal{M}$ be an ideal of a Lie ring A. By P^* we denote the set of all $x \in A$ such that $xP = 0$. If $a \in A$ and $b \in P^*$, then, by the Jacobi identity, $(ab)P \subseteq (bP)a + (Pa)b$. But $bP = 0$ since $b \in P^*$. Moreover, $Pa \subseteq P$, whence $(Pa)b = 0$. Therefore $(ab)P = 0$, i.e. P^* is an ideal of A. Now take any $a \in A$ and consider the derivation defined as $\delta_a(x) = xa$ for $x \in A$. We have $\delta_a P \subseteq P$ since P is an ideal of A. From the fact that P is complete we infer that δ_a induces an inner derivation on P, i.e. there is such a $p_0 \in P$ that for any $p \in P$ we have $\delta_a(p) = pa = pp_0$. We set $b = a - p_0$. Then $pb = p(a - p_0) = pa - pp_0 = 0$ for any $p \in P$. Therefore $b \in P^*$ and $a = b + p_0$. Hence $A = P^* + P$. Now let $c \in P \cap P^*$. Then $cp = 0$ for any $p \in P$. Therefore c belongs to the centre of P, whence $c = 0$ since P is complete. Thus we have proved that $A = P^* \oplus P$, i.e. P is a retract of A. Now let M be such an $\mathcal{M}$-maximal ideal of A that $P \cap M = 0$. Then $P \cup M = A$, whence $A = P \oplus M$. If $x \in M$ then $xP \subseteq M \cap P = 0$, whence $x \in P^*$, i.e. $M \subseteq P^*$. Conversely, let $x \in P^*$. Then $x = p_0 + m$, where $p_0 \in P$, $m \in M$. For any $p \in P$ we have $o = xp = p_0p + mp$. But $mp \in M \cap P = 0$. Then $p_0p = 0$ for any $p \in P$, which means that p_0 belongs to the centre of P. Since P is complete, we have $p_0 = 0$. Therefore $x = m \in M$ and $P^* \subseteq M$.

The author does not know whether the class $\mathcal{M}$ of simple Lie rings satisfies condition 4.5.II.

PROPOSITION 4.7. *Let (l, δ) be an ideal of an object a and let (q, σ) be an $\mathcal{M}$-maximal ideal of l. If $\mathcal{M}$ is a modular class, then there is a unique $\mathcal{M}$-maximal ideal $(m, \mu_m) \in M_a \setminus \bar{N}$ such that $(q, \sigma\delta) = (m, \mu_m) \cap (l, \delta)$, where $\bar{N}$ is the set of all $\mathcal{M}$-maximal ideals of a containing (l, δ) and $M_a \setminus \bar{N}$ is the set-theoretical complement of $\bar{N}$ in M_a.*

Proof. Let $\theta\colon l \to b$ be such an epimorphism that $(q, \sigma) = \ker\theta$ and $b \in \mathcal{M}$. By condition 4.5.II, $(q, \sigma\delta)$ is an ideal of a. Let $\lambda\colon a \to c$ be such an epimorphism that $(q, \sigma\delta) = \ker\lambda$. Then $\sigma\delta\lambda = \omega$, whence there is such a $\theta^*\colon b \to c$ that $\delta\lambda = \theta\theta^*$ since θ is an epimorphism. If $(f, \varphi) = \ker\theta^*$ then, by the simplicity of b, we have either $(f, \varphi) = (0, \omega)$ or $(f, \varphi) = (b, \varepsilon_b)$. If $(f, \varphi) = (b, \varepsilon_b)$ then $\delta\lambda = \theta\theta^* = \theta\omega = \omega$. Hence there is such a $\bar{\sigma}\colon l \to q$ that $\bar{\sigma}\sigma\delta = \delta$ since $(q, \sigma\delta) = \ker\lambda$. Thus $\bar{\sigma}\sigma = \varepsilon_l$ since δ is a monomorphism, i.e. $q \sim l\,(\sigma, \bar{\sigma})$, which leads to a contradiction since (q, σ) is an $\mathcal{M}$-maximal ideal of l. Therefore $(f, \varphi) = (0, \omega)$, i.e. θ^*:

