[6] — *Nombre d'éléments indépendants et nombre d'éléments générateurs dans les algèbres abstraites finies*, Annali di Matematica (4) 59 (1962), pp. 1-9.

[7] — *Independence in abstract algebras. Results and problems*, Colloq. Math. 14 (1966), pp. 169-188.

[8] E. Marczewski and K. Urbanik, *Abstract algebras in which all elements are independent*, Bull. Acad. Pol. Sci., Série des sci. math., astr. et phys. 8 (1960), pp. 291-293.

[9] —— *Abstract algebras in which all elements are independent*, Colloq. Math. 9 (1962), pp. 199-207.

[10] J. Płonka, *Diagonal algebras and algebraic independence*, Bull. Acad. Pol. Sci., Série des sci. math., astr. et phys. 12 (1964), pp. 729-733.

[11] E. L. Post, *The two-valued iterative systems of mathematical logic*, Princeton 1941.

[12] S. Świerczkowski, *Algebras which are independently generated by every n elements*, Fund. Math. 49 (1960), pp. 93-104.

[13] K. Urbanik, *On algebraic operations in idempotent algebras*, Coll. Math. 13 (1965), pp. 129-157.

WROCŁAW UNIVERSITY,
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 17. 9. 1965

Equationally compact algebras (I)

by

B. Węglorz (Wrocław)

0. Introduction. This paper gives a study of equationally compact algebras, introduced by J. Mycielski [11], and some generalizations of this notions. The equational compactness is a simple reformulation in the language of general algebras of a definition of J. Łoś [8] of the notion of algebraical compactness of Abelian groups introduced by I. Kaplansky. The definitions of this and related notions are given in Section 1.

The main results of this paper are contained in Section 2 and give a characterization of equationally compact algebras in terms of ultrapowers and retracts. Perhaps the most interesting result is that positive compactness and atomic-compactness coincide.

In Section 3 we add several remarks and propositions concerning equationally and weakly equationally compact algebras of well-known kinds such as linear spaces, groups and modules, and in Section 4 equationally compact Boolean algebras are studied. In Section 5 we prove that equational compactness in general is not elementarily definable and we mention some open problems.

The author is indebted to Jan Mycielski and C. Ryll-Nardzewski for their discussions which improved the theorems and simplified the proofs, and to the first of them for many stimulating questions and help in composition of this paper.

The main results were announced in [17].

1. Preliminaries. For any non-empty sets X and Y, Y^X denotes the set of all functions $f\colon X\to Y$; the cardinality of a set X is denoted by $|X|$; $\omega = \{0, 1, 2, \ldots\}$. $\mathfrak{A} = \langle A, \{F_q\}_{q\in Q}, \{G_r\}_{r\in R}\rangle$ is an *algebraic system* if A is a non-void set, there are maps $f\colon Q\to\omega$ and $g\colon R\to\omega-\{O\}$ such that $F_q\colon A^{f(q)}\to A$ for $f(q)>0$ and $F_q\in A$ for $f(q)=0$, and $G_r\subseteq A^{g(r)}$ for all $r\in R$. The sequence $\tau = \langle Q, f, R, g\rangle$, uniquely determined by $\mathfrak{A}$, is called the *similarity type of* $\mathfrak{A}$. If R is void, then $\mathfrak{A}$ is called an *algebra*. A is called the set of $\mathfrak{A}$. In the sequal we denote algebraic systems by $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ and their sets by $A, B, C, \ldots$, respectively.

If $B\subseteq A$ and, for each F_q, if $b_1, \ldots, b_{f(q)}\in B$ then $F_q(b_1, \ldots, b_{f(q)})\in B$, then $\mathfrak{B} = \langle B, \{F'_q\}_{q\in Q}, \{G'_r\}_{r\in R}\rangle$, where $G'_r = G_r\cap B^{g(r)}$ and F'_q are obtained from F_q by restriction of F_q to $B^{f(q)}$, is an algebraic system. Such a system $\mathfrak{B}$ is called a *subsystem of* $\mathfrak{A}$ and $\mathfrak{A}$ is called an *extension of* $\mathfrak{B}$, in symbols $\mathfrak{B} \subseteq \mathfrak{A}$.

It is easy to see that the intersection of arbitrary subsystems $\mathfrak{B}_i$ $(i \in I)$ of $\mathfrak{A}$ is a subsystem of $\mathfrak{A}$ if $\bigcap_{i \in I} B_i \neq O$. Thus we can talk about the *smallest subsystem of* $\mathfrak{A}$ *containing a given non-empty set* X *of elements of* $\mathfrak{A}$.

With any similarity type τ we correlate the class $L^{(\tau)}$ of all elementary formulas with identity, with the logical symbols: $\wedge$ (and), $\vee$ (or), $\sim$ (non), $=$ (equals), $\bigvee$ (there exists), $\bigwedge$ (for every) (1), the distinct $f(q)$-ary operation symbols $\mathbf{F}_q$ $(q \in Q)$, and $g(r)$-ary predicates $\mathbf{G}_r$ $(r \in R)$ and individual variables x_a, where a is an arbitrary ordinal number (unlike in the usual treatements which suppose that $a < \omega$).

