8.4.4. Negation of condition (4). Let f be defined as follows: $f(X) = \dim X$ if and only if $\dim X \leqslant 0$; $f(X) = \operatorname{in} P_0 X$ if and only if $\dim X > 0$.

8.4.5. Negation of condition (5). Let f be defined as follows: $f(\emptyset) = -1$; $f(X) = \operatorname{in} \overline{T} X + 1$ if and only if $X \neq \emptyset$.

8.4.6. Negation of condition (6). Let f be defined as follows: $f(\emptyset) = -1$; $f(X) = \dim X / \operatorname{in} P_0 X$ if and only if $-1 < \dim X < \infty$; $f(X) = 1$ if and only if $\dim X = \infty$.

8.4.7. Negation of condition (7). Let $f(X) = \operatorname{in} P_0 X$ for all X.

References

[1] J. de Groot, *Topologische Studiën*, Groningen, 1942.

[2] J. de Groot and T. Nishiura, *Inductive compactness as a generalization of semicompactness*, Fund. Math. 58 (1966), pp. 201-218.

[3] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press, 1948.

[4] J. Isbell, *Uniform spaces*, Mathematical Surveys No. 12, Amer. Math. Soc. 1964.

[5] A. Lelek, *Dimension and mappings of spaces with finite deficiency*, Colloq. Math. 12 (1964), pp. 221-227.

[6] K. Menger, *Zur Begründung einer axiomatischen Theorie der Dimension*, Monat. für Math. und Phys. 36 (1929), pp. 193-218.

[7] K. Kuratowski, *Topologie I*, Monografie Matematyczne 20, Warszawa 1958.

WAYNE STATE UNIVERSITY
Detroit, Michigan, U.S.A.

Reçu par la Rédaction le 15. 9. 1965

On some numerical constants associated with abstract algebras

by

K. Urbanik (Wrocław)

1. Introduction. For the terminology and notation used here, see [5]. In particular, for a given abstract algebra $\mathfrak{A} = (A; \mathbf{F})$, where A is a non-void set and $\mathbf{F}$ is a class of fundamental operations, by $\mathbf{A}(\mathfrak{A})$ or $\mathbf{A}(\mathbf{F})$ we shall denote the class of all algebraic operations, i.e. the smallest class, closed under the composition, containing all fundamental operations and all trivial operations $e_k^{(n)}$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ defined by the formula

$$e_k^{(n)}(x_1, x_2, \ldots, x_n) = x_k .$$

The subclass of all n-ary algebraic operations in $\mathfrak{A}$ will be denoted by $\mathbf{A}^{(n)}(\mathfrak{A})$ or $\mathbf{A}^{(n)}(\mathbf{F})$ $(n \geqslant 0)$. Two algebras $(A; \mathbf{F}_1)$ and $(A; \mathbf{F}_2)$ having the same class of algebraic operations will be treated here as identical. If a non-void subset B of A is closed with respect to $\mathbf{F}$, then the algebra $(B; \mathbf{F})$ is called a *subalgebra* of the algebra $(A; \mathbf{F})$. An algebra $(A; \mathbf{G})$ is called a *reduct* of the algebra $(A; \mathbf{F})$ if $\mathbf{A}(\mathbf{G}) \subset \mathbf{A}(\mathbf{F})$. Further, by $\mathfrak{A}^{(n)}$ we shall denote the algebra of all n-ary algebraic operations in the algebra $\mathfrak{A}$.

In his study of certain numerical constants associated with abstract algebras, E. Marczewski introduced the order of enlargeability (called by him the *degree of extendability*) of abstract algebras (see [7], p. 182). We recall his definition of this concept. Let $\mathfrak{A} = (A; \mathbf{F})$. We say that a non-negative integer n belongs to the set $N(\mathfrak{A})$ if for every family $\mathbf{G}$ of operations in the set A the equation $\mathbf{A}^{(n)}(\mathbf{F}) = \mathbf{A}^{(n)}(\mathbf{G})$ implies the inclusion $\mathbf{A}(\mathbf{F}) \supset \mathbf{A}(\mathbf{G})$. In other words, $n \in N(\mathfrak{A})$ if and only if for every family $\mathbf{G}$ satisfying the condition $\mathbf{A}^{(n)}(\mathbf{F}) = \mathbf{A}^{(n)}(\mathbf{G})$ the algebra $(A; \mathbf{G})$ is a reduct of the algebra $(A; \mathbf{F})$. Further, let $\varepsilon(\mathfrak{A})$ be the smallest integer belonging to $N(\mathfrak{A})$ if the set $N(\mathfrak{A})$ is non-void and let $\varepsilon(\mathfrak{A}) = \infty$ in the opposite case. The quantity $\varepsilon(\mathfrak{A})$ is called the *order of enlargeability* of the algebra $\mathfrak{A}$. It is evident that

(i) *For an algebra* $\mathfrak{A} = (A; \mathbf{F})$ *the inequality* $\varepsilon(\mathfrak{A}) > k$ *holds if and only if there exists an operation* f *in* A *such that* $\mathbf{A}^{(k)}(\mathbf{F}) = \mathbf{A}^{(k)}(\mathbf{F} \cup \{f\})$ *and* $f \notin \mathbf{A}(\mathbf{F})$.

In this paper we shall prove that for many finitely generated algebras usually treated in mathematics, such as Abelian groups, vector spaces, Boolean algebras, etc., the order of enlargeability is simply the minimal number of generators. Now we shall define a substitute of the minimal number of generators for algebras which are not finitely generated.

Let $\gamma(\mathfrak{A})$ be the minimal number of generators of a finitely generated algebra $\mathfrak{A}$ (see [6]). We assume here that algebras whose all elements are algebraic constants, i.e. the values of constant algebraic operations, are generated by the empty set. Thus $\gamma(\mathfrak{A}) = 0$ if and only if all elements of $\mathfrak{A}$ are algebraic constants. Further, we put $\gamma(\mathfrak{A}) = \infty$ for algebras which are not finitely generated.

For any subalgebra $\mathfrak{B}$ of the algebra $\mathfrak{A}$ we put

$$\gamma(\mathfrak{B}, \mathfrak{A}) = \min \gamma(\mathfrak{C}),$$

where the minimum is extended over all subalgebras $\mathfrak{C}$ of the algebra $\mathfrak{A}$ containing the subalgebra $\mathfrak{B}$. Further, we put

$$\gamma_0(\mathfrak{A}) = \sup \gamma(\mathfrak{B}, \mathfrak{A}),$$

where the supremum is extended over all finitely generated subalgebras $\mathfrak{B}$ of the algebra $\mathfrak{A}$. Of course, for each algebra $\mathfrak{A}$ we have the inequality $\gamma_0(\mathfrak{A}) \leqslant \gamma(\mathfrak{A})$. Moreover, the inequality $\gamma_0(\mathfrak{A}) < \gamma(\mathfrak{A})$ may happen. For instance, if $\mathfrak{A}$ is the additive group of rationals, then $\gamma_0(\mathfrak{A}) = 1$ and $\gamma(\mathfrak{A}) = \infty$.

It is clear that for finitely generated algebras $\mathfrak{A}$ the equation $\gamma_0(\mathfrak{A}) = \gamma(\mathfrak{A})$ holds. Further, it is very easy to prove that

(ii) *For all reducts $\mathfrak{B}$ of the algebra $\mathfrak{A}$ the inequality $\gamma_0(\mathfrak{B}) \geqslant \gamma_0(\mathfrak{A})$ holds.*

Moreover,

(iii) *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are algebras defined on the same set and $\mathbf{A}^{(n)}(\mathfrak{A}_1) = \mathbf{A}^{(n)}(\mathfrak{A}_2)$ for all $n \leqslant \gamma_0(\mathfrak{A}_1)$, then $\gamma_0(\mathfrak{A}_1) = \gamma_0(\mathfrak{A}_2)$.*

In the sequel we shall write ε and γ_0 instead of $\varepsilon(\mathfrak{A})$ and $\gamma_0(\mathfrak{A})$ respectively when no confusion will arise.

2. p-enlargement of algebras. An algebra $\mathfrak{B}$ is said to be the *p-enlargement* of an algebra $\mathfrak{A}$ ($p = 1, 2, \ldots$), in symbols $\mathfrak{B} = \mathfrak{E}_p(\mathfrak{A})$, if both algebras $\mathfrak{A}$ and $\mathfrak{B}$ are defined on the same set and the class $\mathbf{A}(\mathfrak{B})$ consists of all operations whose compositions with operations from $\mathbf{A}^{(p)}(\mathfrak{A})$ belong to $\mathbf{A}^{(p)}(\mathfrak{A})$. Of course, for any index p the algebra $\mathfrak{A}$ is a reduct of the algebra $\mathfrak{E}_p(\mathfrak{A})$. Moreover,

(i) *If two algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are defined on the same set and $\mathbf{A}^{(p)}(\mathfrak{A}_1) = \mathbf{A}^{(p)}(\mathfrak{A}_2)$, then $\mathfrak{E}_p(\mathfrak{A}_1) = \mathfrak{E}_p(\mathfrak{A}_2)$.*

Hence we get the equation

$$\mathfrak{E}_p\big(\mathfrak{E}_p(\mathfrak{A})\big) = \mathfrak{E}_p(\mathfrak{A}) \qquad (p = 1, 2, \ldots). \tag{2.1}$$

THEOREM 2.1. *For each algebra $\mathfrak{A}$ we have the formula*

$$\gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big) = \min\big(p+1, \gamma_0(\mathfrak{A})\big) \qquad (p = 1, 2, \ldots).$$

Proof. First consider the case $\gamma_0(\mathfrak{A}) \leqslant p+1$. We have to prove the formula $\gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big) = \gamma_0(\mathfrak{A})$. Suppose the contrary, $\gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big) \neq \gamma_0(\mathfrak{A})$. Since the algebra $\mathfrak{A}$ is a reduct of the algebra $\mathfrak{E}_p(\mathfrak{A})$, we have, by proposition (ii) in Section 1, the inequality $\gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big) < \gamma_0(\mathfrak{A})$. Thus $\gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big) \leqslant p$ and, consequently, by the definition of the p-enlargement, $\mathbf{A}^{(n)}\big(\mathfrak{E}_p(\mathfrak{A})\big) = \mathbf{A}^{(n)}(\mathfrak{A})$ for all $n \leqslant \gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big)$. But this contradicts proposition (iii) in Section 1. Thus $\gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big) = \gamma_0(\mathfrak{A})$ whenever $\gamma_0(\mathfrak{A}) \leqslant p+1$.

Now consider the case $\gamma_0(\mathfrak{A}) > p+1$. We have to prove the formula $\gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big) = p+1$. Suppose that $\gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big) \leqslant p$. Then, by the definition of the p-enlargement, the equation $\mathbf{A}^{(n)}\big(\mathfrak{E}_p(A)\big) = \mathbf{A}^{(n)}(\mathfrak{A})$ holds for all $n \leqslant \gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big)$. Consequently, by proposition (iii) in Section 1, $\gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big) = \gamma_0(\mathfrak{A})$ which gives the contradiction. Thus

$$\gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big) \geqslant p+1. \tag{2.2}$$

Suppose that $\gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big) > p+1$. Then there exist elements $a_1, a_2, \ldots, a_{p+2}$ such that each subalgebra of the algebra $\mathfrak{E}_p(\mathfrak{A})$ containing the elements $a_1, a_2, \ldots, a_{p+2}$ is generated by at least $p+2$ elements. Put

$$f(a_1, a_2, \ldots, a_{p+1}) = a_{p+2} \tag{2.3}$$

and $f(x_1, x_2, \ldots, x_{p+1}) = x_1$ in the opposite case. Since the elements $a_1, a_2, \ldots, a_{p+1}$ do not belong to a subalgebra of the algebra $\mathfrak{E}_p(\mathfrak{A})$ generated by less than $p+1$ elements, we infer that for any system $g_1, g_2, \ldots, g_{p+1}$ of operations from $\mathbf{A}^{(p)}\big(\mathfrak{E}_p(\mathfrak{A})\big)$ and, consequently, from $\mathbf{A}^{(p)}(\mathfrak{A})$ the equation

$$f\big(g_1(x_1, x_2, \ldots, x_p), g_2(x_1, x_2, \ldots, x_p), \ldots, g_{p+1}(x_1, x_2, \ldots, x_p)\big) = g_1(x_1, x_2, \ldots, x_p)$$

holds. Thus, by the definition of the p-enlargement, the $(p+1)$-ary operation f is algebraic in the algebra $\mathfrak{E}_p(\mathfrak{A})$. Hence and from (2.3) it follows that the elements $a_1, a_2, \ldots, a_{p+2}$ belong to the subalgebra of the algebra $\mathfrak{E}_p(\mathfrak{A})$ generated by the elements $a_1, a_2, \ldots, a_{p+1}$ which gives the contradiction. Consequently, the inequality $\gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big) \leqslant p+1$ is true. Hence and from (2.2) we get the formula $\gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big) = p+1$ whenever $\gamma_0(\mathfrak{A}) > p+1$. The theorem is thus proved.