$b \to c$ is a monomorphism. The subobject (b, θ^*) of c, as the image of the ideal (l, δ) by λ, is a simple $\mathcal{M}$-ideal of c because of (C_7). Therefore, by 4.5.I and Proposition 4.3, there is such an epimorphism $\pi\colon c \to b$ that $\theta^*\pi = \varepsilon_b$. Then we have $\theta = \theta\varepsilon_b = \theta\theta^*\pi = \delta\lambda\pi$. If we take $(m_0, \mu_0) = \ker\lambda\pi$, then $(m_0, \mu_0) \in M_a$ since by Theorem 1.3 the map $\lambda\pi\colon a \to b$ is an epimorphism and $b \in \mathcal{M}$. Suppose that $(m_0, \mu_0) \in \overline{N}$. Then $\overline{\delta}\mu_0 = \delta$ for some $\overline{\delta}\colon l \to m_0$. Hence $\theta = \delta\lambda\pi = \overline{\delta}\mu_0\lambda\pi = \overline{\delta}\omega = \omega$, which is impossible since $\ker\theta = (q, \sigma) \neq (l, \varepsilon_l)$. Therefore $(m_0, \mu_0) \in M \setminus \overline{N}$. We set $(r, \varrho) = (m_0, \mu_0) \frown (l, \delta)$. Let $\varrho_1\colon r \to l$ be such a map that $\varrho_1\delta = \varrho$. Then by Proposition 2.4 $(r, \varrho_1) = \ker\delta\lambda\pi = \ker\theta = (q, \sigma)$. Therefore $(r, \varrho_1\delta) = (q, \sigma\delta)$, i.e. $(r, \varrho) = (q, \sigma\delta)$.

Now let us take any $(m, \mu_m) \in M \setminus \overline{N}$. By (C_6) we have $\mu_m\lambda = \tau\nu$, where $\tau\colon m \to m'$ is an epimorphism and $\nu\colon m' \to c$ a monomorphism. By (C_7) and Theorem 1.4, (m', ν) is a maximal ideal of c. We shall show that (m', ν) is an $\mathcal{M}$-maximal ideal of c. Indeed, let $\alpha\colon c \to c'$ be such an epimorphism that $(m', \nu) = \ker\alpha$. Then we have $\mu_m\lambda\alpha = \tau\nu\alpha = \tau\omega = \omega$. Therefore there is such a $\xi\colon a_m \to c'$ that $\alpha_m\xi = \lambda\alpha$ since $\alpha_m\colon a \to a_m$ is the epimorphism with the kernel (m, μ_m). The map ξ is an epimorphism since $\alpha_m\xi = \lambda\alpha$ and α_m and, by Theorem 1.3, $\lambda\alpha$ are epimorphisms. But, on the other hand, $\xi \neq \omega$ since otherwise $\lambda\alpha = \omega$ and $c' = 0$, which is impossible. Then $\ker\xi \neq (a_m, \varepsilon)$, whence $\ker\xi = (0, \omega)$ since a_m is simple. Therefore ξ is also a monomorphism, i.e. $a_m \sim c'(\xi, \xi^{-1})$ and $c' \in \mathcal{M}$.

Now we shall show that for $(m, \mu_m) \in M \setminus \overline{N}$ from $(m, \mu_m) \frown (l, \delta) = (q, \sigma\delta)$ it follows that $(m', \nu) \frown (b, \theta^*) = (0, \omega)$. Indeed, since (b, θ^*) is a simple ideal of c, we have either $(m', \nu) \frown (b, \theta^*) = (0, \omega)$ or $(m', \nu) \frown (b, \theta^*) = (b, \theta^*)$. Let us suppose the second case. Then $(b, \theta^*) \leqslant (m', \nu)$. Since $(m, \mu_m) \geqslant (q, \sigma\delta) = \ker\lambda$; therefore (m, μ_m) is a complete counterimage of (m', ν) by the epimorphism λ. Moreover, (l, δ) is a complete counterimage of (b, θ^*) by λ. Then applying Theorem 1.4 we get $(l, \delta) \leqslant (m, \mu_m)$, which is impossible since $(m, \mu_m) \in M \setminus \overline{N}$.

Now let $(m_1, \mu_1), (m_2, \mu_2) \in M \setminus \overline{N}$ be such ideals that $(m_1, \mu_1) \frown (l, \delta) = (m_2, \mu_2) \frown (l, \delta) = (q, \sigma\delta)$. Then $(m_1', \nu_1) \frown (b, \theta^*) = (m_2', \nu_2) \frown (b, \theta^*) = (0, \omega)$, where (m_1', ν_1), (m_2', ν_2) are the images of (m_1, μ_1) and (m_2, μ_2) by λ. Since (m_1', ν_1) and (m_2', ν_2) are $\mathcal{M}$-maximal ideals of c, we have $(m_1', \nu_1) = (m_2', \nu_2)$ because of 4.5.I. Therefore applying Theorem 1.1 we get $(m_1, \mu_1) = (m_2, \mu_2)$.

THEOREM 4.8. *Let $U_{\mathcal{M}}$ be a Brown-McCoy radical property and let a be an object of $\mathfrak{K}$. Then the ideal (k_M, μ_M) is the $U_{\mathcal{M}}$-radical of a.*

Proof. At first we shall show that (k_M, μ_M) is a $U_{\mathcal{M}}$-ideal of a. Indeed, let us suppose that there is such an epimorphism $\theta\colon k_M \to b$, $\theta \neq \omega$, that $b \in \mathcal{M}$. Then $(q, \sigma) = \ker\theta$ is an $\mathcal{M}$-maximal ideal of k_M. Applying Proposition 4.7 we find that there is such an ideal $(m_0, \mu_0) \in M_a$

that $(q, \sigma\mu_M) = (m_0, \mu_0) \frown (k_M, \mu_M)$, which is impossible since (k_M, μ_M) is contained in each $(m, \mu_M) \in M$.

