(2) every model of θ is isomorphic to a system of the form $\langle B, \varrho, \varepsilon_\varrho, S_\mu\rangle_{\mu<\eta}$ where ϱ is a (non-zero) ordinal.

Next let $\varkappa = 2\exp(2\exp(2\exp\pi))$ and $\mathfrak{m} = \beth_\varkappa$ and finally let δ be the ordinal $\mathfrak{m}+1$. In [2] we proved that the Hanf-number (for single sentences) of $\mathbf{L}_{\alpha\omega}$ is smaller than or equal to $\mathfrak{m}$. Let $\mathbf{P}$ be the unary relation symbol occurring in θ whose interpretation is the well-ordered set, i.e. ϱ in (1) and (2). Then by essentially relativizing the quantifiers to the predicate $\mathbf{P}$ we obtain by the methods used in [2] that there exists a sentence ψ of $\mathbf{L}_{\alpha\omega}$ such that:

(3) there exists C, S_μ, R such that $\mathfrak{A}_0 = \langle C, \delta, \varepsilon_\delta, S_\mu, R_\xi\rangle_{\mu<\eta,\xi<\lambda}$ is a model of $\theta\wedge\psi$,

(4) any model of $\theta\wedge\psi$ must be isomorphic to a system of the form $\langle D, \varrho, \varepsilon_\varrho, U_\mu, V_\xi\rangle_{\xi<\lambda,\mu<\eta}$ where $|\varrho| \leqslant \mathfrak{m}$.

Let $\mathbf{F}$ be a binary relation symbol not occurring in $\theta\wedge\psi$ and let χ be a sentence which expresses the condition that (the interpretation of) $\mathbf{F}$ be a (1-1) function from the universe onto (the interpretation of) $\mathbf{P}$, for example let χ be the sentence

$$(\forall xy)(\mathbf{F}xy \to \mathbf{P}y) \wedge (\forall x)(\mathrm{E}y)\mathbf{F}xy \wedge (\forall xyz)(\mathbf{F}xy \wedge \mathbf{F}xz \to y = z)\,,$$
$$(\forall xyz)(\mathbf{F}xy \wedge \mathbf{F}zy \to x = z) \wedge (\forall y)\big(\mathbf{P}y \to (\mathrm{E}x)(\mathbf{F}xy)\big)\,.$$

Finally let Θ be the sentence $\theta\wedge\Psi\wedge\chi$. Suppose that $\langle B, A, \ldots\rangle$ is a model of Θ. Then from χ it follows that A and B are of the same cardinally. Thus it follows from (4) that Θ does not have arbitrarily large models. Thus in order to complete the proof of the theorem it suffices to show that Θ has a model of cardinallity $\mathfrak{m}$. Let $\mathfrak{A}_0$ be the model of $\theta\wedge\Psi$ mentioned in (3). Since $\max(\mathfrak{m}, \alpha) = \mathfrak{m}$, we can apply the downward Lowenheim-Skolem Theorem for $\mathbf{L}_{\alpha\omega}$ (cf. [1], Theorem 2, p. 34, modified for the languages $\mathbf{L}_{\alpha\omega}$) to obtain a subsystem $\mathfrak{B}$ of $\mathfrak{A}_0$ whose universe includes δ and such $\mathfrak{B}$ is a model of $\theta\wedge\Psi$ of cardinality $\mathfrak{m}$. It follows then that $\mathfrak{B}$ is a model of $\theta\wedge\Psi$ of the form $\langle B, \delta, \varepsilon_\varrho, S'_\mu, R'_\xi\rangle_{\mu<\eta,\xi<\lambda}$ where δ and B are both of cardinality $\mathfrak{m}$. It is now clear how to add an extra relation to $\mathfrak{B}$ in order to obtain a model of Θ.

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Characterizations of weakly modular lattices *

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This paper deals with the characterizations of weakly modular lattices. Defining ψ, the weakly modular congruence on a lattice L to be that congruence generated by its ineffective intervals, we show that the intersection of all maximal congruences on any lattice L contains ψ in general (theorem 2) and equals ψ when L is semidiscrete (theorem 8). As a consequence of theorem 2, we arrive at a characterization of semi-discrete weakly modular lattices (theorem 4). Next we prove that the quotient of a weakly modular lattice by a separable congruence is weakly modular (theorem 6). This enables us to give a characterization theorem for semi-discrete lattices—viz. theorem 7 which states that "any semi-discrete lattice is a subdirect union of simple lattices if and only if it is weakly modular". We next prove that L/ψ is weakly modular if the weakly modular congruence ψ on the lattice L is separable.

We start with

DEFINITION 1. Let L be any lattice and ψ be the join in $\theta(L)$ (the lattice of congruences on L) of all congruences generated by the ineffective intervals (cf. [3]) of L. ψ is called the *weakly modular congruence on* L.

The weakly modular congruence on a weakly modular lattice L is the *null congruence on* L.