A relationship between the concepts of the order of enlargeability and the p-enlargement is given by the following simple theorem.

THEOREM 2.2. *The inequality* $\varepsilon(\mathfrak{A}) \leqslant p$ *holds if and only if* $\mathfrak{A} = \mathfrak{E}_p(\mathfrak{A})$.

From the above theorem and from formula (2.1) we deduce the following theorems.

THEOREM 2.3. *For all algebras* $\mathfrak{A}$ *the inequality* $\varepsilon\big(\mathfrak{E}_p(\mathfrak{A})\big) \leqslant p$ *holds.*

THEOREM 2.4. *If* $p \geqslant 2$ *and* $\mathfrak{E}_{p-1}(\mathfrak{A}) \neq \mathfrak{E}_p(\mathfrak{A})$, *then* $\varepsilon\big(\mathfrak{E}_p(\mathfrak{A})\big) = p$.

3. Algebras of algebraic operations. Let $\mathfrak{A}^{(n)}$ denote the algebra of all n-ary algebraic operations in the algebra $\mathfrak{A}$ (see [5], Section 1.2).

THEOREM 3.1. *If* $n \geqslant \gamma_0(\mathfrak{A})$, *then* $\varepsilon(\mathfrak{A}^{(n)}) \geqslant \varepsilon(\mathfrak{A})$.

Proof. Each operation $f \in \mathbf{A}^{(m)}(\mathfrak{A})$ $(m = 0, 1, \dots)$ induces an m-ary operation $\hat{f}$ belonging to $\mathbf{A}^{(m)}(\mathfrak{A}^{(n)})$ by means of the formula

$$\hat{f}(g_1, g_2, \dots, g_m) = t \qquad (g_1, g_2, \dots, g_m \in \mathbf{A}^{(n)}(\mathfrak{A}))\,,$$

where

$$t(x_1, x_2, \dots, x_n) = f\big(g_1(x_1, x_2, \dots, x_n), g_2(x_1, x_2, \dots, x_n), \dots, g_m(x_1, x_2, \dots, x_n)\big)\,.$$

Moreover, by the definition of the algebra of n-ary algebraic operations each operation from $\mathbf{A}^{(m)}(\mathfrak{A}^{(n)})$ is of the form $\hat{g}$ where $g \in \mathbf{A}^{(m)}(\mathfrak{A})$.

Let us introduce the notation

$$\varepsilon(\mathfrak{A}^{(n)}) = p\,. \tag{3.1}$$

Since in the case $p = \infty$ the theorem is obvious, we may assume that $p < \infty$. Suppose that for an m-ary operation h the composition $h(v_1, v_2, \dots, v_m)$ with arbitrary operations $v_1, v_2, \dots, v_m \in \mathbf{A}^{(p)}(\mathfrak{A})$ belongs to $\mathbf{A}^{(p)}(\mathfrak{A})$. By (3.1) there exists an operation $g \in \mathbf{A}^{(m)}(\mathfrak{A})$ such that $\hat{g}(g_1, g_2, \dots, g_m) = h(g_1, g_2, \dots, g_m)$ for all operations $g_1, g_2, \dots, g_m \in \mathbf{A}^{(n)}(\mathfrak{A})$. Consequently,

$$\begin{aligned} &g\big(g_1(x_1, x_2, \dots, x_n), g_2(x_1, x_2, \dots, x_n), \dots, g_m(x_1, x_2, \dots, x_n)\big) \\ &\quad = h\big(g_1(x_1, x_2, \dots, x_n), g_2(x_1, x_2, \dots, x_n), \dots, g_m(x_1, x_2, \dots, x_n)\big) \end{aligned} \tag{3.2}$$

for all operations $g_1, g_2, \dots, g_m \in \mathbf{A}^{(n)}(\mathfrak{A})$. Let $a_1, a_2, \dots, a_m$ be an arbitrary m-tuple of elements of the algebra $\mathfrak{A}$. Since $n \geqslant \gamma_0(\mathfrak{A})$, there exist an n-tuple $b_1, b_2, \dots, b_n$ of elements of the algebra $\mathfrak{A}$ and operations $h_1, h_2, \dots, h_m \in \mathbf{A}^{(n)}(\mathfrak{A})$ such that $a_j = h_j(b_1, b_2, \dots, b_n)$ $(j = 1, 2, \dots, m)$. Taking into account (3.2), we get the equation $g(a_1, a_2, \dots, a_m) = h(a_1, a_2, \dots, a_m)$. Thus $g = h$ and, consequently, $h \in \mathbf{A}^{(m)}(\mathfrak{A})$. Hence we obtain the inequality $\varepsilon(\mathfrak{A}) \leqslant p$ which, by (3.1), completes the proof of the theorem.

4. Some subalgebras and reducts. Now we shall give a relation between the order of enlargeability of an algebra and the order of enlargeability of some its subalgebras and reducts.

THEOREM 4.1. *If* $\mathfrak{B}$ *is a subalgebra of* $\mathfrak{A}$ *such that each algebraic operation in* $\mathfrak{B}$ *has a unique extension to an algebraic operation in* $\mathfrak{A}$, *then*

$$\varepsilon(\mathfrak{A}) \leqslant \max\big(\varepsilon(\mathfrak{B}), \gamma_0(\mathfrak{A})\big)\,.$$

Proof. Put $p = \max\big(\varepsilon(\mathfrak{B}), \gamma_0(\mathfrak{A})\big)$. Of course, it suffices to consider the case $p < \infty$. Moreover, if $p = 0$, then all elements of $\mathfrak{A}$ are algebraic constants. Consequently, $\mathfrak{A} = \mathfrak{B}$ which implies the assertion of the theorem. Thus we may assume that $p \geqslant 1$. Suppose that $\gamma_0(\mathfrak{B}) > p$. Then there exist elements $d_1, d_2, \dots, d_{p+1}$ in $\mathfrak{B}$ which do not belong to a subalgebra of $\mathfrak{B}$ generated by less than $p+1$ elements. Put $h(d_1, d_2, \dots, d_{p+1}) = d_{p+1}$ and $h(x_1, x_2, \dots, x_{p+1}) = x_1$ otherwise. Of course, the operation h is non-trivial and, by $\varepsilon(\mathfrak{B}) \leqslant p$, algebraic in $\mathfrak{B}$. Moreover, $h(g_1, g_2, \dots, g_{p+1}) = g_1$ in $\mathfrak{B}$ for all $g_1, g_2, \dots, g_{p+1} \in \mathbf{A}^{(p)}(\mathfrak{A})$. Hence and from $\gamma_0(\mathfrak{A}) \leqslant p$ it follows that h is a restriction to $\mathfrak{B}$ of the trivial operation $e_1^{(p+1)}$ which gives a contradiction. Thus $\gamma_0(\mathfrak{B}) \leqslant p$.

Let f be an operation algebraic in $\mathfrak{E}_p(\mathfrak{A})$. From the inequalities $\gamma_0(\mathfrak{B}) \leqslant p$ and $\varepsilon(\mathfrak{B}) \leqslant p$ it follows that the restriction of f to $\mathfrak{B}$ is algebraic. Let f_0 be the extension of this restriction to an algebraic operation in $\mathfrak{A}$. By the definition of p-enlargement, $f(g_1, g_2, \dots, g_m) \in \mathbf{A}^{(p)}(\mathfrak{A})$ whenever $g_1, g_2, \dots, g_m \in \mathbf{A}^{(p)}(\mathfrak{A})$. Moreover,

$$f(g_1, g_2, \dots, g_m) = f_0(g_1, g_2, \dots, g_m) \tag{4.1}$$

in the subalgebra $\mathfrak{B}$. Hence, by the uniqueness of the extension of the algebraic operations from $\mathfrak{B}$ to $\mathfrak{A}$, we get equation (4.1) in the algebra $\mathfrak{A}$. Since $p \geqslant \gamma_0(\mathfrak{A})$, we hawe $f = f_0$ and, consequently, the operation f is algebraic in the algebra $\mathfrak{A}$. Hence we get the equation $\mathfrak{A} = \mathfrak{E}_p(\mathfrak{A})$ which, according to Theorem 2.2, implies the inequality $\varepsilon(\mathfrak{A}) \leqslant p$. The theorem is thus proved.

THEOREM 4.2. *Let* $\mathfrak{B} = (A; \mathbf{F})$ *and* $\mathfrak{A} = (A; \mathbf{F} \cup \{c_1, c_2, \dots, c_k\})$, *where* $c_1, c_2, \dots, c_k$ *are constant operations. If* $\mathbf{A}(\mathfrak{A}) \cap \mathbf{A}\big(\mathfrak{E}_p(\mathfrak{B})\big) = \mathbf{A}(\mathfrak{B})$, *then* $\varepsilon(\mathfrak{B}) \leqslant \max\big(p, \varepsilon(\mathfrak{A}) + k\big)$.

Proof. Put $r = \max\big(p, \varepsilon(\mathfrak{A}) + k\big)$ and $s = \varepsilon(\mathfrak{A})$. Of course, it suffices to consider the case $r < \infty$. Let f be an arbitrary operation algebraic in the algebra $\mathfrak{E}_r(\mathfrak{B})$. Since for every operation g from $\mathbf{A}^{(s)}(\mathfrak{A})$ there exists an operation h in $A^{(s+k)}(\mathfrak{B})$ such that

$$g(x_1, x_2, \dots, x_s) = h(x_1, x_2, \dots, x_s, c_1, c_2, \dots, c_k)$$

and $r \geqslant s + k$, we infer that $f(g_1, g_2, \dots, g_m) \in \mathbf{A}^{(s)}(\mathfrak{A})$ whenever $g_1, g_2, \dots, g_m \in \mathbf{A}^{(s)}(\mathfrak{A})$. Consequently, the operation f is algebraic in $\mathfrak{A}$, because $s = \varepsilon(\mathfrak{A})$.

Moreover, by the inequality $r \geqslant p$, the operation f is also algebraic in $\mathfrak{E}_p(\mathfrak{B})$. Hence, by the assumption of the theorem, we infer that the operation f is algebraic in $\mathfrak{B}$. Thus $\mathfrak{B} = \mathfrak{E}_r(\mathfrak{B})$, which, by Theorem 2.2, implies the inequality $\varepsilon(\mathfrak{B}) \leqslant r$. The theorem is thus proved.

An operation f is said to be *idempotent* if $f(x, x, \ldots, x) = x$ for all elements x. Given an algebra $\mathfrak{A}$, by $\mathfrak{I}(\mathfrak{A})$ we shall denote the *maximal idempotent reduct* of $\mathfrak{A}$, i.e. the reduct for which the class of algebraic operations is the class of all idempotent algebraic operations in $\mathfrak{A}$. It is evident that $\mathbf{A}(\mathfrak{A}) \cap \mathbf{A}\big(\mathfrak{E}_1(\mathfrak{I}(\mathfrak{A}))\big) = \mathbf{A}\big(\mathfrak{I}(\mathfrak{A})\big)$. Hence and from Theorem 4.2 we get the following theorem.

THEOREM 4.3. *If $\mathfrak{A} = (A; \mathbf{F} \cup \{c_1, c_2, \ldots, c_k\})$, where $c_1, c_2, \ldots, c_k$ are constant operations and all operations from* $\mathbf{F}$ *are idempotent then $\varepsilon\big(\mathfrak{I}(\mathfrak{A})\big) \leqslant \varepsilon(\mathfrak{A}) + k$.*

5. Some examples. Now we shall give some examples of abstract algebras which will be used in the further considerations.

(i) *The algebras* $\mathfrak{C}_q$. Let C_q $(q = 1, 2, \ldots, \infty)$ be the set of all q-tuples $(k_1, k_2, \ldots, k_q)$ of non-negative integers different from the q-tuple $(0, 0, \ldots, 0)$. Of course, the usual addition

$$(k_1, k_2, \ldots, k_q) + (s_1, s_2, \ldots, s_q) = (k_1 + s_1, k_2 + s_2, \ldots, k_q + s_q)$$

and the scalar multiplication $n(k_1, k_2, \ldots, k_q) = (nk_1, nk_2, \ldots, nk_q)$ by positive integers are well defined on C_q. We introduce the notation

$$e_{jq} = (\delta_{j1}, \delta_{j2}, \ldots, \delta_{jq}) \quad (j = 1, 2, \ldots, q),$$

where $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jj} = 1$. Further, let $D_1 = \{0\}$ and let D_q $(q = 2, 3, \ldots, \infty)$ be the set of all finite sums $e_{j_1q} + e_{j_2q} + \ldots + e_{j_sq}$, where $1 \leqslant j_1 < j_2 < \ldots < j_s \leqslant q$ and $s \geqslant 2$.