Now let us suppose that there is an $U_{\mathcal{M}}$-ideal (l, δ) of a which is not contained in (k_M, μ_M). Then there is an $(m, \mu_m) \in M$ which does not contain (l, δ). Therefore $\delta\alpha_m \neq \omega$, whence by Proposition 4.1, $\delta\alpha_m\colon l \to a_m$ is an epimorphism. But it is impossible since (l, δ) is $U_{\mathcal{M}}$-ideal and $a_m \in \mathcal{M}$.

THEOREM 4.9. *If $U_{\mathcal{M}}$ is a Brown-McCoy radical property, then each $U_{\mathcal{M}}$-semisimple object can be subdirectly embedded into a direct product of simple objects from $\mathcal{M}$.*

Proof. Let a be a $U_{\mathcal{M}}$-semisimple object. Consider the direct product $g_M(\pi_m)$ and the epimorphisms $\alpha_m\colon a \to a_m$, where $(m, \mu_m) \in M$. Then there is such a map $\gamma\colon a \to g_M$ that $\gamma\pi_m = \alpha_m$ for each $(m, \mu_m) \in M$. By Proposition 2.1, $(k_M, \mu_M) = \ker\gamma$. But, by Theorem 4.8, $(k_M, \mu_M) = (0, \omega)$ since a is $U_{\mathcal{M}}$-semisimple. Therefore γ is a monomorphism, i.e. the object a is subdirectly embedded into g_M.

THEOREM 4.10. *Let $U_{\mathcal{M}}$ be a Brown-McCoy radical property. If (l, δ) is an ideal of an object a, then $(l_U, \overline{\sigma}_U\delta) = (l, \delta) \frown (a_U, \sigma_U)$, where (a_U, σ_U) is the $U_{\mathcal{M}}$-radical of a and $(l_U, \overline{\sigma}_U)$ is the $U_{\mathcal{M}}$-radical of the object l.*

Proof. By Theorem 4.8, $(l_U, \overline{\sigma}_U)$ is the intersection of all $\mathcal{M}$-maximal ideals of the object l. Then applying Proposition 4.7 we have $(l_U, \overline{\sigma}_U\delta) = (l, \delta) \frown (k_{M\setminus\overline{N}}, \mu_{M\setminus\overline{N}})$, where $\overline{N}$ is the set of all $\mathcal{M}$-maximal ideals of a containing (l, δ). Then taking into account that $(l, \delta) = (l, \delta) \frown (k_{\overline{N}}, \mu_{\overline{N}})$ we get $(l_U, \overline{\sigma}_U\delta) = (l, \delta) \frown (k_{\overline{N}}, \mu_{\overline{N}}) \frown (k_{M\setminus\overline{N}}, \mu_{M\setminus\overline{N}}) = (l, \delta) \frown (k_M, \mu_M) = (l, \delta) \frown (a_U, \sigma_U)$ since, by Theorem 4.8, $(k_M, \mu_M) = (a_U, \sigma_U)$.

COROLLARY 4.11. *Each ideal of a $U_{\mathcal{M}}$-radical object is $U_{\mathcal{M}}$-ideal.*

Proof. If (l, δ) is an ideal of a $U_{\mathcal{M}}$-radical object a, then by Theorem 4.10 $(l_U, \overline{\sigma}_U\delta) = (l, \delta) \frown (a_U, \sigma_U) = (l, \delta)$ since $(a_U, \sigma_U) = (a, \varepsilon_a)$.

COROLLARY 4.12. *Each ideal of a $U_{\mathcal{M}}$-semisimple object is also $U_{\mathcal{M}}$-semisimple.*

Proof. If (l, δ) is an ideal of a $U_{\mathcal{M}}$-semisimple object a, then $(l_U, \overline{\sigma}_U\delta) = (l, \delta) \frown (a_U, \sigma_U) = (0, \omega)$ since $(a_U, \sigma_U) = (0, \omega)$.

§ 5. Subdirect products of simple objects. Throughout the sequel $U_{\mathcal{M}}$ always denotes a Brown-McCoy radical property defined by a modular class $\mathcal{M}$. We shall consider $U_{\mathcal{M}}$-semisimple objects, i.e. (by Theorem 4.9) subdirect products of simple objects from $\mathcal{M}$.