For any class $K \subseteq L^{(\tau)}$ and any $\mathfrak{A}$ of type τ, we denote by $K(\mathfrak{A})$ the class of all formulas which can be obtained from formulas in K by substituting some elements of A for some free variables. Thus we have $K \subseteq K(\mathfrak{A}) \subseteq L^{(\tau)}(\mathfrak{A})$. Formulas of $L^{(\tau)}(\mathfrak{A})$ are called *formulas with constants in* $\mathfrak{A}$.

The satisfaction of a set Σ of formulas of $L^{(\tau)}(\mathfrak{A})$ by a system $\{a_a\}_{a \in S}$ of elements of $\mathfrak{A}$, where S is the set of indices of the free variables of Σ, is defined in the natural way (see [14]); for the notion of elementary extensions and elementary subsystems of algebraic systems, see also [14].

Let $K \subseteq L^{(\tau)}$. An algebraic system $\mathfrak{A}$ of type τ is called K*-compact* (or *weakly* K*-compact*) if and only if each set $\Sigma \subseteq K(\mathfrak{A})$ (or $\Sigma \subseteq K$) is satisfiable in $\mathfrak{A}$ whenever each finite subset of Σ is satisfiable in $\mathfrak{A}$.

If K consists of all equations, then K-compactness (weak K-compactness) is called *equational compactness* (*weak equational compactness*); if K consists of all atomic formulas (i.e. formulas of the form $\mathbf{G}_r(\vartheta_1, \ldots, \vartheta_{g(r)})$ or $\vartheta = \vartheta'$, where $\vartheta_1, \ldots, \vartheta_{g(r)}, \vartheta, \vartheta'$ are terms) this is called *atomic-compactness* (*weak atomic compactness*); if K consists of all positive formulas (i.e. formulas which do not contain the symbol $\sim$) (2), this is called *positive compactness* (*weak positive compactness*); if $K = L^{(\tau)}$, this is called *elementary compactness* (*weak elementary compactness*) (3). All those notions are due to Jan Mycielski (for some properties weaker than equational and elementary compactness, see also his paper [11]).

An algebraic system $\mathfrak{A}$ is called *pure* (*weakly pure*) *in the algebraic system* $\mathfrak{B}$ or $\mathfrak{B}$ is called a *pure* (*weakly pure*) *extension of* $\mathfrak{A}$ if it is a subsystem of $\mathfrak{B}$ and any finite set of atomic formulas with constants (without constants) in $\mathfrak{A}$ which is satisfiable in $\mathfrak{B}$ is also satisfiable in $\mathfrak{A}$.

(1) It is important for the futher definitions that $\Rightarrow$ and $\Leftrightarrow$ are not here.

(2) The semantical theory of positive formulas was studied by R. C. Lyndon [7].

(3) Of course, a system $\mathfrak{A}$ is elementarily compact (weakly elementarily compact) if and only if $|A| < \aleph_0$.

Of course, each elementary subsystem of a system is a pure subsystem of this system. It is easy to see that in case Abelian groups the notion of pure subsystem coincides with the usual notion of pure subgroup.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebraic systems of similarity type τ. A function $h: A \to B$ is called a *homomorphism of* $\mathfrak{A}$ *into* $\mathfrak{B}$ if for every atomic formula $\Phi(x_{a_1}, \ldots, x_{a_n})$ if $\Phi(a_{a_1}, \ldots, a_{a_n})$ holds in $\mathfrak{A}$ for $a_{a_1}, \ldots, a_{a_n} \in A$, then $\Phi(h(a_{a_1}), \ldots, h(a_{a_n}))$ holds in $\mathfrak{B}$. If, moreover, $\mathfrak{B}$ is a subsystem of $\mathfrak{A}$ and h restricted to B is the identity map, then h is called a *retraction of* $\mathfrak{A}$ *onto* $\mathfrak{B}$ and $\mathfrak{B}$ is called a *retract of* $\mathfrak{A}$.

If $\mathfrak{A}$ and $\mathfrak{B}$ are algebras, then those notions of homomorphism and retraction coincide with the usual homomorphism and retraction of the theory of general algebras.

An algebraic system $\mathfrak{A}$ is called an *absolute retract in a class* K of algebraic systems of the same similarity type if for each algebraic system $\mathfrak{B} \in K$, such that $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A}$ is a retract of $\mathfrak{B}$.

We note the following three propositions which are due to Jan Mycielski (for a part of 1.1 and 1.2, see his paper [11]).

PROPOSITION 1.1. *The direct product of atomic compact (weak atomic compact) similar algebraic systems is atomic compact (weak atomic compact).*

PROPOSITION 1.2. *A retract of an atomic (positively) compact algebraic systems is atomic (positively) compact. If* $\mathfrak{B} \subseteq \mathfrak{A}$ *and* $h: \mathfrak{A} \to \mathfrak{B}$ *is a homomorphism then weak atomic compactness of any one of the three systems* $h(\mathfrak{A})$ *or* $\mathfrak{B}$ *or* $\mathfrak{A}$ *implies the same property for remaining two systems* (4).

PROPOSITION 1.3. *An algebraic system* $\mathfrak{A}$ *is a retract of* $\mathfrak{B}$ *if and only if each set of atomic formulas with constants in* $\mathfrak{A}$ *which is satisfiable in* $\mathfrak{B}$ *is also satisfiable in* $\mathfrak{A}$.