Let $\mathbf{F}_q$ $(q = 1, 2, \ldots, \infty)$ be the class of all operations f on C_q defined as

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} c_j x_j + c_0 d,$$

where $c_0, c_1, \ldots, c_n$ are non-negative integers, $d \in D_q$, and at least one coefficient c_j $(1 \leqslant j \leqslant n)$ is greater than 1 if $c_0 = 0$ or $q = 1$.

Put $\mathfrak{C}_q = (C_q; \mathbf{F}_q)$ $(q = 1, 2, \ldots, \infty)$ and $\mathfrak{C}_0 = (C_1; \mathbf{F}_1 \cup \{e\})$, where e is the constant operation on C_1 equal to e_{11} everywhere. Of course, all elements of the algebra $\mathfrak{C}_0$ are algebraic constants and, consequently, $\gamma_0(\mathfrak{C}_0) = 0$. Moreover, it is easy to verify that $e_{1q}, e_{2q}, \ldots, e_{qq}$ are the only generators of the algebra $\mathfrak{C}_q$ $(q = 1, 2, \ldots, \infty)$. Hence it follows that $\gamma_0(\mathfrak{C}_q) = q$ $(q = 1, 2, \ldots, \infty)$.

For each index q, the operation

$$f_n(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} x_j$$

is well defined on the set C_q. Moreover, it is easy to verify that $f_n \in \mathbf{A}\big(\mathfrak{E}_{n-1}(\mathfrak{C}_q)\big)$ and $f_n \notin \mathbf{A}\big(\mathfrak{E}_n(\mathfrak{C}_q)\big)$. Thus

$$(5.1) \qquad \mathfrak{E}_{n-1}(\mathfrak{C}_q) \neq \mathfrak{E}_n(\mathfrak{C}_q) \quad (n = 2, 3, \ldots;\ q = 0, 1, \ldots, \infty)$$

and $\mathfrak{C}_q \neq \mathfrak{E}_n(\mathfrak{C}_q)$ $(n = 2, 3, \ldots;\ q = 0, 1, \ldots, \infty)$. Hence and from Theorem 2.2 we get the inequality $\varepsilon(\mathfrak{C}_q) > n$ $(n = 2, 3, \ldots)$ which implies the formula $\varepsilon(\mathfrak{C}_q) = \infty$ $(q = 0, 1, \ldots,)$.

(ii) *The algebras* $\mathfrak{C}_{p,q}$. Let $p = 1, 2, \ldots, q = 0, 1, \ldots$ and $p \geqslant q-1$. We put $\mathfrak{C}_{p,q} = \mathfrak{E}_p(\mathfrak{C}_q)$. Since $\gamma_0(\mathfrak{C}_q) = q$ and $p+1 \geqslant q$, we have, by Theorem 2.1, the formula $\gamma_0(\mathfrak{C}_{p,q}) = q$. Moreover, from (5.1) we get, in view of Theorem 2.4, the formula $\varepsilon(\mathfrak{C}_{p,q}) = p$ for $p \geqslant 2$. Put $f_1(x) = x + e_{11}$, $f_2(e_{12}) = e_{12} + e_{22}$ and $f_2(x) = x$ in the opposite case. It is easy to verify that $\mathbf{A}^{(0)}(\mathbf{F}_1) = \mathbf{A}^{(0)}(\mathbf{F}_1 \cup \{f_1\})$, $\mathbf{A}^{(0)}(\mathbf{F}_1 \cup \{e\}) = \mathbf{A}^{(0)}(\mathbf{F}_1 \cup \{f_1, e\})$ and $\mathbf{A}^{(0)}(\mathbf{F}_2) = \mathbf{A}^{(0)}(\mathbf{F}_2 \cup \{f_2\})$. On the other hand the operation f_1 is not algebraic in $\mathfrak{C}_{1,0}$ and $\mathfrak{C}_{1,1}$ and the operation f_2 is not algebraic in $\mathfrak{C}_{1,2}$. Thus $\varepsilon(\mathfrak{C}_{2,q}) \geqslant 1$ $(q = 0, 1, 2)$. Hence and from Theorem 2.3 we get the formula $\varepsilon(\mathfrak{C}_{1,q}) = 1$ $(q = 0, 1, 2)$.

We can now summarize the discussion of constants ε and γ_0 for the algebras $\mathfrak{C}_q$ and $\mathfrak{C}_{p,q}$:

$$(5.2) \quad \varepsilon(\mathfrak{C}_q) = \infty, \quad \gamma_0(\mathfrak{C}_q) = q \quad (q = 0, 1, \ldots, \infty),$$

$$(5.3) \quad \varepsilon(\mathfrak{C}_{p,q}) = p, \quad \gamma_0(\mathfrak{C}_{p,q}) = q \quad (p = 1, 2, \ldots;\ q = 0, 1, \ldots;\ p \geqslant q-1).$$

Let $\mathfrak{C}$ be a complete algebra over a set C, i.e. the algebra for which $\mathbf{A}(C)$ is the class of all operations on C. Then we have the formula

$$(5.4) \qquad \varepsilon(\mathfrak{C}) = \gamma_0(\mathfrak{C}) = 0.$$

Further, let $\mathfrak{D}$ be the algebra over the set $\{0, 1\}$ such that $\mathbf{A}(\mathfrak{D})$ consists of all operations f satisfying the condition $f(0, 0, \ldots, 0) = 0$. Then

$$(5.5) \qquad \varepsilon(\mathfrak{D}) = 0, \quad \gamma_0(\mathfrak{D}) = 1.$$

6. Description of all pairs (ε, γ_0). Now we shall give a description of all possible pairs (ε, γ_0) for abstract algebras.

LEMMA 6.1. *If $\varepsilon(\mathfrak{A}) = 0$, then $\gamma_0(\mathfrak{A}) \leqslant 1$.*

Proof. Suppose that $\gamma_0(\mathfrak{A}) \geqslant 2$. Consequently, there exists a pair a_1, a_2 of elements of the algebra $\mathfrak{A}$ which does not belong to a subalgebra of $\mathfrak{A}$ generated by less than two elements. Put $f(a_1) = a_2$, $f(a_2) = a_1$ and $f(x) = x$

in the opposite case. Let $\mathbf{G}_0$ be the family of all constant algebraic operations in A. Put $\mathbf{G} = \mathbf{G}_0 \cup \{f\}$. Taking into account that the elements a_1 and a_2 are not algebraic constants in $\mathfrak{A}$, we infer that $\mathbf{A}^{(0)}(\mathfrak{A}) = \mathbf{A}^{(0)}(\mathbf{G})$. Hence and from the equation $\varepsilon(\mathfrak{A}) = 0$ it follows that the operation f is algebraic in the algebra $\mathfrak{A}$. Consequently, the element a_1 belongs to the subalgebra of $\mathfrak{A}$ generated by the element a_2 which gives a contradiction. The lemma is thus proved.

THEOREM 6.1. *The set of all possible pairs (ε, γ_0) is the set of all pairs (p, q), where $p, q = 0, 1, \ldots, \infty$ and $p \geqslant q-1$. Consequently, for each algebra the inequality*

$$\varepsilon \geqslant \gamma_0 - 1 \tag{6.1}$$

holds.

Proof. By formulas (5.2), (5.3), (5.4), and (5.5) to prove the theorem it suffices to prove inequality (6.1). Of course, we may assume that ε is finite. Moreover, by Lemma 6.1, we may assume that $\varepsilon \geqslant 1$. Let $\mathfrak{A}$ be an algebra and $p = \varepsilon(\mathfrak{A})$ $(1 \leqslant p < \infty)$. Then, by Theorem 2.2, $\mathfrak{A} = \mathfrak{E}_p(\mathfrak{A})$. Hence and from Theorem 2.1 we get the inequality $\gamma_0(\mathfrak{A}) = \gamma_0(\mathfrak{E}_p(\mathfrak{A})) \leqslant p+1 = \varepsilon(\mathfrak{A})+1$ which implies inequality (6.1). The theorem is thus proved.

7. A class of algebras. We say that an algebra $\mathfrak{A}$ *has the property* $(*)$ if two operations f and g from $\mathbf{A}^{(n)}(\mathfrak{A})$ $(n \geqslant 1)$ are identical whenever $f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n)$ for all sequences $a_1, a_2, \ldots, a_n$ of elements of $\mathfrak{A}$ containing at most two different elements.

Many algebras usually treated in mathematics, such as Abelian groups, vector spaces, Boolean algebras, etc., have the property $(*)$. Obviously, each two-element algebra has the property $(*)$.

THEOREM 7.1. *If an algebra $\mathfrak{A}$ has the property $(*)$ and $p \geqslant \max(3, \gamma_0(\mathfrak{A}))$, then the algebra $\mathfrak{E}_p(\mathfrak{A})$ has the property $(*)$.*

Proof. Suppose that $f, g \in \mathbf{A}^{(n)}(\mathfrak{E}_p(\mathfrak{A}))$ and

$$f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n) \tag{7.1}$$

for all sequences $a_1, a_2, \ldots, a_n$ containing at most two different elements. We shall prove the equation $f = g$ by induction with respect to n. For $n \leqslant p$ this equation is obvious because of the formula $\mathbf{A}^{(p)}(\mathfrak{E}_p(\mathfrak{A})) = \mathbf{A}^{(p)}(\mathfrak{A})$ and the property $(*)$ of $\mathfrak{A}$.

Suppose that $n > p$ and for operations from $\mathbf{A}^{(n-1)}(\mathfrak{E}_p(\mathfrak{A}))$ the statement is true. By (7.1) and the inductive assumption we have the equation

$$f(x_1, x_2, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n) = g(x_1, x_2, \ldots, x_{i-1}, x_j, x_{i+1} \ldots, x_n) \tag{7.2}$$

for all $i < j$ $(i, j = 1, 2, \ldots, n)$. Let $h \in \mathbf{A}^{(2)}(\mathfrak{E}_p(\mathfrak{A}))$. From (7.2) and the inequality $n > p \geqslant 3$ it follows that the equation

$$f(h(x_2, x_3), x_2, x_3, \ldots, x_n) = g(h(x_2, x_3), x_2, x_3, \ldots, x_n) \tag{7.3}$$

holds whenever the sequence $x_2, x_3, \ldots, x_n$ contains at most two different elements. Since both compositions in (7.3) belong to $\mathbf{A}^{(n-1)}(\mathfrak{E}_p(\mathfrak{A}))$, we infer, by the inductive assumption, that equation (7.3) holds for all elements $x_2, x_3, \ldots, x_n$ and all operations $h \in \mathbf{A}^{(2)}(\mathfrak{E}_p(\mathfrak{A}))$.

Let $h_1, h_2 \in \mathbf{A}^{(2)}(\mathfrak{E}_p(\mathfrak{A}))$ and

$$f_1(x_2, x_3, \ldots, x_n) = f(h_1(x_2, x_3), x_3, x_4, h_2(x_2, x_3), x_5, \ldots, x_n),$$

$$g_1(x_2, x_3, \ldots, x_n) = g(h_1(x_2, x_3), x_3, x_4, h_2(x_2, x_3), x_5, \ldots, x_n).$$

Of course, $f_1, g_1 \in \mathbf{A}^{(n-1)}(\mathfrak{E}_p(\mathfrak{A}))$ and, by (7.2) and (7.3), the equations

$$f_1(x_2, x_3, x_3, x_5, \ldots, x_n) = g_1(x_2, x_3, x_3, x_5, \ldots, x_n),$$
$$f_1(x_2, x_2, x_4, x_5, \ldots, x_n) = g_1(x_2, x_2, x_4, x_5, \ldots, x_n),$$
$$f_1(x_2, x_3, x_2, x_5, \ldots, x_n) = g_1(x_2, x_3, x_2, x_5, \ldots, x_n)$$

hold. These equations show that

$$f_1(x_2, x_3, \ldots, x_n) = g_1(x_2, x_3, \ldots, x_n) \tag{7.4}$$

whenever at least two elements among x_2, x_3, x_4 are identical. Consequently, equation (7.4) holds for all sequences $x_2, x_3, \ldots, x_n$ containing at most two different elements which, by the inductive assumption, implies the equation $f_1 = g_1$. Thus

$$\begin{aligned} f(h_1(x_2, x_3), x_3, x_4, h_2(x_2, x_3), x_5, \ldots, x_n) \\ = g(h_1(x_2, x_3), x_3, x_4, h_2(x_2, x_3), x_5, \ldots, x_n) \end{aligned} \tag{7.5}$$

for all elements $x_2, x_3, \ldots, x_n$ and all operations $h_1, h_2 \in \mathbf{A}^{(2)}(\mathfrak{E}_p(\mathfrak{A}))$.