PROPOSITION 5.1. *If (l, δ) is an ideal of a $U_{\mathcal{M}}$-semisimple object a, then the object l can be subdirectly embedded into the direct product $g_{M\setminus\overline{N}}(\pi_m)$, where $\overline{N}$ is the set of all ideals from M_a containing (l, δ).*

Proof. If we consider the maps $\alpha_m\colon a \to a_m$, where $(m, \mu_m) \in M \setminus \overline{N}$, then there is a unique map $\gamma\colon a \to g_{M\setminus\overline{N}}$ such that $\gamma\pi_m = \alpha_m$ for each

$(m,\mu_m)\in M\setminus\overline{N}$. Moreover, $\delta a_m\neq\omega$ for $(m,\mu_m)\in M\setminus\overline{N}$ since in the opposite case $\delta a_m=\omega$ the ideal (m,μ_m) would contain (l,δ), i.e. $(m,\mu_m)\in\overline{N}$, which is impossible. Hence by Proposition 4.1, $\delta a_m\colon l\to a_m$ are epimorphisms. Now let $(f,\varphi)=\ker\delta\gamma$. We have $\varphi\delta a_m=\varphi\delta\gamma\pi_m=\omega\pi_m=\omega$ for $(m,\mu_m)\in M\setminus\overline{N}$. Therefore the subobject $(f,\varphi\delta)$ of a is contained in each $(m,\mu_m)\in M\setminus\overline{N}$, i.e. $(f,\varphi\delta)\leqslant(k_{M\setminus\overline{N}},\mu_{M\setminus\overline{N}})$. But, on the other hand, $(f,\varphi\delta)\leqslant(l,\delta)\leqslant(k_{\overline{N}},\mu_{\overline{N}})$. Therefore $(f,\varphi\delta)\leqslant(k_{M\setminus\overline{N}},\mu_{M\setminus\overline{N}})\cap(k_{\overline{N}},\mu_{\overline{N}})=(0,\omega)$ since a is $U_{\mathcal{M}}$-semisimple. Hence $(f,\varphi)=(0,\omega)$, i.e. $\delta\gamma$ is a monomorphism.

By Proposition 5.1 and Theorem 3.4 we can get another proof of Corollary 4.12.

Definition 5.2. An ideal (l,δ) of an object a will be called $\mathcal{M}$-*representable* if (l,δ) can be represented as the intersection of all $\mathcal{M}$-maximal ideals containing it.

Definition 5.3. Let (k_N,μ_N), where $N\subseteq M$, be an $\mathcal{M}$-representable ideal of a $U_{\mathcal{M}}$-semisimple object a. Then the ideal $(k_N^*,\mu_N^*)=(k_{M\setminus\overline{N}},\mu_{M\setminus\overline{N}})$ will be called the *annihilator* of (k_N,μ_N) in a.

In the category of associative rings the annihilator A^*, thus defined, of the ideal A of a $U_{\mathcal{M}}$-semisimple ring R consists of such elements $x\in R$ that $xA=Ax=0$ (cf. [12]).

Proposition 5.4. *Let (m_0,μ_0) be an $\mathcal{M}$-maximal ideal of a $U_{\mathcal{M}}$-semisimple object a. Then $(m_0^*,\mu_0^*)\neq(0,\omega)$ if and only if there is such a monomorphism $\sigma_0\colon a_{m_0}\to a$ that*

(i) $\sigma_0 a_{m_0}=\varepsilon_{a_{m_0}}$;

(ii) $\sigma_0 a_m=\omega$ *for* $(m,\mu_m)\in L$, *where* $L=M\setminus\{(m_0,\mu_0)\}$. *Moreover, if $(m_0^*,\mu_0^*)\neq(0,\omega)$, then (m_0^*,μ_0^*) is a simple $\mathcal{M}$-ideal of a.*

Proof. Let us assume that the conditions are satisfied. Then $\sigma_0\neq\omega$ since otherwise $\varepsilon_{a_{m_0}}\neq\omega$, whence $a_{m_0}=0$, which is impossible. Therefore $(a_{m_0},\sigma_0)\neq(0,\omega)$. The subobject (a_{m_0},σ_0) of a is contained in each $(m,\mu_m)\in L$ because of (ii). Therefore $(0,\omega)\neq(a_{m_0},\sigma_0)\leqslant(k_L,\mu_L)=(m_0^*,\mu_0^*)$. Conversely, let us assume that $(m_0^*,\mu_0^*)\neq(0,\omega)$. Then $\mu_0^*a_{m_0}\neq\omega$ since in the opposite case (m_0^*,μ_0^*) would be contained in (m_0,μ_0), whence, by the $U_{\mathcal{M}}$-semisimplicity of a, $(m_0^*,\mu_0^*)=(k_L,\mu_L)=(k_L,\mu_L)\cap(m_0,\mu_0)=(k_M,\mu_M)=(0,\omega)$, which is impossible. Therefore applying Proposition 4.1 we find that the map $\mu_0^*a_{m_0}\colon m_0^*\to a_{m_0}$ is an epimorphism. We shall show that $\mu_0^*a_{m_0}$ is also a monomorphism. Indeed, for each $(m,\mu_m)\in L$ there is such a $\bar{\mu}_m\colon m_0^*\to m$ that $\bar{\mu}_m\mu_m=\mu_0^*$. If $(f,\varphi)=\ker\mu_0^*a_{m_0}$ then $\varphi\mu_0^*a_m=\varphi\bar{\mu}_m\mu_m a_m=\varphi\bar{\mu}_m\omega=\omega$. Moreover, $\varphi\mu_0^*a_{m_0}=\omega$. Then $(f,\varphi\mu_0^*)\leqslant(k_M,\mu_M)=(0,\omega)$. Hence $(f,\varphi)=(0,\omega)$, i.e. $\mu_0^*a_{m_0}$ is a monomorphism. Therefore $m_0^*\sim a_{m_0}$ (ξ,ξ^{-1}), where $\xi=\mu_0^*a_{m_0}$. We set $\sigma_0=\xi^{-1}\mu_0^*\colon a_{m_0}\to a$. Then $\sigma_0 a_{m_0}=\xi^{-1}\mu_0^*a_{m_0}=\xi^{-1}\xi=\varepsilon$ and $\sigma_0 a_m=\xi^{-1}\mu_0^*a_m=\xi^{-1}\bar{\mu}_m\mu_m a_m=\omega$ for $(m,\mu_m)\in L$.

