a weak equational compactification has such a compactification in the smallest equational class containing $\mathfrak{A}$ [16].

Beside the K-compactness ($K \subseteq L^{(\tau)}$), we can consider for every cardinal $\mathfrak{m}$, a weaker notion:

An algebraic system $\mathfrak{A}$ of type τ is called *K-$\mathfrak{m}$-compact* if the condition of compactness holds for sets of formulas having at most the cardinality $\mathfrak{m}$. This notion was considered in [11] and [12]; compare also [6]. I do not know whether $\mathfrak{m}^+$-completeness for Boolean algebras ($\mathfrak{m}^+$ denotes the succesor of $\mathfrak{m}$) implies equational $\mathfrak{m}$-compactness or conversely?

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[16] For equational compactifications the answer is affirmative. The proof of this fact will be published in my paper — *Equationally compact algebras* (III) — in Fundamenta Mathematicae.

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An addition to "On defining well-orderings"

by

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In the paper *On defining well-orderings* [2] we proved that the class $\mathbf{W}$ of well-orderings is not a $\mathbf{PC}_{\Delta}$-class of any infinitary first-order language of the type $\mathbf{L}_{a\omega}$. The addition that we wish to make is to prove that $\mathbf{W}$ is not even a *relativized* $\mathbf{PC}_{\Delta}$-class (i.e. that for all a, $\mathbf{W} \notin \mathbf{RPC}_{\Delta}(\mathbf{L}_{a\omega})$; (cf. definition below).

The method used to show that for all a, $\mathbf{W} \notin \mathbf{RPC}(\mathbf{L}_{a\omega})$ (this clearly suffices in order to prove that for all a, $\mathbf{W} \notin \mathbf{RPC}_{\Delta}(\mathbf{L}_{a\omega})$) is basically the same as that used in [2]. That is, from the assumption that $\mathbf{W} \in \mathbf{RPC}(\mathbf{L}_{a\omega})$ we obtain a sentence Θ of $\mathbf{L}_{a\omega}$ which has a model of cardinality greater (or possibly equal) to the Hanf-number for $\mathbf{L}_{a\omega}$ but which does not have arbitrary large models. [1]

DEFINITION. *Suppose that* $\mathbf{K}$ *is a class of relational systems of the type* $\langle A, R\rangle$ *where* $R \subseteq A^2$, *then*:

(i) "$\mathbf{K}$ is a relativized $\mathbf{PC}_{\Delta}$-class of $\mathbf{L}_{a\omega}$", *in symbols*: $\mathbf{K} \in \mathbf{RPC}_{\Delta}(\mathbf{L}_{a\omega})$, *just in case that there exist a set* T *of sentences of* $\mathbf{L}_{a\omega}$ *such that* $\mathbf{K}$ *consists exactly of those systems* $\langle A, R\rangle$ *for which there exists a set* $B \supseteq A$ *and relations* S_μ *on* B *such that* $\langle B, A, R, S_\mu\rangle_{\mu<\eta}$ *is a model of* T;

(ii) "$\mathbf{K}$ is a relativized $\mathbf{PC}$-class of $\mathbf{L}_{a\omega}$", *in symbols*: $\mathbf{K} \in \mathbf{RPC}(\mathbf{L}_{a\omega})$, *just in case that for some sentence* $\theta \in \mathbf{L}_{a\omega}$, $\mathbf{K}$ *consists exactly of those systems* $\langle A, R\rangle$ *for which there exists a set* $B \supseteq A$ *and relations* S_μ *on* B *such that* $\langle B, A, R, S_\mu\rangle_{\mu<\eta}$ *is a model of* θ.

Note that if for all a, $\mathbf{W} \notin \mathbf{RPC}(\mathbf{L}_{a\omega})$, then for all a, $\mathbf{W} \notin \mathbf{RPC}_{\Delta}(\mathbf{L}_{a\omega})$.

THEOREM. *There does not exist a cardinal* a *such that* $\mathbf{W} \in \mathbf{RPC}(\mathbf{L}_{a\omega})$.

Proof. Assume on the contrary that for some a, $\mathbf{W} \in \mathbf{RPC}(\mathbf{L}_{a\omega})$. It is clear that we may assume that a is a successor cardinal, i.e. that for some cardinal π, $a = \pi^+$. The assumption that $\mathbf{W} \in \mathbf{RPC}(\mathbf{L}_{a\omega})$ means that there exists a sentence θ of $\mathbf{L}_{a\omega}$ such that:

(1) to every (non-zero) ordinal ϱ there corresponds a set $B \supseteq \varrho$ and relations S_μ on B such that $\langle B, \varrho, \varepsilon_\varrho, S_\mu\rangle_{\mu<\eta}$ is a model of θ,

[1] For undefined notation, see [2].

(2) every model of θ is isomorphic to a system of the form $\langle B, \varrho, \varepsilon_\varrho, S_\mu\rangle_{\mu<\eta}$ where ϱ is a (non-zero) ordinal.