If $\mathfrak{A}_i$ $(i \in I)$ is a set of similar algebraic systems and $\mathfrak{D}$ is a filter of subsets of I, then $\mathfrak{P}_{i \in I}\mathfrak{A}_i/\mathfrak{D}$ denotes the *reduced direct product*, i.e. the quotient of the direct product $\mathfrak{P}_{i \in I}\mathfrak{A}_i$ by an equivalence $\equiv_{\mathfrak{D}}$ defined as follows:

$$\{a(i)\}_{i \in I} \equiv_{\mathfrak{D}} \{b(i)\}_{i \in I} \text{ if and only if } \{i \in I: a(i) = b(i)\} \in \mathfrak{D}.$$

If $\mathfrak{D}$ is an ultrafilter, then $\mathfrak{P}_{i \in I}\mathfrak{A}_i/\mathfrak{D}$ is called *an ultraproduct*. If for each $i \in I$, $\mathfrak{A}_i = \mathfrak{A}$ and $\mathfrak{D}$ is an ultrafilter over I, then $\mathfrak{P}_{i \in I}\mathfrak{A}_i/\mathfrak{D}$ is denoted by $\mathfrak{A}^I_{\mathfrak{D}}$ and called the *ultrapower* of $\mathfrak{A}$. For each $a \in A$ we put $h(a) = \{a\}_{i \in I}/\equiv_{\mathfrak{D}}$. It is easy to see that h is an isomorphism of $\mathfrak{A}$ into $\mathfrak{A}^I_{\mathfrak{D}}$. When speaking of ultrapowers we shall identify $\mathfrak{A}$ with $h(\mathfrak{A})$. For a detailed study of reduced products, see [4].

(4) The proof of this proposition is based on the well-known invariance of positive formulas under homomorphism. See Marczewski [10] and l.c. (2).

For the notion of *elementary class* $\mathbf{EC}_\Delta$, *universal class* $\mathbf{UC}_\Delta$, *pseudo-elementary class* $\mathbf{PC}_\Delta$, and *quasi-elementary class* $\mathbf{QC}_\Delta$, see e.g. [4].

Let K be an arbitrary class of similar algebraic systems; then by $\mathfrak{I}K$, $\mathfrak{S}K$, $\mathfrak{P}K$, $\mathfrak{U}K$ we denote the class of all isomorphic images, subsystems, products, and ultraproducts of members of K, respectively. All classes in this paper are supposed to be similar and such that $\mathfrak{I}K \subseteq K$.

2. The main results.

In this section we give several characterizations of positively compact and weakly positively compact algebraic systems.

LEMMA 2.1. *If an atomic compact (weak atomic compact) system $\mathfrak{A}$ is a pure (weakly pure) subsystem of $\mathfrak{B}$, then $\mathfrak{A}$ is a retract (contains a homomorphic image) of $\mathfrak{B}$* (5).

Proof. Let $\mathfrak{A}$ be an atomic compact and pure subsystem of $\mathfrak{B}$. Let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a set of atomic formulas, in which for simplicity we denote the indices of free variables by elements of $\mathfrak{B}$ defined as follows:

$$\Sigma_1 = \{\text{“}\Phi(x_{b_1}, \ldots, x_{b_n})\text{”}: b_1, \ldots, b_n \in B \text{ and } \Phi(b_1, \ldots, b_n) \text{ holds in } \mathfrak{B}\},$$
$$\Sigma_2 = \{\text{“}x_b = b\text{”}: b \in A\}.$$

Clearly Σ is satisfiable in $\mathfrak{B}$ since it suffices to put $x_b = b$ for each $b \in B$. Thus each finite subsystem of Σ is satisfiable in $\mathfrak{B}$. Since $\mathfrak{A}$ is a pure subsystem of $\mathfrak{B}$ and constants of Σ belong to $\mathfrak{A}$, thus each finite subsystem of Σ is satisfiable in $\mathfrak{A}$. But $\mathfrak{A}$ is atomic compact; thus Σ is satisfiable by a system $\{a_b\}_{b\in B}$ of elements of $\mathfrak{A}$. The mapping $h: B \to A$ defined by $h(b) = a_b$ $(b \in B)$ is a homomorphism, because $\{a_b\}_{b\in B}$ satisfies Σ_1 in $\mathfrak{A}$, and it is a retraction since $\{a_b\}_{b\in B}$ satisfies also Σ_2.

In the "weakly" case, the proof can be obtained by restricting our consideration to the set Σ_1 only.