Let $h_1, h_2, h_3 \in \mathbf{A}^{(2)}(\mathfrak{E}_p(\mathfrak{A}))$ and

$$f_2(x_2, x_3, \ldots, x_n) = f(h_1(x_2, x_3), x_4, h_2(x_2, x_3), h_3(x_2, x_3), x_5, \ldots, x_n),$$

$$g_2(x_2, x_3, \ldots, x_n) = g(h_1(x_2, x_3), x_4, h_2(x_2, x_3), h_3(x_2, x_3), x_5, \ldots, x_n).$$

Obviously, $f_2, g_2 \in \mathbf{A}^{(n-1)}(\mathfrak{E}_p(\mathfrak{A}))$ and, by (7.5), the equations

$$f_2(x_2, x_2, x_4, x_5, \ldots, x_n) = g_2(x_2, x_2, x_4, x_5, \ldots, x_n),$$
$$f_2(x_2, x_3, x_2, x_5, \ldots, x_n) = g_2(x_2, x_3, x_2, x_5, \ldots, x_n),$$
$$f_2(x_2, x_3, x_3, x_5, \ldots, x_n) = g_2(x_2, x_3, x_3, x_5, \ldots, x_n)$$

hold. Hence it follows that

$$f_2(x_2, x_3, \ldots, x_n) = g_2(x_2, x_3, \ldots, x_n) \tag{7.6}$$

whenever at least two elements among x_2, x_3, x_4 are identical. Consequently, equation (7.6) holds for all sequences $x_2, x_3, \ldots, x_n$ containing at most two different elements which implies, by the inductive assumption, the equation $f_2 = g_2$. Thus

$$f\big(h_1(x_2, x_3), x_4, h_2(x_2, x_3), h_3(x_2, x_3), x_5, \ldots, x_n\big) \\ = g\big(h_1(x_2, x_3), x_4, h_2(x_2, x_3), h_3(x_2, x_3), x_5, \ldots, x_n\big)$$

for all elements $x_2, x_3, \ldots, x_n$ and all operations $h_1, h_2, h_3 \in \mathbf{A}^{(2)}\big(\mathfrak{E}_p(\mathfrak{A})\big)$. Hence it follows that for any system $h_1, h_2, \ldots, h_n$ of operations from $A^{(2)}\big(\mathfrak{E}_p(\mathfrak{A})\big)$ the equation

$$f\big(h_1(x, y), h_2(x, y), \ldots, h_n(x, y)\big) = g\big(h_1(x, y), h_2(x, y), \ldots, h_n(x, y)\big) \tag{7.7}$$

is true. Further, let $w_1, w_2, \ldots, w_n$ be an arbitrary system of operations from $\mathbf{A}^{(n-1)}\big(\mathfrak{E}_p(\mathfrak{A})\big)$. The compositions $f(w_1, w_2, \ldots, w_n)$, $g(w_1, w_2, \ldots, w_n)$ belong also to $\mathbf{A}^{(n-1)}\big(\mathfrak{E}_p(\mathfrak{A})\big)$ and, by (7.7), the equation

$$f\big(w_1(x_1, x_2, \ldots, x_{n-1}), w_2(x_1, x_2, \ldots, x_{n-1}), \ldots, w_n(x_1, x_2, \ldots, x_{n-1})\big) \\ = g\big(w_1(x_1, x_2, \ldots, x_{n-1}), w_2(x_1, x_2, \ldots, x_{n-1}), \ldots, w_n(x_1, x_2, \ldots, x_{n-1})\big) \tag{7.8}$$

holds for all sequences $x_1, x_2, \ldots, x_{n-1}$ containing at most two different elements. Hence, by the inductive assumption, equation (7.8) holds for all elements $x_1, x_2, \ldots, x_{n-1}$ and all operations $w_1, w_2, \ldots, w_n$ from $\mathbf{A}^{(n-1)}\big(\mathfrak{E}_p(\mathfrak{A})\big)$.

Let $b_1, b_2, \ldots, b_n$ be an arbitrary sequence of elements of the algebra $\mathfrak{E}_p(\mathfrak{A})$. Since $n > p \geqslant \gamma_0(\mathfrak{A})$, we have, by Theorem 2.1, the inequality $\gamma_0\big(\mathfrak{E}_p(\mathfrak{A})\big) = \gamma_0(\mathfrak{A}) < n$. Consequently, there exist elements $c_1, c_2, \ldots, c_{n-1}$ and operations $v_1, v_2, \ldots, v_n \in \mathbf{A}^{(n-1)}\big(\mathfrak{E}_p(\mathfrak{A})\big)$ such that $b_j = v_j(c_1, c_2, \ldots, c_{n-1})$ $(j = 1, 2, \ldots, n)$. Hence, by formula (7.8), we obtain the equation

$$f(b_1, b_2, \ldots, b_n) = g(b_1, b_2, \ldots, b_n)\,.$$

Thus the operations f and g are identical and, consequently, the algebra $\mathfrak{E}_p(\mathfrak{A})$ has the property $(*)$ which completes the proof.

Since all algebras $\mathfrak{C}_q$ $(q = 0, 1, \ldots, \infty)$ defined in Section 5 (Example (i)) have the property $(*)$ and $\mathfrak{C}_{p,q} = \mathfrak{E}_p(\mathfrak{C}_q)$, we have, by (5.2) and Theorem 7.1, the following

COROLLARY. *The algebras $\mathfrak{C}_{p,q}$ $(q = 0, 1, \ldots; p = 3, 4, \ldots; p \geqslant q)$ have the property* $(*)$.

8. Two-element algebras. In his fundamental paper [11], E. L. Post determined all iterative systems and, consequently, all abstract algebras over a two-element set. We shall quote some of Post's results which will enable us to compute the constants ε and γ_0 for two-element algebras. In this way we shall get other examples of pairs (ε, γ_0) for algebras with the property $(*)$.

Let $T = \{0, 1\}$. The set T will be regarded as a two-element Boolean algebra with 0 as a neutral element. We define auxiliary operations on T in terms of the Boolean operations:

$$p_*(x, y, z) = (x \cap y \cap z) \cup (x \cap y' \cap z') \cup (x' \cap y \cap z') \cup (x' \cap y' \cap z)\,,$$
$$p^*(x, y, z) = (x \cap y) \cup (y \cap z) \cup (x \cap z)\,,$$
$$q_1(x, y, z) = (x \cap y \cap z) \cup (x \cap y' \cap z')\,, \qquad q_2(x, y, z) = x \cap (y \cup z)\,,$$
$$s_1(x, y) = x \cup y\,, \quad s_2(x, y) = x \cap y\,, \quad s_3(x, y) = (x \cap y') \cup (x' \cap y)\,,$$
$$t(x) = x'\,, \qquad u_n(x_1, x_2, \ldots, x_n) = \bigcup_{j=1}^{n} \bigcap_{i \neq j} x_i \qquad (n = 3, 4, \ldots)\,.$$

Moreover, by 0 and 1 we shall denote the constant operations equal to 0 and 1 respectively.

Post's Theorem states that each algebra over T is isomorphic to one of the following algebras:

$$\mathfrak{F} = (T; t, s_1, s_2)\,, \quad \mathfrak{G} = (T; 0, 1, s_1, s_2)\,, \quad \mathfrak{H}_1 = (T; 0, 1)\,, \quad \mathfrak{H}_2 = (T; 0, t)\,.$$
$$\mathfrak{H}_3 = (T; 0, 1, s_2)\,, \quad \mathfrak{H}_4 = (T; t, s_3)\,, \quad \mathfrak{J}_1 = (T; s_2, s_3)\,, \quad \mathfrak{J}_2 = (T; s_2, p_*)\,,$$
$$\mathfrak{K} = (T; t, p_*, p^*)\,, \quad \mathfrak{L}_1 = (T; 0)\,, \quad \mathfrak{L}_2 = (T; 0, s_1)\,, \quad \mathfrak{L}_3 = (T; s_3)\,,$$
$$\mathfrak{L}_4 = (T; 0, s_1, s_2)\,, \quad \mathfrak{M}_1 = (T; t_1)\,, \quad \mathfrak{M}_2 = (T; t, p_*)\,, \quad \mathfrak{M}_3 = (T; 0, s_2)\,,$$
$$\mathfrak{N}_1 = (T; 0, q_1)\,, \quad \mathfrak{N}_2 = (T; 0, q_2)\,, \quad \mathfrak{N}_{1,n} = (T; 0, q_1, u_n) \quad (n = 3, 4, \ldots)\,,$$
$$\mathfrak{N}_{2,n} = (T; 0, q_2, u_n) \quad (n = 3, 4, \ldots)\,, \quad \mathfrak{P} = (T; p_*, p^*)\,, \quad \mathfrak{P}_* = (T; p_*)\,,$$
$$\mathfrak{P}^* = (T; p^*)\,, \quad \mathfrak{Q}_1 = (T; s_2)\,, \quad \mathfrak{Q}_2 = (T; s_1, s_2)\,, \quad \mathfrak{R}_1 = (T; q_1)\,,$$
$$\mathfrak{R}_2 = (T; q_2)\,, \quad \mathfrak{R}_{1,n} = (T; q_1, u_n) \quad (n = 3, 4, \ldots)\,,$$
$$\mathfrak{R}_{2,n} = (T; q_2, u_n) \quad (n = 3, 4, \ldots) \text{ and } \mathfrak{T}\,,$$

where $\mathfrak{T}$ is the trivial algebra over the set T, i.e. an algebra for which all algebraic operations are trivial.

Taking into account the inclusion relation between the algebras defined above, we can determine their order of enlargeability. We note that the order of enlargeability for the algebras $\mathfrak{T}$, $\mathfrak{P}_*$, $\mathfrak{P}^*$, and $\mathfrak{P}$ can be also obtained by means of results contained in [8] or [9]. The following table gives the constants ε and γ_0 for all algebras in question:

Algebras	ε	γ_0
$\mathfrak{F}$	0	0
$\mathfrak{G}$	1	0
$\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3, \mathfrak{H}_4$	2	0
$\mathfrak{I}_1$	0	1
$\mathfrak{K}$	1	1
$\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3, \mathfrak{L}_4$	2	1
$\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3$	3	1
$\mathfrak{N}_{1,n}, \mathfrak{N}_{2,n}$ $(n = 3, 4, \dots)$	n	1
$\mathfrak{N}_1, \mathfrak{N}_2$	∞	1
$\mathfrak{I}_2$	1	2
$\mathfrak{P}$	2	2
$\mathfrak{T}, \mathfrak{P}_*, \mathfrak{P}^*, \mathfrak{Q}_1, \mathfrak{Q}_2$	3	2
$\mathfrak{R}_{1,n}, \mathfrak{R}_{2,n}$ $(n = 3, 4, \dots)$	n	2
$\mathfrak{R}_1, \mathfrak{R}_2$	∞	2

The above table shows that for any pair p, q $(p, q = 0, 1, 2; p \geqslant q-1)$ there exists an algebra $\mathfrak{A}$ on the set T for which $\varepsilon(\mathfrak{A}) = p$ and $\gamma_0(\mathfrak{A}) = q$. Moreover, there are only four non-isomorphic algebras on the set T for which the order of enlargeability is infinite. It is well known (see [3] and [2]) that there exist non-denumerably many non-isomorphic algebras over a finite set containing at least three elements. Since the class of all n-ary operations on such a set is finite, we infer that there are only finitely many algebras for which $\varepsilon = n$. Thus the class of algebras with finite order of enlargeability is at most denumerable. Hence it follows that there exist non-denumerably many non-isomorphic algebras with infinite order of enlargeability defined on a finite set containing at least three elements.

9. Description of all pairs (ε, γ_0) for algebras with the property $(*)$. From the table in the preceding section it follows that

$$\varepsilon(\mathfrak{I}_1) = 0, \quad \gamma_0(\mathfrak{I}_1) = 1 \quad \text{and} \quad \varepsilon(\mathfrak{I}_2) = 1, \quad \gamma_0(\mathfrak{I}_2) = 2.$$

Moreover, each algebra over the set T satisfying the inequality $\varepsilon < \gamma_0$ is either isomorphic to $\mathfrak{I}_1$ or to $\mathfrak{I}_2$. Now we shall prove more general result.