The subobjects (m_0^*,μ_0^*) and (a_{m_0},σ_0) are equivalent since $\xi\sigma_0=\mu_0^*$ and $\xi^{-1}\mu_0^*=\sigma_0$.

Proposition 5.5. *Let (l,δ) be an ideal of a $U_{\mathcal{M}}$-semisimple object a. If (p,ϱ) is a simple $\mathcal{M}$-ideal of the object l, then there is such an ideal $(m_0,\mu_0)\in M\setminus\overline{N}$ that $(p,\varrho\delta)\leqslant(m_0^*,\mu_0^*)$, where $\overline{N}$ is the set of all ideals from M containing (l,δ).*

Proof. By condition 4.5.I there is such an $\mathcal{M}$-maximal ideal (q,σ) of l that $(q,\sigma)\cap(p,\varrho)=(0,\omega)$. Then by Proposition 4.7, there is such a $(m_0,\mu_0)\in M\setminus\overline{N}$ that $(m_0,\mu_0)\cap(l,\delta)=(q,\sigma\delta)$. Take any $(m,\mu_m)\in L$, where $L=(M\setminus\overline{N})\setminus\{(m_0,\mu_0)\}$. We set $(r,\nu)=(m,\mu_m)\cap(l,\delta)$. Then $\nu_1\delta=\nu$ for some $\nu_1\colon r\to l$. By Propositions 2.4 and 4.1, (r,ν_1) is an $\mathcal{M}$-maximal ideal of l. Let us suppose that $(r,\nu_1)\cap(p,\varrho)=(0,\omega)$. Then by condition 4.5.I, $(r,\nu_1)=(q,\sigma)$, whence $(r,\nu_1\delta)=(q,\sigma\delta)$, i.e. $(r,\nu)=(q,\sigma\delta)$. Applying Proposition 4.7 we get $(m,\mu_m)=(m_0,\mu_0)$, which is impossible since $(m,\mu_m)\in L$. Therefore $(r,\nu_1)\cap(p,\varrho)\neq(0,\omega)$, whence $(r,\nu_1)\cap(p,\varrho)=(p,\varrho)$ since (p,ϱ) is a simple ideal of l. Then $(p,\varrho)\leqslant(r,\nu_1)$ and $(p,\varrho\delta)\leqslant(r,\nu)=(m,\mu_m)\cap(l,\delta)$. Therefore $(p,\varrho\delta)\leqslant(m,\mu_m)$ for any $(m,\mu_m)\in L$. Hence $(p,\varrho\delta)\leqslant(k_L,\mu_L)\cap(l,\delta)\leqslant(k_L,\mu_L)\cap(k_{\overline{N}},\mu_{\overline{N}})=(k_{L\cup\overline{N}},\mu_{L\cup\overline{N}})=(m_0^*,\mu_0^*)$.

Definition 5.6. Let a be a $U_{\mathcal{M}}$-semisimple object and let D_a be the set of all ideals $(m,\mu_m)\in M_a$ such that $(m^*,\mu_m^*)\neq(0,\omega)$. The object a will be called *special* if $(k_D,\mu_D)=(0,\omega)$. If, however, D_a is the empty set, i.e. if $(k_D,\mu_D)=(a,\varepsilon_a)$, then the object a will be called *completely non-special*.

These notions for the category of rings were considered in [7] and [12].

Recently Tsalenko [13] introduced the following notion of the special subdirect sum. An object a is said to be the *special subdirect* sum of objects a_i, $i\in I$, if

(1) There is such a family of maps $\sigma_i\colon a_i\to a$, $\tau_i\colon a\to a_i$, $i\in I$, that $\sigma_i\tau_i=\varepsilon_{a_i}$ and $\sigma_i\tau_j=\omega$ for $i\neq j$, $i,j\in I$;

(2) If $\alpha\tau_i=\beta\tau_i$ for each $i\in I$, where $\alpha\colon b\to a$, $\beta\colon b\to a$ then $\alpha=\beta$.