LEMMA 2.2. *Let Σ be a set of formulas of $L^{(\tau)}(\mathfrak{A})$. If each finite subset of Σ is satisfiable in $\mathfrak{A}$, then there is an ultrapower of $\mathfrak{A}$ in which Σ is satisfiable.*

Proof. Let $I = \{\Theta \subseteq \Sigma: |\Theta| < \aleph_0\}$. For each $\varphi \in \Sigma$, let $D_\varphi = \{\Theta \in I: \varphi \in \Theta\}$. Let us observe that $D_\varphi \neq O$ for each $\varphi \in \Sigma$ (since $\{\varphi\} \in D_\varphi$) and that finite intersections of the sets D_φ are non-empty. Thus the smallest $\mathfrak{D}_0$ containing the family $\{D_\varphi: \varphi \in \Sigma\}$ is proper. Let $\mathfrak{D}$ be an ultrafilter containing $\mathfrak{D}_0$. We will prove that Σ is satisfiable by a system of elements of $\mathfrak{A}^I_{\mathfrak{D}}$. For each $\Theta \in I$, let the system $\{a_\alpha(\Theta)\}_{\alpha\in S}$ satisfy Θ in $\mathfrak{A}$, where S is the set of indices of free variables in Σ. Hence, by the theorem of Łoś, $\{\{a_\alpha(\Theta)\}_{\Theta\in I}/\equiv_{\mathfrak{D}}\}_{\alpha\in S}$ satisfies Σ in $\mathfrak{A}^I_{\mathfrak{D}}$.

(5) This lemma, in the case of algebras, is due to Jan Mycielski and C. Ryll-Nardzewski.

This lemma can be obtained as a simple consequence of some theorems of Keisler [6] by using Theorem 3 of [11]. We due the above proof to C. Ryll-Nardzewski. This proof is analogous to the proof of Theorem 2.10 in [4]. Originally, the proofs of Theorems 2.3 and 2.4 below were obtained using some results of Keisler, but Lemma 2.2 is much simpler.

THEOREM 2.3. (6) *The following conditions are equivalent:*

(i) *$\mathfrak{A}$ is positively compact;*

(ii) *$\mathfrak{A}$ is atomic compact;*

(iii) *$\mathfrak{A}$ is a retract of every algebraic system in which $\mathfrak{A}$ is pure;*

(iv) *$\mathfrak{A}$ is a retract of every elementary extension of $\mathfrak{A}$;*

(v) *$\mathfrak{A}$ is a retract of every ultrapower of $\mathfrak{A}$.*

Proof. (i) trivially implies (ii), (ii) implies (iii) by Lemma 2.1; (iii) trivially implies (iv) and (iv) trivially implies (v) (by a theorem of Łoś on ultrapowers (see e.g. [4])).

Now we show that (v) implies (i). Let Σ be an arbitrary set of positive formulas of $L^{(\tau)}(\mathfrak{A})$ such that every finite subset of Σ is satisfiable in $\mathfrak{A}$. By Lemma 2.2, there is a ultrapower $\mathfrak{A}^I_{\mathfrak{D}}$ of $\mathfrak{A}$ in which it is satisfied by a system $\{a_\alpha\}_{\alpha\in S}$ of elements of $\mathfrak{A}^I_{\mathfrak{D}}$ (S denotes the set of indices of free variables in Σ). But, by (v), there is a retraction h of $\mathfrak{A}^I_{\mathfrak{D}}$ onto $\mathfrak{A}$, and by Marczewski's theorem (7) the system $\{h(a_\alpha)\}_{\alpha\in S}$ satisfies Σ in $\mathfrak{A}$. Thus, since Σ was arbitrary, $\mathfrak{A}$ is positively compact and (v) implies (i).

If $\mathfrak{A}$ is an algebra, then atomic formulas are equations, thus atomic compactness and equational compactness coincide. Hence Theorem 2.3 gives a characterization of equationally compact algebras. For similar results, see [12]. Additional characterizations of equationally compact Boolean algebras will be given in Theorem 4.1 below. For Abelian groups, two parts of Theorem 2.3 were known. Łoś [8] proved that *an Abelian group $\mathfrak{G}$ is equationally compact if and only if $\mathfrak{G}$ is a direct summand of each Abelian group in which $\mathfrak{G}$ is a pure subgroup*. And it was shown by Balcerzyk [1] and Gacsályi [5] that *a subgroup $\mathfrak{G}$ of an Abelian group $\mathfrak{H}$ is a retract of $\mathfrak{H}$ if and only if it is a summand of $\mathfrak{D}$*. Thus the equivalence of (ii) and (iii) follows. Also the equivalence of (ii) and (v) for Abelian groups was established by Łoś [9]. Another characterization of equationally compact Abelian groups was given by Balcerzyk [2]. By his result, *$\mathfrak{G}$ is such if and only if every set of equations*

$$\{\text{“}x_0 - a_n = n!x_n\text{”}: n = 1, 2, \ldots\} \quad (a_n \in G)$$

(6) The actual formulation of theorem 2.3 and its clause (i) was proposed by Jan Mycielski.

(7) See footnote (4).

is satisfiable in $\mathfrak{G}$ whenever every finite subset of this set is satisfiable (thus countable sets of equations are sufficient!).

THEOREM 2.4. *The following conditions are equivalent:*

(i) $\mathfrak{A}$ *is weakly atomic compact;*

(ii) $\mathfrak{A}$ *contains a homomorphic image of every algebraic systems in which* $\mathfrak{A}$ *is weakly pure;*

(iii) $\mathfrak{A}$ *contains a homomorphic image of every elementary extension of* $\mathfrak{A}$;

(iv) $\mathfrak{A}$ *contains a homomorphic image of every ultrapower of* $\mathfrak{A}$.

The proof is analogous to that of Theorem 2.3; it uses the "weakly" part of Lemmas 2.1 and 2.2.

From those theorems we obtain a corollary which is important for applications.