THEOREM 9.1. *Each algebra with the property $(*)$ satisfying the inequality $\varepsilon < \gamma_0$ is either isomorphic to the algebra $\mathfrak{I}_1$ or to the algebra $\mathfrak{I}_2$.*

Proof. Let $\mathfrak{A}$ be an algebra with the property $(*)$ for which the inequality $\varepsilon(\mathfrak{A}) < \gamma_0(\mathfrak{A})$ holds. Of course, to prove the theorem it suffices to prove that $\mathfrak{A}$ is a two-element algebra. Let us suppose that the algebra $\mathfrak{A}$ contains at least three elements. Setting $m = \max\big(3, \varepsilon(\mathfrak{A})+1\big)$ and taking into account the inequality $\varepsilon(\mathfrak{A}) < \gamma_0(\mathfrak{A})$, we infer that there exists a sequence $b_1, b_2, \dots, b_m$ of elements of the algebra $\mathfrak{A}$ which belongs to no subalgebra of the algebra $\mathfrak{A}$ generated by $\varepsilon(\mathfrak{A})$ elements. We define an m-ary operation f as follows:

$$f(b_1, b_2, \dots, b_m) = b_m \tag{9.1}$$

and $f(x_1, x_2, \dots, x_m) = x_1$ in the opposite case. Hence and from the definition of the sequence $b_1, b_2, \dots, b_m$ we get the formula

$$f\big(g_1(x_1, x_2, \dots, x_k), g_2(x_1, x_2, \dots, x_k), \dots, g_m(x_1, x_2, \dots, x_k)\big) = g_1(x_1, x_2, \dots, x_k)$$

for all operations $g_1, g_2, \dots, g_m$ from $\mathbf{A}^{(k)}(\mathfrak{A})$ $\big(k \leqslant \varepsilon(\mathfrak{A})\big)$. Consequently, by the definition of the order of enlargeability, the operation f is algebraic in $\mathfrak{A}$. Since $f(a_1, a_2, \dots, a_m) = a_1$ for all sequences $a_1, a_2, \dots, a_m$ containing at most two different elements, we have, by the property $(*)$, the equation $f = e_1^{(m)}$ which contradicts (9.1). Thus $\mathfrak{A}$ is an at most two-element algebra. Since for one-element algebras the equation $\varepsilon(\mathfrak{A}) = \gamma_0(\mathfrak{A}) = 0$ holds, we conclude that $\mathfrak{A}$ is a two-element algebra which completes the proof.

THEOREM 9.2. *The set of all possible pairs (ε, γ_0) for algebras with the property $(*)$ non-isomorphic to one of the exceptional algebras $\mathfrak{I}_1$ and $\mathfrak{I}_2$ is the set of all pairs (p, q), where $p, q = 0, 1, \dots, \infty$ and $p \geqslant q$.*

Proof. The algebras $\mathfrak{C}_q$ and $\mathfrak{C}_{p,q}$ $(q = 0, 1, \dots, \infty;\ p = 3, 4, \dots;\ p \geqslant q)$ defined in Section 5 have the property $(*)$ (see Corollary to Theorem 7.1) and, obviously, are not isomorphic to the exceptional algebras $\mathfrak{I}_1$ and $\mathfrak{I}_2$. Moreover, by (5.2) and (5.3), $\varepsilon(\mathfrak{C}_q) = \infty$, $\varepsilon(\mathfrak{C}_{p,q}) = p$ and $\gamma_0(\mathfrak{C}_q) = \gamma_0(\mathfrak{C}_{p,q}) = q$. Further, the table in Section 8 shows that for each pair (p, q) $(p, q = 0, 1, 2;\ p \geqslant q)$ there exists a two-element algebra $\mathfrak{A}$ neither isomorphic to $\mathfrak{I}_1$ nor to $\mathfrak{I}_2$ for which $\varepsilon(\mathfrak{A}) = p$ and $\gamma_0(\mathfrak{A}) = q$. Finally, by Theorem 9.1, for all algebras with the property $(*)$ non-isomorphic to one of the algebras $\mathfrak{I}_1$ and $\mathfrak{I}_2$ the inequality $\varepsilon \geqslant \gamma_0$ holds which completes the proof.

10. Boolean algebras and their reducts. Let 0 denote the neutral element of a Boolean algebra and let $1 = 0'$. It is clear that the algebraic operations in a Boolean algebra are Boolean polynomials. For more detailed treatment of Boolean algebras from the point of view of abstract algebras, we refer to the paper [4].

Given a reduct $\mathfrak{A}$ of a Boolean algebra, by $\mathfrak{A}_0$ we shall denote the two-element subalgebra of $\mathfrak{A}$ with the carrier $\{0, 1\}$. It is evident that each algebraic operation in $\mathfrak{A}$ is uniquely determined by its restriction to $\mathfrak{A}_0$. Hence it follows that each reduct of a Boolean algebra has the property $(*)$. Moreover, we have the following Lemma.

LEMMA 10.1. *Let $\mathfrak{A}$ be a reduct of a Boolean algebra $\mathfrak{B}$. If $f \in \mathbf{A}(\mathfrak{B})$ and its restriction to $\{0, 1\}$ belongs to $\mathbf{A}(\mathfrak{A}_0)$, then $f \in \mathbf{A}(\mathfrak{A})$.*

LEMMA 10.2. *If $\mathfrak{A}$ is a reduct of a Boolean algebra, then* $\varepsilon(\mathfrak{A}) \geqslant \varepsilon(\mathfrak{A}_0)$.

Proof. Let $\mathfrak{A} = (A; \mathbf{F})$, $\mathfrak{A}_0 = (\{0,1\}; \mathbf{F}_0)$ and let $\mathfrak{A}$ be a reduct of a Boolean algebra $\mathfrak{B}$. For every non-negative integer $n < \varepsilon(\mathfrak{A}_0)$ there exists an operation f_0 on the set $\{0,1\}$ such that $f_0 \notin \mathbf{A}(\mathfrak{A}_0)$ and $\mathbf{A}^{(n)}(\mathbf{F}_0) = \mathbf{A}^{(n)}(\mathbf{F}_0 \cup \{f_0\})$. Since each operation on the set $\{0,1\}$ is a Boolean polynomial, the operation f_0 has a unique extension to a Boolean polynomial f on the set A. Of course, $f \in \mathbf{A}(\mathfrak{B})$. Moreover, by Lemma 10.1, $f \notin \mathbf{A}(\mathfrak{A})$ and $\mathbf{A}^{(n)}(\mathbf{F}) = \mathbf{A}^{(n)}(\mathbf{F} \cup \{f\})$. Hence, by the definition of the order of enlargeability, we get the inequality $\varepsilon(\mathfrak{A}) > n$ which completes the proof.

The following theorem is a consequence of Theorems 4.1 and 9.1, and Lemma 10.2.

THEOREM 10.1. *If $\mathfrak{A}$ is a reduct of a Boolean algebra different from the exceptional algebras $\mathfrak{J}_1$ and $\mathfrak{J}_2$, then*

$$\varepsilon(\mathfrak{A}) = \max\big(\varepsilon(\mathfrak{A}_0), \gamma_0(\mathfrak{A})\big).$$

We note that the constants $\varepsilon(\mathfrak{A}_0)$ can be obtained from the table in Section 8. In particular, if $\mathfrak{A}$ is a Boolean algebra, then $\varepsilon(\mathfrak{A}_0) = \varepsilon(\mathfrak{F}) = 0$ which yields the following corollary.

COROLLARY. *For Boolean algebras $\mathfrak{A}$ the formula $\varepsilon(\mathfrak{A}) = \gamma_0(\mathfrak{A})$ is true.*

11. Distributive lattices. Let $\mathfrak{A}$ be a distributive lattice. It is known (see [1], Chapter IX, Sections 9 and 10) that each n-ary ($n \geqslant 1$) algebraic operation in $\mathfrak{A}$ is of the form

$$f(x_1, x_2, \ldots, x_n) = \bigcup_{S \in F} \bigcap_{j \in S} x_j, \tag{11.1}$$

where F is a family of non-void subsets of the set of indices $\{1, 2, \ldots, n\}$ such that $U \in F$ and $V \supset U$ implies the relation $V \in F$. Given a non-void subset U of the set $\{1, 2, \ldots, n\}$, we put $x_j(U) = x$ if $j \in U$ and $x(U) = y$ if $j \notin U$. By (11.1) we have the inclusion $f\big(x_1(U), x_2(U), \ldots, x_n(U)\big) \supset x$ if and only if $U \in F$. Hence it follows that distributive lattices have the property $(*)$.

Further, since each algebraic unary operation in $\mathfrak{A}$ is trivial, we infer that the condition $\gamma_0(\mathfrak{A}) \leqslant 1$ implies that $\mathfrak{A}$ is a one-element algebra and, consequently, $\varepsilon(\mathfrak{A}) = \gamma_0(\mathfrak{A}) = 0$. The following theorem gives the order of enlargeability in the remaining case $\gamma_0(\mathfrak{A}) \geqslant 2$.

THEOREM 11.1. *For distributive lattices $\mathfrak{A}$ we have the formula* $\varepsilon(\mathfrak{A}) = \gamma_0(\mathfrak{A})$ *if* $\gamma_0(\mathfrak{A}) \geqslant 3$ *and* $\varepsilon(\mathfrak{A}) = 3$ *if* $\gamma_0(\mathfrak{A}) = 2$.

Proof. If $\gamma_0(\mathfrak{A}) = \infty$, then, by Theorem 6.1, $\varepsilon(\mathfrak{A}) = \infty$. Suppose that $\gamma_0(\mathfrak{A}) < \infty$. Since each sublattice of $\mathfrak{A}$ generated by $\gamma_0(\mathfrak{A})$ elements has at most $2^{2^{\gamma_0(\mathfrak{A})}}$ elements (see [1], Chapter IX, Section 10), we infer that the lattice $\mathfrak{A}$ is finite. Consequently, it has a zero-element 0 and a unit element 1. From the assumption $\gamma_0(\mathfrak{A}) \geqslant 2$ it follows that $0 \neq 1$. Moreover, the set $\{0,1\}$ is closed under lattice operations and each algebraic operation in $\mathfrak{A}$ is uniquely determined by its restriction to $\{0,1\}$. Further, it is easy to verify that the subalgebra of $\mathfrak{A}$ with the carrier $\{0,1\}$ is equal to the algebra $\mathfrak{Q}_2$ defined in Section 8. Hence, in view of Theorem 4.1, the inequality $\varepsilon(\mathfrak{A}) \leqslant \max\big(\varepsilon(\mathfrak{Q}_2), \gamma_0(\mathfrak{A})\big)$ follows. Since $\varepsilon(\mathfrak{Q}_2) = 3$ (see the table in Section 8), we have the inequality

$$\varepsilon(\mathfrak{A}) \leqslant \max\big(3, \gamma_0(\mathfrak{A})\big). \tag{11.2}$$

Moreover, the lattice $\mathfrak{A}$ has the property $(*)$ and is not isomorphic to the exceptional algebras $\mathfrak{J}_1$ and $\mathfrak{J}_2$. Thus, by Theorem 9.1, the inequality $\varepsilon(\mathfrak{A}) \geqslant \gamma_0(\mathfrak{A})$ holds. Hence and from (11.2) we get the formula $\varepsilon(\mathfrak{A}) = \gamma_0(\mathfrak{A})$ if $\gamma_0(\mathfrak{A}) \geqslant 3$.

Suppose that $\gamma_0(\mathfrak{A}) = 2$. Then it easy to verify that the algebra $\mathfrak{A}$ has either two elements of four elements. In the first case we have the equation $\mathfrak{A} = \mathfrak{Q}_2$ which implies $\varepsilon(\mathfrak{A}) = 3$. In the remaining case the algebra $\mathfrak{A}$ is isomorphic to the algebra $(A; \cap, \cup)$ where A is the family of all subsets of a two-element set. Hence it follows that the algebra $\mathfrak{A}$ is a reduct of a Boolean algebra and, consequently, by Theorem 10.1, $\varepsilon(\mathfrak{A}) = \max\big(\varepsilon(\mathfrak{Q}_2), \gamma_0(\mathfrak{A})\big) = 3$. The theorem is thus proved.

12. Conditions for the equation $\varepsilon = \gamma_0$. An algebra $\mathfrak{A}$ is said to have the *property* $(**)$ if for any n-ary operation f ($n > 3$) such that the composition $f(g_1, g_2, \ldots, g_n)$ belongs to $\mathbf{A}^{(n-1)}(\mathfrak{A})$ whenever $g_1, g_2, \ldots, g_n \in \mathbf{A}^{(n-1)}(\mathfrak{A})$ there exists an operation $f_0 \in \mathbf{A}^{(n)}(\mathfrak{A})$ satisfying the equation

$$f(a_1, a_2, \ldots, a_n) = f_0(a_1, a_2, \ldots, a_n)$$

for all sequences $a_1, a_2, \ldots, a_n$ containing at most two different elements.