LEMMA 1. *Let L be a weakly modular lattice and I be a prime interval of L such that θ_I, the congruence generated by I, is a separable congruence on L. Then there exists a maximal congruence on L not annuling I.*

Proof. As θ_I is separable and L weakly modular, θ_I is complemented (cf. [2]). Let Φ be the complement of θ_I. Also Φ is defined by $x \equiv y\ (\Phi)$ if and only if the interval $(x+y, xy)$ consists of single point congruence classes under θ_I (cf. [2]). That is, $x \equiv y\ (\Phi)$ if and only if the interval $(x+y, xy)$ contains no nontrivial interval J with the property J is a lattice translate of I. But then Φ is a maximal congruence on L. For if ζ strictly contains Φ, it annuls at least one J with the property that J

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is a lattice translate of I. L being weakly modular, J is effective; and as I is prime, one has that I is a lattice translate of J. So ζ annuls I and hence ζ contains θ_I also; hence ζ is the universal congruence on L.

THEOREM 1. *If L is a weakly atomic, weakly modular lattice and any congruence generated by a prime interval on L is separable, then any separable congruence on L is an intersection of maximal congruences on L.*

Proof. Let ζ be a separable congruence on L. Let η be the intersection of all maximal congruences on L containing ζ. Let J be an interval of L not annulled by ζ. Since ζ is separable, there exists a subinterval J_1 of J consisting of one element congruence classes under ζ. As L is weakly atomic, J_1 contains a prime interval I not annulled by ζ.

Let θ be the maximal congruence on L not annuling I, which exists by lemma 1. Now θ does not annul J; also $\zeta \cup \theta$ does not annul I, as I is prime and each of ζ and θ does not annul I. Therefore $\zeta \cup \theta$ is not the universal congruence on L. Now the maximality of θ implies $\zeta \subseteq \theta$. Thus $\eta \subseteq \theta$. Hence J is not annulled by η. As the choice of J is arbitrary, any interval not annulled by ζ is not annulled by η. This means $\eta \subseteq \zeta$; thus $\eta = \zeta$ as trivially $\zeta \subseteq \eta$.

COROLLARY. *If L is a weakly atomic, distributive lattice, then any separable congruence on L is an intersection of maximal congruences on L.*

Proof. For when L is distributive, any congruence generated by a prime interval is separable; and the rest of the proof follows from theorem 1 above.

THEOREM 2. *Let L be an arbitrary lattice; then the intersection of all maximal congruences L contains ψ, the weakly modular congruence on L.*

Proof. It is sufficient to show that any maximal congruence θ on L annuls every ineffective interval of L. Let I be an ineffective interval of L; then there exists an interval J in L such that I is a lattice translate of J but no subinterval of J is a lattice translate of I.

Suppose that θ does not annul I; then θ cannot annul J. Let θ_I be the congruence generated by I. Then $\theta \vee \theta_I$, as it annulls I, strictly contains θ. But $\theta \vee \theta_I$ cannot annul J, for θ does not annul J and J consists of single point congruence classes under θ_I. Then $\theta \vee \theta_I$ is not the universal congruence on L. Hence θ is not a maximal congruence on L. Thus every maximal congruence on L annulls all ineffective intervals of L. Hence the conclusion.

THEOREM 3. *If L is a weakly atomic lattice and any congruence generated by a prime interval on L is separable, then the following statements are equivalent.*

(i) *L is weakly modular.*

(ii) *Any separable congruence on L is an intersection of maximal congruences on L.*

(iii) *The null congruence on L is the intersection of maximal congruence on L.*

Proof. (i) implies (ii) by theorem 1.

(ii) implies (iii) trivially.

(iii) implies (i) by theorem 2.

As a particular case of theorem 3, we get

THEOREM 4. *If L is a semi-discrete lattice, then the following statements are equivalent.*

(i) *L is weakly modular.*

(ii) *Any congruence on L is an intersection of maximal congruences on L.*

(iii) *The null congruence on L is an intersection of maximal congruence on L.*

Proof. The theorem follows from the fact that any congruence of a semi-discrete lattice is separable and a semi-discrete lattice is weakly atomic.

As a corollary to theorem 4, we get

THEOREM 5. *If L is a semi-discrete, weakly modular lattice, then every homomorphic image of L is a semi-discrete weakly modular lattice.*

Proof. Let θ be any congruence on L. Any maximal congruence containing θ on L goes over to a maximal congruence on L/θ. Thus the intersection of maximal congruences of L/θ is the null congruence on L/θ; as the intersection of all maximal congruences containing θ on L is θ. Hence L/θ is weakly modular by theorem 4, as L/θ is semi-discrete, being a homomorphic image of a semi-discrete lattice. Hence the conclusion.