This special subdirect sum will be denoted by $a=\sum_{i\in I}a_i(\sigma_i,\tau_i)$.

Theorem 5.7. *A $U_{\mathcal{M}}$-semisimple object a is special if and only if a is a special subdirect sum of objects from $\mathcal{M}$.*

Proof. Let a be a special $U_{\mathcal{M}}$-semisimple object. By Theorem 2.3 the object a can be subdirectly embedded into the direct product $g_{D_a}(\pi_m)$ by a monomorphism $\gamma\colon a\to g_D$. Then $\gamma\pi_m=a_m\colon a\to a_m$ and $(m,\mu)=\ker a_m$. Let $\alpha\colon b\to a$, $\beta\colon b\to a$ be such maps that $\alpha a_m=\beta a_m$ for each $(m,\mu_m)\in D$. Then $\alpha\gamma\pi_m=\beta\gamma\pi_m$, whence by the uniqueness of $\alpha\gamma$ we have $\alpha\gamma=\beta\gamma$; therefore $\alpha=\beta$ since γ is a monomorphism. But, on the other hand, by Proposition 5.4, for each $(m, \mu_m) \in D$ there is such a monomorphism $\sigma_m\colon a_m \to a$, that $\sigma_m a_m = \varepsilon$ and $\sigma_m a_{m'} = \omega$ for $(m, \mu_m) \neq (m', \mu_{m'})$. Therefore $a = \sum_{(m,\mu_m)\in D} a_m(\sigma_m, a_m)$. Conversely, let $a = \sum_{i\in I} a_i(\sigma_i, \tau_i)$, where $a_i \in \mathcal{M}$. Consider the direct product $g = \prod_{i\in I} a_i(\pi_i)$ and the maps $\tau_i\colon a \to a_i$. Then there is a unique map $\gamma\colon a \to g$ such that $\gamma\pi_i = \tau_i$ for $i \in I$. If we set $(k, \mu) = \ker\gamma$, then $\mu\tau_i = \mu\gamma\pi_i = \omega\pi_i = \omega$. But, on the other hand, $\omega\tau_i = \omega$ for $i \in I$. Therefore applying condition (2) we get $\mu = \omega$, i.e. γ is a monomorphism. Moreover, by Proposition 4.3 each of the maps τ_i is an epimorphism since $\varepsilon_{a_i} = \sigma_i\tau_i$. Therefore the object a can be subdirectly embedded into the direct product g, whence, by Theorem 3.4, a is $U_{\mathcal{M}}$-semisimple. The fact that a is special follows immediately from condition (1) and Proposition 5.4.

THEOREM 5.8. *A $U_{\mathcal{M}}$-semisimple object is special if and only if each of its non-zero $\mathcal{M}$-representable ideals contains a simple $\mathcal{M}$-ideal.*

Proof. Let a be a special $U_{\mathcal{M}}$-semisimple object and let $(k_N, \mu_N) \neq (0, \omega)$, where $N \subseteq M_a$. Then there is an ideal $(m_0, \mu_0) \in D_a$ which does not contain (k_N, μ_N) since otherwise (k_N, μ_N) would be contained in each $(m, \mu_m) \in D_a$, i.e. $(k_N, \mu_N) \leqslant (k_D, \mu_D) = (0, \omega)$, which is impossible. Therefore $(m_0, \mu_0) \in M\setminus\overline{N}$, whence $\overline{N} \subseteq L$, where $L = M\setminus\{(m_0, \mu_0)\}$. Then $(0, \omega) \neq (m_0^*, \mu_0^*) = (k_L, \mu_L) \leqslant (k_N, \mu_N)$. By Proposition 5.4, (m_0^*, μ_0^*) is a simple $\mathcal{M}$-ideal of a. Conversely, let us suppose that $(k_D, \mu_D) \neq (0, \omega)$. If the object k_D contains a simple $\mathcal{M}$-ideal (p, ϱ), then by Proposition 5.5 there is such an ideal $(m_0, \mu_0) \in M\setminus\overline{D}$ that $(0, \omega) \neq (p, \varrho\mu_D) \leqslant (m_0^*, \mu_0^*)$. Hence $(m_0^*, \mu_0^*) \neq (0, \omega)$, which is impossible since $(m_0, \mu_0) \in M\setminus\overline{D}$.

This Theorem for the category of rings was proved in [7].