THEOREM 12.1. *If an algebra has the properties $(*)$ and $(**)$, then $\varepsilon = \gamma_0$ for $\gamma_0 \geqslant 3$ and $\varepsilon \leqslant 3$ for $\gamma_0 \leqslant 2$.*

Proof. Suppose that the algebra $\mathfrak{A}$ has the properties $(*)$ and $(**)$. Put $p = \max\big(3, \gamma_0(\mathfrak{A})\big)$. Let f be an n-ary operation from $\mathbf{A}\big(\mathfrak{E}_p(\mathfrak{A})\big)$. If $n \leqslant p$, then, by the definition of the p-enlargement, $f \in \mathbf{A}(\mathfrak{A})$. Suppose that $n > p$. Then, by the property $(**)$ of the algebra $\mathfrak{A}$, there exists an n-ary operation f_0 belonging to $\mathbf{A}(\mathfrak{A})$ such that

$$f(a_1, a_2, \ldots, a_n) = f_0(a_1, a_2, \ldots, a_n) \tag{12.1}$$

for all sequences $a_1, a_2, \ldots, a_n$ containing at most two different elements. Since, by Theorem 7.1, the algebra $\mathfrak{E}_p(\mathfrak{A})$ has also the property $(*)$, formula (12.1) implies the equation $f = f_0$. Thus $f \in \mathbf{A}(\mathfrak{A})$ and, consequently, $\mathfrak{A} = \mathfrak{E}_p(\mathfrak{A})$. Hence and from Theorem 2.2 we get the inequality $\varepsilon(\mathfrak{A}) \leqslant p$. Thus $\varepsilon(\mathfrak{A}) \leqslant 3$ for $\gamma_0(\mathfrak{A}) \leqslant 2$ and

$$\varepsilon(\mathfrak{A}) \leqslant \gamma_0(\mathfrak{A}) \quad \text{for} \quad \gamma_0(\mathfrak{A}) \geqslant 3. \tag{12.2}$$

In the case $\gamma_0(\mathfrak{A}) \geqslant 3$ the algebra $\mathfrak{A}$ is not isomorphic to the exceptional two-element algebras $\mathfrak{J}_1$ and $\mathfrak{J}_2$. Consequently, by Theorem 9.1, $\varepsilon(\mathfrak{A}) \geqslant \gamma_0(\mathfrak{A})$. Hence and from (12.2) we get the formula $\varepsilon(\mathfrak{A}) = \gamma_0(\mathfrak{A})$ for $\gamma_0(\mathfrak{A}) \geqslant 3$. The theorem is thus proved.

In the next sections we shall give some applications of the above theorem to various classes of abstract algebras. Now we shall prove a theorem which gives a sufficient condition for (*) and (**).

THEOREM 12.2. *Suppose that the algebra $\mathfrak{A}$ contains an algebraic constant c such that for every system $f_1, f_2, \ldots, f_n \in \mathbf{A}^{(1)}(\mathfrak{A})$ $(n \geqslant 2)$ there exists one and only one operation $h \in \mathbf{A}^{(n)}(\mathfrak{A})$ for which the equations*

$$h(u_{1j}, u_{2j}, \ldots, u_{nj}) = f_j(x) \qquad (j = 1, 2, \ldots, n)$$

hold, where $u_{jj} = x$ and $u_{ij} = c$ if $i \neq j$ $(i, j = 1, 2, \ldots, n)$. Then the algebra $\mathfrak{A}$ has the properties (*) and (**),

Proof. Since, by the assumption, each n-ary algebraic operation in $\mathfrak{A}$ is uniquely determined by its values on the sequences $u_{1j}, u_{2j}, \ldots, u_{nj}$ $(j = 1, 2, \ldots, n)$ defined by the formula

$$(12.3) \qquad u_{jj} = x \quad \text{and} \quad u_{ij} = c \text{ if } i \neq j \quad (i, j = 1, 2, \ldots, n),$$

we infer that the algebra $\mathfrak{A}$ has the property (*).

Suppose now that $n > 3$ and f is an n-ary operation such that for all operations $g_1, g_2, \ldots, g_n \in \mathbf{A}^{(n-1)}(\mathfrak{A})$ the composition $f(g_1, g_2, \ldots, g_n)$ belongs to $\mathbf{A}^{(n-1)}(\mathfrak{A})$. Put

$$(12.4) \qquad f_j(x) = f(u_{1j}, u_{2j}, \ldots, u_{nj}) \qquad (j = 1, 2, \ldots, n),$$

where the quantities u_{ij} are defined by formula (12.3). Of course, $f_j \in \mathbf{A}^{(1)}(\mathfrak{A})$ $(j = 1, 2, \ldots, n)$. Consequently, there exists an operation $h \in \mathbf{A}^{(n)}(\mathfrak{A})$ such that

$$(12.5) \qquad h(u_{1j}, u_{2j}, \ldots, u_{nj}) = f_j(x) \qquad (j = 1, 2, \ldots, n)$$

for all sequences $u_{1j}, u_{2j}, \ldots, u_{nj}$ defined by formula (12.3). Given $1 \leqslant i < j \leqslant n$ we put

$$v_{ij}(x, y) = f(z_1, z_2, \ldots, z_n), \qquad w_{ij}(x, y) = h(z_1, z_2, \ldots, z_n),$$

where $z_i = x$, $z_j = y$ and $z_k = c$ if $k \neq i, j$. Since $n > 3$, both operations v_{ij} and w_{ij} are algebraic. Moreover, by (12.4) and (12.5), $v_{ij}(x, c) = w_{ij}(x, c)$ and $v_{ij}(c, y) = w_{ij}(c, y)$ which implies the equation

$$(12.6) \qquad v_{ij} = w_{ij} \qquad (i, j = 1, 2, \ldots, n;\ i < j).$$

Further, setting

$$f_{rs}(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_{r-1}, x_s, x_{r+1}, \ldots, x_n),$$
$$h_{rs}(x_1, x_2, \ldots, x_n) = h(x_1, x_2, \ldots, x_{r-1}, x_s, x_{r+1}, \ldots, x_n)$$

for each pair of indices $r < s$ $(r, s = 1, 2, \ldots, n)$, we get algebraic operations satisfying the conditions

$$f_{js}(u_{1j}, u_{2j}, \ldots, u_{nj}) = f(c, c, \ldots, c),$$
$$h_{js}(u_{1j}, u_{2j}, \ldots, u_{nj}) = h(c, c, \ldots, c),$$
$$f_{rj}(u_{1j}, u_{2j}, \ldots, u_{nj}) = v_{rj}(x, x),$$
$$h_{rj}(u_{1j}, u_{2j}, \ldots, u_{nj}) = w_{rj}(x, x)$$

and for $j \neq r, s$ the conditions

$$f_{rs}(u_{1j}, u_{2j}, \ldots, u_{nj}) = f(u_{1j}, u_{2j}, \ldots, u_{nj}),$$
$$h_{rs}(u_{1j}, u_{2j}, \ldots, u_{nj}) = h(u_{1j}, u_{2j}, \ldots, u_{nj}),$$

where the sequences $u_{1j}, u_{2j}, \ldots, u_{nj}$ are defined by formula (12.3). Hence and from (12.4), (12.5), and (12.6) we get the equations

$$f_{rs}(u_{1j}, u_{2j}, \ldots, u_{nj}) = h_{rs}(u_{1j}, u_{2j}, \ldots, u_{nj}) \qquad (j, r, s = 1, 2, \ldots, n;\ r < s).$$

Since both operations f_{rs} and h_{rs} are algebraic and, consequently, uniquely determined by their values on the sequences $u_{1j}, u_{2j}, \ldots, u_{nj}$ $(j = 1, 2, \ldots, n)$, the last equation implies the identity $f_{rs} = h_{rs}$ $(r, s = 1, 2, \ldots, n;\ r < s)$. In other words,

$$f(x_1, x_2, \ldots, x_n) = h(x_1, x_2, \ldots, x_n)$$

whenever the sequence $x_1, x_2, \ldots, x_n$ contains at least two identical elements. Hence, taking into account the inequality $n > 3$ we get the equation

$$f(a_1, a_2, \ldots, a_n) = h(a_1, a_2, \ldots, a_n)$$

for all sequences $a_1, a_2, \ldots, a_n$ containing at most two different elements. Thus algebra $\mathfrak{A}$ has the property (**) which completes the proof.

13. Unary algebras. An algebra is said to be *unary* if all its algebraic operations essentially depend on at most one variable. It is obvious that each unary algebra has the property (*). Now we shall prove a less obvious lemma.

LEMMA 13.1. *Unary algebras have the property* (**).

Proof. Let f be an n-ary operation $(n > 3)$ such that the composition $f(g_1, g_2, \ldots, g_n)$ with arbitrary $(n-1)$-ary algebraic operations $g_1, g_2, \ldots, g_n$ is also algebraic. In particular, the operation

$$h(x) = f(x, x, \ldots, x)$$

is algebraic. Since all algebraic operations essentially depend on at most one variable, we infer that there exist indices p, q, r, s $(2 \leqslant p, q, r, s \leqslant n)$

for which the following equations are true:

$$f(x_2, x_2, x_3, x_4, \ldots, x_n) = h(x_p)\,, \tag{13.1}$$

$$f(x_3, x_2, x_3, x_4, \ldots, x_n) = h(x_q)\,, \tag{13.2}$$

$$f(x_4, x_2, x_3, x_4, \ldots, x_n) = h(x_r)\,, \tag{13.3}$$

$$f(x_2, x_3, x_3, x_4, \ldots, x_n) = h(x_s)\,. \tag{13.4}$$

First consider the case of constant operation h: $h(x) \equiv c$. Setting $f_0(x_1, x_2, \ldots, x_n) \equiv c$, we get an n-ary algebraic operation satisfying, in view of (13.1), (13.2), and (13.4), the equation

$$f(x_1, x_2, \ldots, x_n) = f_0(x_1, x_2, \ldots, x_n)$$

for all sequences $x_1, x_2, \ldots, x_n$ such that x_1, x_2, x_3 consists of at most two different elements. Consequently, in this case the algebra has the property (**).

Now suppose that the operation h is not constant. In this case the indices p, q, r and s are uniquely determined by the conditions (13.1)-(13.4). Setting $x_2 = x_3$ into (13.1) and (13.2), we infer that $h(x_p) = h(x_q)$ whenever $x_2 = x_3$. Hence it follows that the indices p and q satisfy the condition

$$p = q \neq 2 \quad \text{or} \quad p, q \in \{2, 3\}\,. \tag{13.5}$$

In the same way setting $x_2 = x_3$ into (13.1) and (13.4) we get the condition

$$p = s \neq 2 \quad \text{or} \quad p, s \in \{2, 3\}\,. \tag{13.6}$$

Further, setting $x_2 = x_4$ into (13.1) and (13.3), we obtain the condition

$$p = r \neq 2 \quad \text{or} \quad p, r \in \{2, 4\}\,. \tag{13.7}$$

Setting $x_3 = x_4$ into (13.2) and (13.3), we get the further condition

$$q = r \neq 3 \quad \text{or} \quad q, r \in \{3, 4\}\,. \tag{13.8}$$

Finally, setting $x_2 = x_3$ into (13.3) and $x_2 = x_4$ into (13.4), we get the condition

$$r = s \neq 2 \text{ or } r = 2 \quad \text{and} \quad s = 3 \text{ or } r = 4 \text{ and } s = 2\,. \tag{13.9}$$

Suppose that $p \neq 2$. If $p \neq q$, then, by (13.5), the equations $p = 3$ and $q = 2$ hold. Consequently, $p \notin \{2, 4\}$ which, by condition (13.7), implies the equation $p = r = 3$. Since also $q \notin \{3, 4\}$, we have, by (13.8), the equation $q = r = 2$ which gives the contradiction. Thus

$$p = q \quad \text{if} \quad p \neq 2\,. \tag{13.10}$$

If $p \neq 2$ and $p \neq r$, then, by (13.7), the equations $p = 4$ and $r = 2$ hold. Hence and from (13.10) we get the inequality $q \neq r$. This inequality and (13.8) imply the relation $r \in \{3, 4\}$ which contradicts the formula $r = 2$. Consequently,

$$p = r \quad \text{if} \quad p \neq 2\,. \tag{13.11}$$

If $p \neq 2$ and $p \neq s$, then, by (13.6), we have equations $p = 3$ and $s = 2$. Hence and from (13.11) we get the formula $r = 3$ which together with the formula $s = 2$ contradicts (13.9). Thus $p = s$ whenever $p \neq 2$ and, consequently, by (13.10) and (13.11),

$$p = q = r = s \quad \text{if} \quad p \neq 2\,. \tag{13.12}$$

Now consider the case $p = 2$ and $p \neq q$. Then, by (13.5), we have the formula $q = 3$. Hence, by (13.8), we get the relation $r \in \{3, 4\}$ which in particular, implies the inequality $p \neq r$. Thus, by (13.7), $r \in \{2, 4\}$ which together with the relation $r \in \{3, 4\}$ implies the formula $r = 4$. Further, by (13.6), we have the relation $s \in \{2, 3\}$. Consequently, $r \neq s$ and, by (13.9), $s = 2$. Thus we have proved the following statement:

$$q = 3\,,\ r = 4 \ \textit{and}\ s = 2 \ \textit{if}\ p = 2 \ \textit{and}\ p \neq q\,. \tag{13.13}$$

Finally consider the case $p = q = 2$. Then, by (13.8), the equation $q = r$ holds. Hence and from (13.9) the formula $s = 3$ follows. Thus

$$r = 2 \ \textit{and}\ s = 3 \quad \textit{if} \quad p = q = 2\,. \tag{13.14}$$

Now we define an n-ary algebraic operation f_0 as follows: $f_0(x_1, x_2, \ldots, x_n) = h(x_p)$ if $p \neq 2$, $f_0(x_1, x_2, \ldots, x_n) = h(x_1)$ if $p = 2$ and $p \neq q$ and $f_0(x_1, x_2, \ldots, x_n) = h(x_2)$ if $p = q = 2$. From equations (13.1), (13.2), (13.4) and formulas (13.12), (13.13), (13.14) it follows that the equation

$$f(x_1, x_2, \ldots, x_n) = f_0(x_1, x_2, \ldots, x_n)$$

holds for all sequences $x_1, x_2, \ldots, x_n$ such that x_1, x_2, x_3 contains at most two different elements. Thus the algebra in question has the property (**) which completes the proof of the lemma.