COROLLARY 2.5. *Let K be a class of algebraic systems closed under formation of ultrapowers and let $\mathfrak{A} \in K$ be an absolute retract in K* (see § 1) *or $\mathfrak{A}$ contains a homomorphic image of each $\mathfrak{B} \in K$ which is an extension of $\mathfrak{A}$. Then $\mathfrak{A}$ is positively compact or weakly atomic compact respectively.*

THEOREM 2.6. *Let $K_0 = \{\mathfrak{A}_1, \ldots, \mathfrak{A}_n\}$ be a finite set of weakly atomic compact algebraic systems and K be the smallest universal class* ($K \in \mathbf{UC}_\Delta$) *containing K_0. Then for every $\mathfrak{A} \in K$ there is a weakly atomic compact system $\mathfrak{B} \in K$ such that $\mathfrak{A} \subseteq \mathfrak{B}$.*

Proof. By a theorem of Łoś (7), we have $K = \mathcal{S}\mathcal{U}K_0$. Since K_0 is finite, $\mathcal{U}K_0$ contains algebraic systems isomorphic with ultrapowers of algebraic systems in K_0 only. By Theorem 2.4 ((i)$\Rightarrow$(iv)) and Proposition 1.2, we see that all algebraic systems of $\mathcal{U}K_0$ are weakly atomic compact. Thus Theorem 2.6 follows.

3. Applications to modules. Let K be an arbitrary class of algebraic systems. We recall that $\mathfrak{A}$ is *injective* in K if $\mathfrak{A} \in K$ and for every homomorphism $h: \mathfrak{B} \to \mathfrak{A}$, where $\mathfrak{B} \in K$ and every extension $\mathfrak{C} \in K$ of $\mathfrak{B}$, h can be extended to a homomorphism $h': \mathfrak{C} \to \mathfrak{A}$.

PROPOSITION 3.1. *If K is an arbitrary class of algebraic systems and $\mathfrak{A}$ is injective in K, then $\mathfrak{A}$ is an absolute retract in K. Thus, if moreover $K = \mathcal{U}K$ and $\mathfrak{A}$ is injective in K, then $\mathfrak{A}$ is positively compact.*

Since we can put $\mathfrak{A} = \mathfrak{B}$ and for h the identity map.

Corollary 2.5 and Proposition 3.1 can be applied for the proofs that some algebras are equationally compact or weakly equationally compact.

First we recall two known results for modules over an associative

(8) See e.g. [4], Theorems 1.15 and 2.15.

ring with unity (see [3] I, § 3, Theorems 3.2 and 3.3) Now $\mathfrak{R}$ is supposed to be such a ring and let M be the class of all left modules over $\mathfrak{R}$ (9).

THEOREM 3.2. *The module $\mathfrak{A} \in M$ is injective in M if and only if for any left ideal J of $\mathfrak{R}$ and any homomorphism $h: \mathfrak{J} \to \mathfrak{A}$ (where $\mathfrak{J}$ denotes the ideal J treated as a left module over $\mathfrak{R}$) there is an element a of $\mathfrak{A}$ such that $h(r) = r \cdot a$, for each $r \in J$.*

THEOREM 3.3. *Each $\mathfrak{A} \in M$ has an extension $\mathfrak{B} \in M$ which is injective in M.*

From Theorem 3.2, we see at once that *each linear space over any field is an equationally compact algebra* (10). Also, we see that *every module $\mathfrak{A} \in M$ can be embedded into an equationally compact one.* But somewhat more will be proved in the next theorem. Let us recall that an algebra $\mathfrak{A}$ is *topologically compact* if we can introduce in $\mathfrak{A}$ a compact Hausdorff topology such that all operations in $\mathfrak{A}$ are continuous (11).

The following theorem solves a problem of Mycielski, asking if every equationally compact algebra is a retract of a compact topological algebra (see [11], P 484) in the case of linear spaces over a field.

THEOREM 3.4. *Each $\mathfrak{A} \in M$ has an extension $\mathfrak{B} \in M$ which is topologically compact. Thus if $\mathfrak{R}$ is a field, then every linear space over $\mathfrak{R}$ is a retract of a topologically compact linear space.*

Proof. Let $\mathfrak{A} = \langle A, +, \{F_r\}_{r \in R}\rangle$ be an arbitrary left module over $\mathfrak{R}$ (the operations F_r are defined by $F_r(x) = r \cdot x$, for all $r \in R$ and $x \in A$); then $\mathfrak{A}^+ = \langle A, +\rangle$ is an Abelian group and, for each $r \in R$, F_r can be treated as an endomorphism of $\mathfrak{A}^+$. Moreover, $\mathfrak{A}^+$ with the discrete topology is a locally compact Abelian group; hence it is isomorphic to a dense subgroup of the Bohr compactificaton $\widetilde{\mathfrak{A}}^+$ of $\mathfrak{A}^+$. Since, for each $r \in R$, F_r is continuous on $\mathfrak{A}^+$, there is an endomorphism $\widetilde{F}_r$ of $\widetilde{\mathfrak{A}}^+$ which is continuous on $\widetilde{\mathfrak{A}}^+$ and is an extension of F_r. But this implies that $\widetilde{\mathfrak{A}} = \langle \widetilde{A}, +, \{F_r\}_{r \in R}\rangle$ is a topologically compact module which is an extension of $\mathfrak{A}$.