THEOREM 5.9. *A $U_{\mathcal{M}}$-semisimple object is completely non-special if and only if it contains no simple $\mathcal{M}$-ideal.*

Proof. Let a be a completely non-special object and let (p, ϱ) be a simple $\mathcal{M}$-ideal of a. By Proposition 5.5 applied to the ideal (a, ε_a) we find that there is such an ideal $(m_0, \mu_0) \in M$ that $(0, \omega) \neq (p, \varrho) \leqslant (m_0^*, \mu_0^*)$. Therefore $(m_0^*, \mu_0^*) \neq (0, \omega)$, which is impossible since a is completely non-special. Conversely, let us assume that an object a is not completely non-special. Then there is such an ideal $(m, \mu_m) \in M$ that $(m^*, \mu_m^*) \neq (0, \omega)$. By Proposition 5.4 (m^*, μ_m^*) is a simple $\mathcal{M}$-ideal of a.

DEFINITION 5.10. Let a be a $U_{\mathcal{M}}$-semisimple object. The ideal (k_P, μ_P), where $P = M\setminus\overline{D}$ will be called the *special part* of the object a, and the ideal (k_D, μ_D) will be called the *completely non-special part* of a.

THEOREM 5.11. *If (k_P, μ_P) is the special part and (k_D, μ_D) the completely non-special part of a $U_{\mathcal{M}}$-semisimple object a, then the object k_P is special and the object k_D completely non-special.*

Proof. By Corollary 4.12, the objects k_P and k_D are $U_{\mathcal{M}}$-semisimple.

If (r, ϱ) is an $\mathcal{M}$-representable non-zero ideal of the object k_P, then by Proposition 4.7 $(r, \varrho\mu_P) = (k_P, \mu_P) \cap (k_{\overline{N}}, \mu_{\overline{N}}) = (k_{P\cup\overline{N}}, \mu_{P\cup\overline{N}})$, where $\overline{N}$ is the set of such $(m, \mu_m) \in M\setminus\overline{P}$ that $(m, \mu_m) \geqslant (r, \varrho\mu_P)$. Let us suppose that $D \subseteq \overline{P \cup \overline{N}}$. Then $\overline{D} \subseteq \overline{P \cup \overline{N}}$. Using the fact that $(r, \varrho) \neq (0, \omega)$ we obtain $M \neq \overline{P \cup \overline{N}} = \overline{D} \cup \overline{P \cup \overline{N}} \supseteq \overline{D} \cup \overline{P} \cup \overline{N} \supseteq \overline{D} \cup \overline{P} = \overline{D} \cup \overline{M\setminus\overline{D}} \supseteq \overline{D} \cup (M\setminus\overline{D}) = M$, which leads to a contradiction. Therefore there is such an ideal $(m_0, \mu_0) \in D$ that $(m_0, \mu_0) \notin \overline{P \cup \overline{N}}$ hence $L \supseteq \overline{P \cup \overline{N}}$, where $L = M\setminus\{m_0, \mu_0\}$. Then we have $(0, \omega) \neq (m_0^*, \mu_0^*) = (k_L, \mu_L) \leqslant (r, \varrho\mu_P)$. Applying Theorem 5.8 we find that k_P is special.

Now let (p, ϱ) be a simple $\mathcal{M}$-ideal of the object k_D. Then by Proposition 5.5 there is such an ideal $(m_0, \mu_0) \in M\setminus\overline{D}$ that $(0, \omega) \neq (p, \varrho\mu_D) \leqslant (m_0^*, \mu_0^*,)$ which leads to a contradiction. Therefore applying Theorem 5.9 we find that k_D is completely non-special.

The author does not know whether the structure $\mathcal{M}$-space M_a of an object a is a topological T_1-space, i.e. whether $\overline{\overline{N}_1 \cup \overline{N}_2} \subseteq \overline{N}_1 \cup \overline{N}_2$, where $\overline{N}_1, \overline{N}_2 \subseteq M_a$. The converse inclusion and the conditions: $\overline{\{(m, \mu_m)\}} = \{(m, \mu_m)\}$ for $(m, \mu_m) \in M$; $\overline{\overline{N}} = \overline{N}$ for $N \subseteq M$; and $\overline{\emptyset} = \emptyset$, where $\emptyset$ is the empty set, are obvious.

THEOREM 5.12. *Let us assume that the structure $\mathcal{M}$-space of a $U_{\mathcal{M}}$-semisimple object a is a T_1-space. Take any $\mathcal{M}$-representable ideal (k_N, μ_N), $N \subseteq M$, of a. If the object k_N is special, then $(k_N, \mu_N) \leqslant (k_P, \mu_P)$, where $P = M\setminus\overline{D}$; if, however, the object k_N is completely non-special, then $(k_N, \mu_N) \leqslant (k_D, \mu_D)$.*