The above lemma and Theorem 12.1 imply the following theorem.

THEOREM 13.1. *For unary algebras with $\gamma_0 \geqslant 3$ the equation $\varepsilon = \gamma_0$ holds.*

It should be noted that the assumption $\gamma_0 \geqslant 3$ in Theorem 13.1 is essential. In fact, for the trivial algebra $\mathfrak{T}$ over the set $\{0, 1\}$ we have the formulas $\varepsilon = 3$ and $\gamma_0 = 2$ (see the table in Section 8).

We finish this section by a proof of Świerczkowski's Theorem on algebras in which all elements are independent. We use here Marczewski's definition of independence in abstract algebras (see [5]). The following simple lemma was proved in [8]:

All elements of an n-element algebra are independent if and only if all n-ary algebraic operations are trivial.

The following theorem proved by S. Świerczkowski in [12] (Theorem 1) is a simple consequence of the above lemma and Theorem 13.1.

Świerczkowski's Theorem. *If all elements in an at least three-element algebra are independent, then the algebra is trivial.*

Indeed, let $\mathfrak{A}$ be an at least three-element algebra in which all elements are independent. Denote by $\mathfrak{B}$ the trivial algebra defined on the same set as the algebra $\mathfrak{A}$. Of course, the inclusion $\mathbf{A}(\mathfrak{B}) \subset \mathbf{A}(\mathfrak{A})$ holds. Moreover, by the quoted above lemma, $\mathbf{A}^{(n)}(\mathfrak{B}) = \mathbf{A}^{(n)}(\mathfrak{A})$ for all integers n which are not greater then the number of elements of the algebra $\mathfrak{A}$, i.e. for all $n \leqslant \gamma_0(\mathfrak{B})$. Since, by Theorem 13.1, $\varepsilon(\mathfrak{B}) = \gamma_0(\mathfrak{B})$, we have the inclusion $\mathbf{A}(\mathfrak{B}) \supset \mathbf{A}(\mathfrak{A})$. Thus $\mathbf{A}(\mathfrak{B}) = \mathbf{A}(\mathfrak{A})$ and, consequently, $\mathfrak{B} = \mathfrak{A}$ which completes the proof of Świerczkowski's Theorem.

14. Diagonal algebras. An algebra with one n-ary fundamental operation d is called *diagonal* if the following postulates are satisfied:

(i) $d(x, x, \ldots, x) = x$,

(ii) $d\big(d(x_1^1, \ldots, x_n^1), \ldots, d(x_1^n, \ldots, x_n^n)\big) = d(x_1^1, x_2^2, \ldots, x_n^n)$.

The class of diagonal algebras was introduced by J. Płonka in [10]. He proved also a representation theorem for diagonal algebras (see [10], Theorem 1). From this representation theorem it follows that diagonal algebras have the property (*). The property (**) for diagonal algebras was proved in [13] (Lemma 15). Hence and from Theorem 12.1 we get the following theorem.

Theorem 14.1. *For diagonal algebras with $\gamma_0 \geqslant 3$ the equation $\varepsilon = \gamma_0$ holds.*

We have seen that the assumption $\gamma_0 \geqslant 3$ in Theorem 13.1 is essential. Since trivial algebras are diagonal, the assumption $\gamma_0 \geqslant 3$ in Theorem 14.1 is essential too.

15. A class of semigroups. We say that a semigroup satisfies *the condition* (o) if for every term $x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$, where $k_1, k_2, \ldots, k_n$ are positive integers and $n \geqslant 2$, essentially depending on the variable x_1 the induced term $x^{k_1}y^{k_2+\ldots+k_n}$ also essentially depends on the variable x. Further, we say that a semigroup satisfies *the condition* (oo) if the equation $x^py^q = x^ry^s$, where p, q, r, s are positive integers and both terms x^py^q, x^ry^s essentially depend on the variable x, implies the equation $x^p = x^r$. It is clear that commutative semigroup satisfying conditions (o) and (oo) have the property (*).

Lemma 15.1. *Commutative semigroups satisfying conditions* (o) *and* (oo) *have the property* (**).

Proof. Let $n > 3$ and let f be an n-ary operation such that the composition $f(g_1, g_2, \ldots, g_n)$ is algebraic for all algebraic $(n-1)$-ary operations $g_1, g_2, \ldots, g_n$. Put

$$h_j(x, y) = f(z_1, z_2, \ldots, z_n) \quad (j = 1, 2, \ldots, n),$$

where $z_j = x$ and $z_i = y$ if $i \neq j$ $(i = 1, 2, \ldots, n)$. Since $n > 3$, the binary operations h_j $(j = 1, 2, \ldots, n)$ are algebraic. Further, for every pair $i < j$ $(i, j = 1, 2, \ldots, n)$ we define an algebraic operation f_{ij} by the formula

$$f_{ij}(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n).$$

Of course, the operation f_{ij} does not depend on the variable x_i. Moreover,

$$f_{ij}(u_{1r}, u_{2r}, \ldots, u_{nr}) = h_r(x, y) \quad (r \neq i, j) \tag{15.1}$$

where $u_{rr} = x$ and $u_{kr} = y$ if $k \neq r$ $(k = 1, 2, \ldots, n)$. For each operation f_{ij} there exist a non-void set of indices $I_{ij} \subset \{1, 2, \ldots, i-1, i+1, \ldots, n\}$ and positive integers $k_r(i, j)$ $(r \in I_{ij})$ such that

$$f_{ij}(x_1, x_2, \ldots, x_n) = \prod_{r \in I_{ij}} x_r^{k_r(i,j)}. \tag{15.2}$$

First consider the case that the operation f_{ij} essentially depends on a variable x_q for certain indices i, j, q such that $q \neq i, j$. Then, by the condition (o) and (15.1), the operation $h_q(x, y)$ essentially depends on the variable x. Let Q be the set of all indices r for which the binary operation $h_r(x, y)$ depends on the variable x. Of course, Q is a non-void set and, by the condition (o) and formula (15.1), $r \in Q$ if and only if f_{ij} depends on the variable x_r for $i, j \neq r$. Furthermore, each operation $h_r(x, y)$ $(r \in Q)$ is either of the form x^{k_r} or of the form $x^{k_r}y^{s_r}$, where k_r and s_r are positive integers. From (15.1), (15.2) and the conditions (o) and (oo) we obtain the formula

$$x^{k_r(i,j)} = x^{k_r} \quad \text{if} \quad r \neq i, j \text{ and } r \in Q. \tag{15.3}$$

Since $n > 3$, for each pair $i < j$ we can choose a disjoint pair of indices $p < q$. Setting $x_j = x$ and $x_s = y$ $(s \neq j;\ s = 1, 2, \ldots, n)$ into f_{ij} and $x_i = x_j = x$, $x_m = y$ $(m \neq i, j;\ m = 1, 2, \ldots, n)$ into f_{pq} we obtain identical expressions. Hence and from (15.2) and (15.3) we get, in view of the conditions (o) and (oo), the formula

$$x^{k_j(i,j)} = \begin{cases} x^{k_i} & \text{if} \quad i \in Q \text{ and } j \notin Q, \\ x^{k_j} & \text{if} \quad i \notin Q \text{ and } j \in Q, \\ x^{k_i+k_j} & \text{if} \quad i \in Q \text{ and } j \in Q. \end{cases}$$

Moreover, if both indices i and j do not belong to Q, then the operation f_{ij} does not depend on both variables x_i and x_j. Consequently, by (15.2) and (15.3), the operation $f(x_1, x_2, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n)$ can be written in the form

$$f(x_1, x_2, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n) = f_0(x_1, x_2, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n)$$

where

$$f_0(x_1, x_2, \ldots, x_n) = \prod_{r\in Q} x^{k_r}$$

and $i<j$ $(i,j=1,2,\ldots,n)$. Hence we get the property $(**)$ because the operation f_0 is algebraic which completes the proof of the lemma in the case where at least one operation f_{ij} essentially depends on a variable x_q with $q\neq i,j$.

Now consider the case where each operation f_{ij} depends on at most one variable x_j. Then the operations f_{ij} are of the form

$$f_{ij}(x_1, x_2, \ldots, x_n) = x_j^{s(i,j)} \qquad (i,j=1,2,\ldots,n;\ i<j),$$

where $s(i,j)$ are positive integers. Since $n>3$ for each pair $i<j$, we can choose a disjoint pair of indices $p<q$. Setting $x_j=x$ and $x_k=y$ $(k=1,2,\ldots,n;\ k\neq j)$ into f_{ij} and $x_i=x_j=x$, $x_m=y$ $(m=1,2,\ldots,n;$ $m\neq i,j)$ into f_{pq}, we obtain identical expressions equal to $x^{s(i,j)}$ and $y^{s(p,q)}$ respectively. Thus all operations $x^{s(i,j)}$ are constant and identical. Consequently,

$$f(x_1, x_2, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n) \equiv c$$

where c is an algebraic constant and $i<j$ $(i,j=1,2,\ldots,n)$. Hence we get the property $(**)$ which completes the proof of the lemma.

From Lemma 15.1 and Theorem 12.1 we get the following theorem.

THEOREM 15.1. *For every commutative semigroup with* $\gamma_0\geqslant 3$ *satisfying the conditions* (o) *and* (oo) *the equation* $\varepsilon=\gamma_0$ *holds.*

We note that each commutative semigroup with a unit element satisfies both conditions (o) and (oo). For instance, the semigroup of all subsets of a fixed set with the union as a semigroup operation satisfies these conditions. Since the two-element semigroup $\mathfrak{Q}_1$ defined on the set $\{0,1\}$ with $x\cup y$ as the semigroup operation satisfies both conditions (o) and (oo) and $\varepsilon(\mathfrak{Q}_1)=3$, $\gamma_0(\mathfrak{Q}_1)=2$ (see the table in Section 8), we infer that the assumption $\gamma_0\geqslant 3$ in Theorem 15.1 is essential.

Now we shall show by a counterexample that each of the conditions (o) and (oo) is also essential.

Let $\mathfrak{A}_1$ be the semigroup of all pairs $\langle E, i\rangle$ where $E\subset\{1,2,\ldots,\}$, $E\neq\emptyset$ and $i=0$ or 1. The semigroup operation is defined by the formula

$$\langle E, i\rangle\langle F, j\rangle = \langle E\cup F, \min(1, |E\cap F|+i+j)\rangle$$

where $|E|$ denotes the number of elements of the set E. It is easy to verify that the semigroup $\mathfrak{A}_1$ is commutative and satisfies the condition (o).

Moreover $\gamma_0(\mathfrak{A}_1)=k$ and the following identities are true in $\mathfrak{A}_1$: $x^n=x^2$ $(n\geqslant 2)$, $x^py^q=x^2y^2$ $(p+q\geqslant 3)$. Further, two operations $x_1x_2\ldots x_k$ and $x_1^2x_2^2\ldots x_k^2$ are different. Setting $f(x_1, x_2, \ldots, x_{k+1}) = x_1x_2\ldots x_k$ if $x_k=x_{k+1}$ and $f(x_1, x_2, \ldots, x_{k+1}) = x_1^2x_2^2\ldots x_k^2$ in the opposite case, we obtain a non-algebraic operation. On the other hand, the composition $f(g_1, g_2, \ldots, g_{k+1})$ with arbitrary k-ary algebraic operations $g_1, g_2, \ldots, g_{k+1}$ is algebraic. Hence we get the inequality $\varepsilon(\mathfrak{A}_1)\geqslant k+1$ which shows that the condition (oo) in Theorem 15.1 is essential.

Let $\mathfrak{A}_2$ be the semigroup of all subsets of the set $\{1,2,\ldots,k\}$ under the semigroup operation xy defined as follows:

$$xy = \begin{cases} x\cup y & \text{if} \quad x\cap y=\emptyset, \\ \emptyset & \text{if} \quad x\cap y\neq\emptyset. \end{cases}$$

It is easy to verify that $\mathfrak{A}_2$ is a commutative semigroup satisfying the condition (oo) and $\gamma_0(\mathfrak{A}_2)=k$. Moreover, the empty set is an algebraic constant and the operation $x_1x_2\ldots x_k$ is not constant. Consequently, setting $f(x_1, x_2, \ldots, x_{k+1}) = x_1x_2\ldots x_k$ if $x_k=x_{k+1}$ and $f(x_1, x_2, \ldots, x_{k+1})=\emptyset$ in the opposite case we get a non-algebraic operation. On the other hand, the composition $f(g_1, g_2, \ldots, g_{k+1})$ with arbitrary k-ary algebraic operations $g_1, g_2, \ldots, g_{k+1}$ is algebraic. Thus $\varepsilon(\mathfrak{A}_2)\geqslant k+1$, which shows that the condition (o) in Theorem 15.1 is also essential.