More generally, we can prove the following

THEOREM 6. *Let L be a weakly modular lattice and θ a separable congruence on L; then L/θ is weakly modular.*

Proof. Let $I = (a, b)$ and $J = (c, d)$ be two intervals not annulled by θ. As θ is separable and I is not annulled by θ, there exists at least one subinterval I_1 of I consisting of single point congruence classes under θ. Let I be a lattice translate of J; then I_1 is a lattice translate of J; for I_1 is a lattice translate of I. Therefore a nontrivial subinterval J_1 of J is a lattice translate of I_1, as I_1 being an interval of weakly modular lattice is effective. Now the pseudo-complement θ' of θ annuls I_1 (cf. [2]) and hence annuls J_1. Therefore J_1 consists of single point congruence classes under θ. Thus there is a nontrivial subinterval J_1 of J consisting of single point congruence classes under θ is a lattice translate of I_1 and hence of I. Thus a nontrivial subinterval J_1 of J is a lattice translate of I and is not annulled by θ. This proves that L/θ is weakly modular.

LEMMA 2. *Any subdirectly irredicible, weakly modular lattice in which all congruences are separable is simple.*

Proof. Let L be a subdirectly irreducible, weakly modular lattice in which all congruences are separable. Let θ be any congruence on L other than the null congruence and the universal congruence on L. θ is complemented (cf. [2]) and the complement θ' is other than the null congruence and the universal congruence on L. Now $\theta \wedge \theta' = 0$ and neither of θ nor θ' equals the null congruence. A contradiction because L is subdirectly irreducible. Hence L is simple.

COROLLARY. *Any semi-discrete, weakly modular subdirectly irreducible lattice is simple.*

THEOREM 7. *Any semi-discrete lattice is weakly modular if and only if it is a subdirect union of simple lattices.*

Proof. Any lattice is a subdirect union of subdirectly irreducible lattices; in particular, any semi-discrete weakly modular lattice L is a subdirect union of subdirectly irreducible weakly modular lattice (by theorem 6). Hence L is a subdirect union of simple lattices, by corollary to lemma 2 above.

To prove the converse, it will suffice to note that the intersection of all maximal congruences on L is the null congruence on L, when L is a subdirect union of simple lattices; then the weak modularity of L follows by theorem 4.

THEOREM 8. *If ψ, the weakly modular congruence on L, is separable, then L/ψ is weakly modular.*

Proof. Let $I = (a, b)$ $(a > b)$ and $J = (c, d)$ $(c > d)$ be two intervals of L not annulled by ψ. As ψ is separable there exists a finite chain C: $a = a_0 \geqslant a_1 \geqslant \ldots \geqslant a_n = b$ such that either (i) $I_i = (a_{i-1}, a_i)$ is annulled by ψ or (ii) $I_i = (a_{i-1}, a_i)$ consists of single point congruence classes under ψ.

Now as I is not annulled by ψ, C contains at least one I_i satisfying (ii). Let J be a lattice translate of I. Then there exists $X_1, X_2, \ldots, X_n$ in L such that $C = f(a, X_1, \ldots, X_n)$ and $d = f(b, X_1, X_2, \ldots, X_n)$ where f is a finite lattice polynomial. Consider the chain $c = d_0 \geqslant d_1 \geqslant \ldots \geqslant d_n = d$, where $d_i = f(a_i, X_1, \ldots, X_n)$. Let $J_i = (d_{i-1}, d_i)$; then J_i is a lattice translate of I_i, for each i. Now as J is not annulled by ψ, at least one J_i is not annulled by ψ. Let J_k be the interval not annulled by ψ. Then I_k, the interval corresponding to J_k, satisfies (ii).

Next, J_k is effective as it is not annulled by ψ. Hence there exists a subinterval N of I_k such that N is a lattice translate of J_k. New N is not annulled by ψ as I_k satisfies (ii). Thus there exists a subinterval N of I such that N is not annulled by ψ and is a lattice translate of J_k and hence a lattice translate of J. This proves that L/ψ is weakly modular.

COROLLARY. *If L is a semi-discrete lattice, then L/ψ is weakly modular.*

This is because every congruence on a semi-discrete lattice is separable, and so is ψ, the weakly modular congruence on L.

Using the above we have the following

THEOREM 9. *In any semi-discrete lattice L, the following conditions hold.*

(i) *The intersection of all maximal congruences on L is the weakly modular congruence on L.*

(ii) *Any congruence θ on L containing ψ is an intersection of maximal congruences on L.*

Proof. There is a natural isomorphism between the lattice of congruences of L/ψ and congruences on L containing ψ. Thus there is a (1-1) correspondence between all maximal congruences of L/ψ and maximal congruences on L, as any maximal congruence on L contains ψ by theorem 2. But L/ψ is weakly modular by corollary to theorem 8. Hence the intersection of maximal congruences on L/ψ is the null congruence on L/ψ by theorem 4, and so the intersection of all maximal congruences L is ψ.

To prove the second assertion it will suffice to note that L/ψ is weakly modular and hence satisfies (ii) by theorem 4.

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