Mycielski's problem for an arbitrary class of modules is not solved.

4. Equationally compact Boolean algebras. By Corollary 2.5, we see that each Boolean algebra is weakly equationally compact. The aim of this section is to prove that for Boolean algebras equational compactness and completeness coincide, and to show the related results.

THEOREM 4.1. *For a Boolean algebra $\mathfrak{A}$ the following conditions are equivalent:*

(9) A left module over a ring $\mathfrak{R}$ is a system having one operation of two variables $x+y$ and the set of operations of one variable $F_r(x) = r \cdot x$, where r is any element of $\mathfrak{R}$.

(10) See footnote (8).

(11) For the definition and properties of topological algebras, see e.g. [11] and [16].

(i) $\mathfrak{A}$ *is complete*;

(ii) $\mathfrak{A}$ *is injective in the class of all Boolean algebras*;

(iii) $\mathfrak{A}$ *is an absolute retract in the class of all Boolean algebras*.

(iv) $\mathfrak{A}$ *is equationally compact*.

Proof. (i)$\Rightarrow$(ii) is known (see e.g. [13]).

(ii)$\Rightarrow$(iii) by Proposition 3.1.

(iii)$\Rightarrow$(iv) by Corollary 2.5.

(iv)$\Rightarrow$(i). Let us suppose that $\mathfrak{A}$ is equationally compact and let $(a_i)_{i\in I}$ be an arbitrary family of elements of $\mathfrak{A}$. Let us write $X = \{c \in A: \bigwedge_{i\in I} c \cap a_i = c\}$. Now, consider a following set of equations having one free variable x_0 only:

$$\Sigma = \{\text{“}x_0 \cap a_i = x_0\text{”}: i \in I\} \cup \{\text{“}x_0 \cup c = x_0\text{”}: c \in X\}.$$

It is easy to see that each finite subset of Σ has a solution in $\mathfrak{A}$, thus by compactness of $\mathfrak{A}$, Σ has a solution in $\mathfrak{A}$. It is easy to verify that this solution is $\bigcap_{i\in I} a_i$. Thus $\mathfrak{A}$ is complete.

The following corollary solves a problem of Mycielski (see [11], P 484) in the case of Boolean algebras.

COROLLARY 4.2. *A Boolean algebra is equationally compact if and only if it is a retract of a compact topological algebra.*

Proof. By Proposition 1.2 and the fact that topologically compact algebras are equationally compact ([12]), all retracts of topologically compact algebras are equationally compact. The converse follows from Theorem 4.1, and from the fact that each algebra $\mathfrak{A}$ is contained in the complete algebra $\mathfrak{S}$ of all subsets of the Stone space of $\mathfrak{A}$, and $\mathfrak{S}$ has a compact topology since it is isomorphic to a product of two-elements Boolean algebras.

5. Other remarks. Let K be an arbitrary class of algebras; denote by K_{TC}, K_{RC}, K_{WRC} classes of those algebras in K which can be endowed with a compact topology ([13]), are equationally compact or weakly equationally compact, respectively.

We have the following example:

EXAMPLE 5.1. *There is an equational class K such that no class K_{TC}, K_{RC}, K_{WRC} is elementary, i.e. $K_{TC}, K_{RC}, K_{WRC} \notin \mathbf{EC}_\Delta$. Moreover, K_{TC}, $K_{RC}, K_{WRC} \notin \mathbf{PC}_\Delta$, $K_{TC}, K_{RC}, K_{WRC} \notin \mathbf{QC}_\Delta$ and $\mathbb{S}K_{TC}, \mathbb{S}K_{RC}, \mathbb{S}K_{WRC} \notin \mathbf{UC}_\Delta$.*

Indeed, let K be an equational class of algebras having two 0-ary operations (i.e. constants), say **0** and **1**, and one 2-ary operation $\bullet$; and K is defined by a single equation $x \bullet x = \mathbf{0}$.

([12]) See [11], Proposition 1.

([13]) See l.c. ([10]).

Let $\mathfrak{M} = \langle\{0, 1, 2, \ldots\}, 0, 1, \cdot\rangle$, where $\cdot$ is defined as follows: $x\cdot y = 0$ if $x = y$ and $x\cdot y = 1$ if $x \neq y$. This algebra was defined by Mycielski in [11] where he proved that there is no weakly equationally compact algebra $\mathfrak{B} \in K$ such that $\mathfrak{M} \subseteq \mathfrak{B}$ ([14]).

Now, let K_0 be the set of all finite subalgebras of $\mathfrak{M}$. It is easy to see that $\mathfrak{M}$ can be embedded in some ultraproduct $\mathfrak{P}$ of members of K_0. Thus such a $\mathfrak{P}$ is not weakly equationally compact, and moreover, there is no weakly equationally compact algebra $\mathfrak{B} \in K$ such that $\mathfrak{P} \subseteq \mathfrak{B}$.

Since $K_0 \subseteq K_{TC} \subseteq K_{RC} \subseteq K_{WRC}$, then none of these classes is closed under formation of ultraproducts and hence $K_{TC}, K_{RC}, K_{WRC} \notin \mathbf{EC}_\Delta$ and $K_{TC}, K_{RC}, K_{WRC} \notin \mathbf{QC}_\Delta$.