16. Modules. In this section unital modules over an associative ring with the unit element, i.e. modules satisfying the condition $1x=x$, will be called briefly *modules*. The class of algebraic operations in a module consists of all homogeneous linear forms. It is clear that modules satisfy the assumptions of Theorem 12.2 if as an algebraic constant c the zero-element is taken. Thus, by Theorem 12.2, all modules have the properties $(*)$ and $(**)$.

THEOREM 16.1. *For every module with* $\gamma_0\geqslant 2$ *the equation* $\varepsilon=\gamma_0$ *holds.*

Proof. For modules with $\gamma_0\geqslant 3$ our statement is a consequence of Theorem 12.1.

Consider the case $\gamma_0=2$. By Theorem 12.1 we have the inequality $\varepsilon\leqslant 3$. To prove the inequality $\varepsilon\leqslant 2$ it suffices to prove that each ternary operation f is algebraic provided the composition $f(g_1, g_2, g_3)$ with arbitrary binary algebraic operations g_1, g_2, g_3 is algebraic. Suppose that an operation f has this property. Then the formula

$$f_0(x,y,z) = f(x,0,0)+f(0,y,0)+f(0,0,z)$$

defines an algebraic operation. Moreover, since algebraic operations in a module are homogeneous linear forms, we have the equations

$$f(x,y,0) = f(x,0,0)+f(0,y,0) = f_0(x,y,0).$$

Hence it follows that for arbitrary unary algebraic operations h_1, h_2 and h_3 the equation

$$f\big(h_1(x), h_2(x), h_3(y)\big) = f_0\big(h_1(x), h_2(x), h_3(y)\big) \tag{16.1}$$

holds whenever $x = 0$ or $y = 0$. Consequently, (16.1) holds for all x and y. Thus, for arbitrary binary algebraic operations g_1, g_2, and g_3, the equation

$$f\big(g_1(x, y), g_2(x, y), g_3(x, y)\big) = f_0\big(g_1(x, y), g_2(x, y), g_3(x, y)\big) \tag{16.2}$$

holds whenever $x = 0$ or $y = 0$. Consequently, equation (16.2) is true for all x and y.

Given arbitrary elements a_1, a_2, a_3 of the module, there exist, by the assumption $\gamma_0 = 2$, a pair b_1, b_2 of elements and binary algebraic operations g_1, g_2, g_3 such that $a_j = g_j(b_1, b_2)$ $(j = 1, 2, 3)$. Hence and from (16.2) we obtain the equation $f(a_1, a_2, a_3) = f_0(a_1, a_2, a_3)$ which implies the identity $f = f_0$. Thus the operation f is algebraic and, consequently, $\varepsilon \leqslant 2$.

It is clear that modules are not isomorphic to the exceptional algebras $\mathfrak{J}_1$ and $\mathfrak{J}_2$ defined in Section 8. Thus, by Theorem 9.1, $\varepsilon \geqslant \gamma_0 = 2$ and, consequently, $\varepsilon = 2$ which completes the proof of the theorem.

We note that the assumption $\gamma_0 \geqslant 2$ in the above theorem is essential. As a counterexample we can take the algebra $\mathfrak{L}_3$ defined in Section 8. Since $\mathfrak{L}_3 = (\{0, 1\}; s_3)$ and $s_3(x, y) = x + y \,(\mathrm{mod}\, 2)$, the algebra $\mathfrak{L}_3$ is a linear space over the two-element field. Further, from the table in Section 8 we obtain the formulas $\varepsilon(\mathfrak{L}_3) = 2$ and $\gamma_0(\mathfrak{L}_3) = 1$.

Since Abelian groups can be regarded as unital modules, the following corollary is a direct consequence of Theorem 16.1.

Corollary. *For Abelian groups with $\gamma_0 \geqslant 2$ the equation $\varepsilon = \gamma_0$ holds.*

17. Maximal idempotent reducts of linear spaces. Let $\mathfrak{A}$ be a linear space over a field. The class of algebraic operations in the maximal idempotent reduct $\mathfrak{I}(\mathfrak{A})$ of the algebra $\mathfrak{A}$ consists of all operations defined as

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} c_j x_j ,$$

where $\sum_{j=1}^{n} c_j = 1$.

Lemma 17.1. *For any linear space we have the formula*

$$\gamma_0\big(\mathfrak{I}(\mathfrak{A})\big) = \gamma_0(\mathfrak{A}) + 1 .$$

Proof. Let $\mathfrak{B}$ be a finitely generated subalgebra of $\mathfrak{I}(\mathfrak{A})$. Then there exists a subalgebra $\mathfrak{B}_0$ of $\mathfrak{A}$ generated by $\gamma_0(\mathfrak{A})$ elements and such that the carrier of $\mathfrak{B}_0$ contains the carrier of $\mathfrak{B}$. Let G be the set of generators of the algebra $\mathfrak{B}_0$. It is clear that $G \cup \{0\}$ is the set of generators of the reduct $\mathfrak{I}(\mathfrak{B}_0)$. Since $\mathfrak{I}(\mathfrak{B}_0)$ is a subalgebra of $\mathfrak{I}(\mathfrak{A})$ and contains the subalgebra $\mathfrak{B}$, we have the inequality

$$\gamma_0\big(\mathfrak{I}(\mathfrak{A})\big) \leqslant \gamma_0(\mathfrak{A}) + 1 .$$

This inequality implies the lemma in the case $\gamma_0\big(\mathfrak{I}(\mathfrak{A})\big) = \infty$. Suppose now that $s = \gamma_0\big(\mathfrak{I}(\mathfrak{A})\big) < \infty$. Let $\mathfrak{C}$ be an arbitrary finitely generated subalgebra of $\mathfrak{A}$ and $\mathfrak{D}$ a subalgebra of $\mathfrak{I}(\mathfrak{A})$ with the same carrier. Of course, the subalgebra $\mathfrak{D}$ is also finitely generated and, consequently, is contained in a subalgebra of $\mathfrak{I}(\mathfrak{A})$ generated by s elements, say $a_1, a_2, \ldots, a_s$. Let $\mathfrak{D}_0$ be the subalgebra of $\mathfrak{A}$ generated by elements $a_1, a_2, \ldots, a_s$. Of course, $\mathfrak{D}_0$ contains the subalgebra $\mathfrak{C}$. Further, taking into account that the zero-element belongs to $\mathfrak{C}$, we get the existence of elements $c_1, c_2, \ldots, c_s$ of the field satisfying the condition $\sum_{j=1}^{s} c_j = 1$ for which $\sum_{j=1}^{s} c_j a_j = 0$. Hence it follows that the subalgebra $\mathfrak{D}_0$ can be generated by $s-1$ elements. Thus

$$\gamma_0(\mathfrak{A}) \leqslant \gamma_0\big(\mathfrak{I}(\mathfrak{A})\big) - 1$$

which completes the proof.

Theorem 17.1. *For any linear space $\mathfrak{A}$ with $\gamma_0(\mathfrak{A}) \geqslant 2$ the formula $\varepsilon\big(\mathfrak{I}(\mathfrak{A})\big) = \gamma_0\big(\mathfrak{I}(\mathfrak{A})\big)$ holds.*

Proof. Let $\mathfrak{I}(\mathfrak{A}) = (A; \mathsf{F})$. It is obvious that $\mathfrak{A} = (A; \mathsf{F} \cup \{0\})$. Hence and from Theorems 4.3, 16.1 and Lemma 17.1 we get the inequality

$$\varepsilon\big(\mathfrak{I}(\mathfrak{A})\big) \leqslant \gamma_0\big(\mathfrak{I}(\mathfrak{A})\big) .$$

Since $\mathfrak{I}(\mathfrak{A})$ together with $\mathfrak{A}$ satisfies condition (*) and is not isomorphic to the exceptional algebras $\mathfrak{J}_1$ and $\mathfrak{J}_2$, we have, according to Theorem 9.1, the converse inequality $\varepsilon\big(\mathfrak{I}(\mathfrak{A})\big) \geqslant \gamma_0\big(\mathfrak{I}(\mathfrak{A})\big)$ which completes the proof.

References

[1] G. Birkhoff, *Lattice theory*, New York, 1948.

[2] A. Hulanicki and S. Świerczkowski, *Number of algebras with a given set of elements*, Bull. Acad. Pol. Sci., Série des sci. math., astr. et phys. 8 (1960), pp. 283-284.

[3] Ю. И. Янов и А. А. Мучник, *О существовании k-значных замкнутых классов, не имеющих конечного базиса*, Доклады Академии Наук СССР 127 (1959), pp. 44-46.

[4] E. Marczewski, *Independence in algebras of sets and Boolean algebras*, Fund. Math. 48 (1960), pp. 135-145.

[5] — *Independence and homomorphisms in abstract algebras*, Fund. Math. 50 (1961), pp. 45-61.

[6] — *Nombre d'éléments indépendants et nombre d'éléments générateurs dans les algèbres abstraites finies*, Annali di Matematica (4) 59 (1962), pp. 1-9.

[7] — *Independence in abstract algebras. Results and problems*, Colloq. Math. 14 (1966), pp. 169-188.

[8] E. Marczewski and K. Urbanik, *Abstract algebras in which all elements are independent*, Bull. Acad. Pol. Sci., Série des sci. math., astr. et phys. 8 (1960), pp. 291-293.

[9] — — *Abstract algebras in which all elements are independent*, Colloq. Math. 9 (1962), pp. 199-207.

[10] J. Płonka, *Diagonal algebras and algebraic independence*, Bull. Acad. Pol. Sci., Série des sci. math., astr. et phys. 12 (1964), pp. 729-733.

[11] E. L. Post, *The two-valued iterative systems of mathematical logic*, Princeton 1941.

[12] S. Świerczkowski, *Algebras which are independently generated by every n elements*, Fund. Math. 49 (1960), pp. 93-104.

[13] K. Urbanik, *On algebraic operations in idempotent algebras*, Coll. Math. 13 (1965), pp. 129-157.

WROCŁAW UNIVERSITY,
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 17. 9. 1965

Equationally compact algebras (I)

by

B. Węglorz (Wrocław)

0. Introduction. This paper gives a study of equationally compact algebras, introduced by J. Mycielski [11], and some generalizations of this notions. The equational compactness is a simple reformulation in the language of general algebras of a definition of J. Łoś [8] of the notion of algebraical compactness of Abelian groups introduced by I. Kaplansky. The definitions of this and related notions are given in Section 1.

The main results of this paper are contained in Section 2 and give a characterization of equationally compact algebras in terms of ultra-powers and retracts. Perhaps the most interesting result is that positive compactness and atomic-compactness coincide.

In Section 3 we add several remarks and propositions concerning equationally and weakly equationally compact algebras of well-known kinds such as linear spaces, groups and modules, and in Section 4 equationally compact Boolean algebras are studied. In Section 5 we prove that equational compactness in general is not elementarily definable and we mention some open problems.

The author is indebted to Jan Mycielski and C. Ryll-Nardzewski for their discussions which improved the theorems and simplified the proofs, and to the first of them for many stimulating questions and help in composition of this paper.

The main results were announced in [17].

1. Preliminaries. For any non-empty sets X and Y, Y^X denotes the set of all functions $f\colon X\to Y$; the cardinality of a set X is denoted by $|X|$; $\omega=\{0,1,2,\ldots\}$. $\mathfrak{A}=\langle A,\{F_q\}_{q\in Q},\{G_r\}_{r\in R}\rangle$ is an *algebraic system* if A is a non-void set, there are maps $f\colon Q\to\omega$ and $g\colon R\to\omega-\{0\}$ such that $F_q\colon A^{f(q)}\to A$ for $f(q)>0$ and $F_q\in A$ for $f(q)=0$, and $G_r\subseteq A^{g(r)}$ for all $r\in R$. The sequence $\tau=\langle Q,f,R,g\rangle$, uniquely determined by $\mathfrak{A}$, is called the *similarity type of* $\mathfrak{A}$. If R is void, then $\mathfrak{A}$ is called an *algebra*. A is called the set of $\mathfrak{A}$. In the sequal we denote algebraic systems by $\mathfrak{A},\mathfrak{B},\mathfrak{C},\ldots$ and their sets by $A,B,C,\ldots$, respectively.

If $B\subseteq A$ and, for each F_q, if $b_1,\ldots,b_{f(q)}\in B$ then $F_q(b_1,\ldots,b_{f(q)})\in B$, then $\mathfrak{B}=\langle B,\{F'_q\}_{q\in Q},\{G'_r\}_{r\in R}\rangle$, where $G'_r=G_r\cap B^{g(r)}$ and F'_q are