Next let $\varkappa = 2\exp(2\exp(2\exp\pi))$ and $\mathfrak{m} = \beth_\varkappa$ and finally let δ be the ordinal $\mathfrak{m}+1$. In [2] we proved that the Hanf-number (for single sentences) of $\mathbf{L}_{a\omega}$ is smaller than or equal to $\mathfrak{m}$. Let $\mathbf{P}$ be the unary relation symbol occurring in θ whose interpretation is the well-ordered set, i.e. ϱ in (1) and (2). Then by essentially relativizing the quantifiers to the predicate $\mathbf{P}$ we obtain by the methods used in [2] that there exists a sentence ψ of $\mathbf{L}_{a\omega}$ such that:

(3) there exists C, S_μ, R such that $\mathfrak{A}_0 = \langle C, \delta, \varepsilon_\delta, S_\mu, R_\xi\rangle_{\mu<\eta,\xi<\lambda}$ is a model of $\theta\wedge\psi$,

(4) any model of $\theta\wedge\psi$ must be isomorphic to a system of the form $\langle D, \varrho, \varepsilon_\varrho, U_\mu, V_\xi\rangle_{\xi<\lambda,\mu<\eta}$ where $|\varrho| \leqslant \mathfrak{m}$.

Let $\mathbf{F}$ be a binary relation symbol not occurring in $\theta\wedge\psi$ and let χ be a sentence which expresses the condition that (the interpretation of) $\mathbf{F}$ be a (1-1) function from the universe onto (the interpretation of) $\mathbf{P}$, for example let χ be the sentence

$$(\forall xy)(\mathbf{F}xy \to \mathbf{P}y)\wedge(\forall x)(\mathrm{E}y)\mathbf{F}xy\wedge(\forall xyz)(\mathbf{F}xy\wedge\mathbf{F}xz\to y = z)\,,$$
$$(\forall xyz)(\mathbf{F}xy\wedge\mathbf{F}zy\to x = z)\wedge(\forall y)\big(\mathbf{P}y\to(\mathrm{E}x)(\mathbf{F}xy)\big)\,.$$

Finally let Θ be the sentence $\theta\wedge\Psi\wedge\chi$. Suppose that $\langle B, A, \ldots\rangle$ is a model of Θ. Then from χ it follows that A and B are of the same cardinally. Thus it follows from (4) that Θ does not have arbitrarily large models. Thus in order to complete the proof of the theorem it suffices to show that Θ has a model of cardinallity $\mathfrak{m}$. Let $\mathfrak{A}_0$ be the model of $\theta\wedge\Psi$ mentioned in (3). Since $\max(\mathfrak{m}, a) = \mathfrak{m}$, we can apply the downward Lowenheim-Skolem Theorem for $\mathbf{L}_{a\omega}$ (cf. [1], Theorem 2, p. 34, modified for the languages $\mathbf{L}_{a\omega}$) to obtain a subsystem $\mathfrak{B}$ of $\mathfrak{A}_0$ whose universe includes δ and such $\mathfrak{B}$ is a model of $\theta\wedge\Psi$ of cardinality $\mathfrak{m}$. It follows then that $\mathfrak{B}$ is a model of $\theta\wedge\Psi$ of the form $\langle B, \delta, \varepsilon_\varrho, S'_\mu, R'_\xi\rangle_{\mu<\eta,\xi<\lambda}$ where δ and B are both of cardinality $\mathfrak{m}$. It is now clear how to add an extra relation to $\mathfrak{B}$ in order to obtain a model of Θ.

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Characterizations of weakly modular lattices *

by

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This paper deals with the characterizations of weakly modular lattices. Defining ψ, the weakly modular congruence on a lattice L to be that congruence generated by its ineffective intervals, we show that the intersection of all maximal congruences on any lattice L contains ψ in general (theorem 2) and equals ψ when L is semidiscrete (theorem 8). As a consequence of theorem 2, we arrive at a characterization of semi-discrete weakly modular lattices (theorem 4). Next we prove that the quotient of a weakly modular lattice by a separable congruence is weakly modular (theorem 6). This enables us to give a characterization theorem for semi-discrete lattices—viz. theorem 7 which states that "any semi-discrete lattice is a subdirect union of simple lattices if and only if it is weakly modular". We next prove that L/ψ is weakly modular if the weakly modular congruence ψ on the lattice L is separable.

We start with

Definition 1. Let L be any lattice and ψ be the join in $\theta(L)$ (the lattice of congruences on L) of all congruences generated by the ineffective intervals (cf. [3]) of L. ψ is called the *weakly modular congruence on L.*

The weakly modular congruence on a weakly modular lattice L is the *null congruence on L.*

Lemma 1. *Let L be a weakly modular lattice and I be a prime interval of L such that θ_I, the congruence generated by I, is a separable congruence on L. Then there exists a maximal congruence on L not annuling I.*

Proof. As θ_I is separable and L weakly modular, θ_I is complemented (cf. [2]). Let Φ be the complement of θ_I. Also Φ is defined by $x\equiv y\ (\Phi)$ if and only if the interval $(x+y, xy)$ consists of single point congruence classes under θ_I (cf. [2]). That is, $x\equiv y\ (\Phi)$ if and only if the interval $(x+y, xy)$ contains no nontrivial interval J with the property J is a lattice translate of I. But then Φ is a maximal congruence on L. For if ζ strictly contains Φ, it annuls at least one J with the property that J

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