Now, suppose e.g. that $\mathbb{S}K_{TC} \in \mathbf{UC}_\Delta$; then $\mathbb{S}K_{TC}$ would be closed under formation of ultraproducts, but this is impossible since we would have $\mathfrak{U}K_0 \subseteq \mathfrak{U}\mathbb{S}K_{TC} = \mathbb{S}K_{TC}$, but $\mathfrak{P} \notin \mathbb{S}K_{TC}$ which is a contradiction. For $\mathbb{S}K_{RC}$ and $\mathbb{S}K_{WRC}$ the proof is the same.

Finally if one of the classes K_{TC}, K_{RC}, K_{WRC} belonged to $\mathbf{PC}_\Delta$, then by a theorem of Łoś and Tarski (see e.g. [15]) the class of subsystems of this class would belong to $\mathbf{UC}_\Delta$, but this was already disproved.

EXAMPLE 5.2. There is a class K of algebras closed under formation of endomorphic images, products, and elementary extensions which is not an elementary class (moreover, $K \notin \mathbf{PC}_\Delta$, $K \notin \mathbf{QC}_\Delta$, and $\mathbb{S}K \notin \mathbf{UC}_\Delta$).

Such is K_{WRC} of Example 5.1.

EXAMPLE 5.3. There is a class K of compact topological algebras closed under topological products (i.e. direct products with the Tychonov product topology), closed subalgebras and continuous homomorphic images, which is not an elementary class (moreover, $K \notin \mathbf{PC}_\Delta$, $K \notin \mathbf{QC}_\Delta$, and $\mathbb{S}K \notin \mathbf{UC}_\Delta$).

Such is K_{TC} of Example 5.1.

Remark 5.4. The assumption of finiteness of class K_0 in Theorem 2.6, is essential since K_0, defined in Example 5.1, satisfies the other assumptions of 2.6.

Example 5.1 suggests the question if there are equational classes K such that some (or all) of the classes $K - K_{TC}$, $K - K_{RC}$, $K - K_{WRC}$ or $K - \mathbb{S}K_{TC}$, $K - \mathbb{S}K_{RC}$, $K - \mathbb{S}K_{WRC}$ are not elementary, but I do not know any such K ([15]).

For any algebra $\mathfrak{A}$, the possibility of imbedding $\mathfrak{A}$ in a compact topological algebra implies that such an algebra can be choosen in the smallest equational class containing $\mathfrak{A}$. I do not know if an algebra having

([14]) By a simple modification of his proof, we can show that even outside of K there is no algebra $\mathfrak{B}$ similar to $\mathfrak{M}$, which is weakly equationally compact and $\mathfrak{M} \subseteq \mathfrak{N}$.

([15]) Now, I have proved that if K is the class of lattices, then $K - K_{TC}$, $K - K_{RC}$, $K - \mathbb{S}K_{TC}$ and $K - \mathbb{S}K_{RC}$ are not elementary. See [18].

a weak equational compactification has such a compactification in the smallest equational class containing $\mathfrak{A}$ [16].

Beside the K-compactness ($K \subseteq L^{(\tau)}$), we can consider for every cardinal $\mathfrak{m}$, a weaker notion:

An algebraic system $\mathfrak{A}$ of type τ is called *K-$\mathfrak{m}$-compact* if the condition of compactness holds for sets of formulas having at most the cardinality $\mathfrak{m}$. This notion was considered in [11] and [12]; compare also [6]. I do not know whether $\mathfrak{m}^+$-completeness for Boolean algebras ($\mathfrak{m}^+$ denotes the succesor of $\mathfrak{m}$) implies equational $\mathfrak{m}$-compactness or conversely?

References

[1] S. Balcerzyk, *Remark on a paper of S. Gacsályi*, Publ. Math. 4 (1956), pp. 357-358.

[2] — *On the algebraically compact groups of I. Kaplansky*, Fund. Math. 44 (1957), pp. 91-93.

[3] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton 1956.

[4] T. Frayne, A. C. Morel and D. S. Scott, *Reduced direct products*, Fund. Math. 51 (1962), pp. 195-228.

[5] S. Gacsályi, *On pure subgroups and direct summands of Abelian groups*, Publ. Math. 4 (1955), pp. 88-92.

[6] H. J. Keisler, *Ultraproducts and saturated models*, Indag. Math. 26 (1964), pp. 178-186.

[7] R. C. Lyndon, *Properties preserved under homomorphism*, Pacyfic Journ. Math. 9 (1959), pp. 143-154.

[8] J. Łoś, *Abelian groups that are direct summands of every Abelian group which contain them as a pure subgroup*, Fund. Math. 44 (1957), pp. 84-90.

[9] — *Generalized limits in algebraically compact groups*, Bull. Acad. Polon. Sci., Seriés sci. math., astr. et phys. 7 (1959), pp. 19-21.

[10] E. Marczewski, *Sur les congruences et les propriétés positives d'algèbres abstraites*, Colloq. Math. 2 (1951), pp. 220-228.

[11] Jan Mycielski, *Some compactifications of general algebras*, Colloq. Math. 13 (1964), pp. 1-9.