Proof. Let us assume that the object k_N is special. We set $W = \overline{D} \cup \overline{N}$. Suppose that $\overline{W} \neq M$, i.e. $(k_W, \mu_W) \neq (0, \omega)$. We have $(k_W, \mu_W) \leqslant (k_N, \mu_N)$, whence there is such a $\mu\colon k_W \to k_N$ that $\mu\mu_N = \mu_W$. The ideal (k_W, μ_W) is an $\mathcal{M}$-representable ideal of the special object k_N, whence by Theorem 5.8, the object k_W contains a simple $\mathcal{M}$-ideal (p, ϱ). Then applying Proposition 5.5 to the ideal (k_W, μ_W), we find that there is such an ideal $(m_0, \mu_0) \in M\setminus\overline{W}$ that $(0, \omega) \neq (p, \varrho\mu_W) \leqslant (m_0^*, \mu_0^*)$. But this is impossible since $D \cap (M\setminus\overline{W}) = D \cap (M\setminus\overline{\overline{D} \cup \overline{N}}) \subseteq \overline{D} \cap \big(M\setminus(\overline{D} \cup \overline{N})\big) \subseteq \overline{D} \cap (M\setminus\overline{D}) = \emptyset$. Therefore $\overline{W} = M$. Hence using the fact that the closure operation is additive we obtain $M = \overline{W} = \overline{\overline{D} \cup \overline{N}} = \overline{D} \cup \overline{N}$. Therefore, $P = M\setminus\overline{D} \subseteq \overline{N}$ i.e. $(k_N, \mu_N) \leqslant (k_P, \mu_P)$.

Now let us suppose that the object k_N is completely non-special. Let $(m, \mu_m) \in D$. If $(m, \mu_m) \in M\setminus\overline{N}$ then $L \supseteq \overline{N}$, where $L = M\setminus\{(m, \mu_m)\}$. Then $(0, \omega) \neq (m^*, \mu_m^*) = (k_L, \mu_L) \leqslant (k_N, \mu_N)$, which is impossible because of Theorem 5.9 and Proposition 5.4. Therefore $D \subset \overline{N}$, i.e. $(k_N, \mu_N) \leqslant (k_D, \mu_D)$.

DEFINITION 5.13. An object a is said to be *strongly $U_{\mathcal{M}}$-semisimple* if each of its ideals is $\mathcal{M}$-representable.

If a is strongly $U_{\mathcal{M}}$-semisimple, then a is $U_{\mathcal{M}}$-semisimple since the ideal $(0, \omega)$ is $\mathcal{M}$-representable.

THEOREM 5.14. *If an object a is strongly $U_{\mathcal{M}}$-semisimple, then*

$$(k_P, \mu_P) \cup (k_D, \mu_D) = (k_{\mathrm{Fr}\overline{D}}, \mu_{\mathrm{Fr}\overline{D}}),$$

where $P = M\setminus\overline{D}$ and $\mathrm{Fr}\,\overline{D} = \overline{D} \cap \overline{M\setminus\overline{D}}$.

Proof. From the fact that a is strongly $U_{\mathcal{M}}$-semisimple we have $(k_P, \mu_P) \cup (k_D, \mu_D) = (k_N, \mu_N)$, where $N \subseteq M$. If $(m, \mu_m) \in \overline{N}$, then $(m, \mu_m) \geqslant (k_D, \mu_D)$ and $(m, \mu) \geqslant (k_P, \mu_P)$. Therefore $(m, \mu_m) \in \overline{D} \cap \overline{M\setminus\overline{D}}$, i.e. $\overline{N} \leqslant \mathrm{Fr}\,\overline{D}$. Conversely, if $(m, \mu_m) \in \overline{D} \cap \overline{M\setminus\overline{D}}$, then $(m, \mu_m) \geqslant (k_D, \mu_D)$ and $(m, \mu_m) \geqslant (k_P, \mu_P)$. Hence $(m, \mu_m) \geqslant (k_N, \mu_N) = (k_D, \mu_D) \cup (k_P, \mu_P)$, i.e. $(m, \mu_m) \in \overline{N}$. Therefore $\mathrm{Fr}\,\overline{D} \leqslant \overline{N}$.

COROLLARY 5.15. *A strongly $U_{\mathcal{M}}$-semisimple object is a direct product of its special and completely non-special part if and only if $\mathrm{Fr}\,D_a = \emptyset$, where $\emptyset$ is the empty set.*

Proof. If $\mathrm{Fr}\,\overline{D} = \emptyset$ then $(k_P, \mu_P) \cup (k_D, \mu_D) = (k_\emptyset, \mu_\emptyset) = (a, \varepsilon_a)$. Therefore by Theorem 2.5 a is a direct product of k_P and k_D, since $(k_P, \mu_P) \cap (k_D, \mu_D) = (k_M, \mu_M) = (0, \omega)$. Conversely, let a be a direct product of k_P and k_D. If $(m, \mu_m) \in \mathrm{Fr}\,\overline{D} = \overline{D} \cap \overline{M\setminus\overline{D}}$ then $(m, \mu_m) \geqslant (k_D, \mu_D) \cup (k_P, \mu_P) = (a, \varepsilon_a)$, which is impossible. Therefore $\mathrm{Fr}\,\overline{D} = \emptyset$.

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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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