[12] — and C. Ryll-Nardzewski, *Equationally compact algebras (II)*, Fund. Math. (in print).

[13] Z. Semadeni, *Projectivity, injectivity and duality*, Rozprawy Mat. 35 (1963).

[14] A. Tarski and R. L. Vaught, *Arithmetical extension of relational systems*, Compositio Math. 13 (1957), pp. 81-102.

[15] R. L. Vaught, *The elementary character of two notions from general algebra*, pp. 226-233, *Essays on the Foundations of Mathematics*, Jerusalem 1961.

[16] B. Węglorz, *Remarks on compactifications of abstract algebras*, Colloq. Math. 14 (1966), Comptes Rendus, Conférence sur l'Algèbre Générale, Varsovie 1964, p. 372.

[17] — *Compactness of algebraic systems*, Bull. Acad. Polon. Sci., Seriés sci. math., astr. et phys. 13 (1965), pp. 705-706.

[18] — *Completeness and compactness of Lattices*, Colloq. Math. (in print).

[16] For equational compactifications the answer is affirmative. The proof of this fact will be published in my paper — *Equationally compact algebras* (III) — in Fundamenta Mathematicae.

Reçu par la Rédaction le 8. 11. 1965

An addition to "On defining well-orderings"

by

E. G. K. Lopez-Escobar (Cambrigde, Mass.)

In the paper *On defining well-orderings* [2] we proved that the class $\mathbf{W}$ of well-orderings is not a $\mathbf{PC}_\Delta$-class of any infinitary first-order language of the type $\mathbf{L}_{\alpha\omega}$. The addition that we wish to make is to prove that $\mathbf{W}$ is not even a *relativized* $\mathbf{PC}_\Delta$-class (i.e. that for all α, $\mathbf{W} \notin \mathbf{RPC}_\Delta(\mathbf{L}_{\alpha\omega})$; (cf. definition below).

The method used to show that for all α, $\mathbf{W} \notin \mathbf{RPC}(\mathbf{L}_{\alpha\omega})$ (this clearly suffices in order to prove that for all α, $\mathbf{W} \notin \mathbf{RPC}_\Delta(\mathbf{L}_{\alpha\omega})$) is basically the same as that used in [2]. That is, from the assumption that $\mathbf{W} \in \mathbf{RPC}(\mathbf{L}_{\alpha\omega})$ we obtain a sentence Θ of $\mathbf{L}_{\alpha\omega}$ which has a model of cardinality greater (or possibly equal) to the Hanf-number for $\mathbf{L}_{\alpha\omega}$ but which does not have arbitrary large models. [1]

DEFINITION. *Suppose that* $\mathbf{K}$ *is a class of relational systems of the type* $\langle A, R\rangle$ *where* $R \subseteq A^2$, *then*:

(i) "$\mathbf{K}$ is a relativized $\mathbf{PC}_\Delta$-class of $\mathbf{L}_{\alpha\omega}$", *in symbols*: $\mathbf{K} \in \mathbf{RPC}_\Delta(\mathbf{L}_{\alpha\omega})$, *just in case that there exist a set* T *of sentences of* $\mathbf{L}_{\alpha\omega}$ *such that* $\mathbf{K}$ *consists exactly of those systems* $\langle A, R\rangle$ *for which there exists a set* $B \supseteq A$ *and relations* S_μ *on* B *such that* $\langle B, A, R, S_\mu\rangle_{\mu<\eta}$ *is a model of* T;

(ii) "$\mathbf{K}$ is a relativized $\mathbf{PC}$-class of $\mathbf{L}_{\alpha\omega}$", *in symbols*: $\mathbf{K} \in \mathbf{RPC}(\mathbf{L}_{\alpha\omega})$, *just in case that for some sentence* $\theta \in \mathbf{L}_{\alpha\omega}$, $\mathbf{K}$ *consists exactly of those systems* $\langle A, R\rangle$ *for which there exists a set* $B \supseteq A$ *and relations* S_μ *on* B *such that* $\langle B, A, R, S_\mu\rangle_{\mu<\eta}$ *is a model of* θ.

Note that if for all α, $\mathbf{W} \notin \mathbf{RPC}(\mathbf{L}_{\alpha\omega})$, then for all α, $\mathbf{W} \notin \mathbf{RPC}_\Delta(\mathbf{L}_{\alpha\omega})$.

THEOREM. *There does not exist a cardinal* α *such that* $\mathbf{W} \in \mathbf{RPC}(\mathbf{L}_{\alpha\omega})$.

Proof. Assume on the contrary that for some α, $\mathbf{W} \in \mathbf{RPC}(\mathbf{L}_{\alpha\omega})$. It is clear that we may assume that α is a successor cardinal, i.e. that for some cardinal π, $\alpha = \pi^+$. The assumption that $\mathbf{W} \in \mathbf{RPC}(\mathbf{L}_{\alpha\omega})$ means that there exists a sentence θ of $\mathbf{L}_{\alpha\omega}$ such that:

(1) to every (non-zero) ordinal ϱ there corresponds a set $B \supseteq \varrho$ and relations S_μ on B such that $\langle B, \varrho, \varepsilon_\varrho, S_\mu\rangle_{\mu<\eta}$ is a model of θ,

[1] For undefined notation, see [2].