# Inductive invariants and dimension theory *

by

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**0. Introduction.** Recently, some interest has been shown in using the beautiful inductive approach of dimension theory to other situations. Some interesting applications and conjectures have resulted. (See [1], [2] and [5].) To clarify our discussion we first give a definition.

DEFINITION. (INDUCTIVE INVARIANT.) All spaces under consideration are separable metrizable spaces. By a topologically closed family $P$ of spaces we mean a family of spaces such that $X \in P$ and $X'$ homeomorphic to $X$ imply $X' \in P$.

The *inductive invariant* $\operatorname{in}P\, X$ induced by the topologically closed family $P$ is defined for every space $X$ as follows:

$\operatorname{in}P\, X = -1$ if and only if $X \in P$.

For each integer $n \geqslant 0$, $\operatorname{in}P\, X \leqslant n$ provided that each point of $X$ has arbitrarily small open neighborhoods $U$ in $X$ such that $\operatorname{in}P\, B \leqslant n-1$, where $B$ is the boundary of $U$.

For each integer $n \geqslant 0$, $\operatorname{in}P\, X = n$ if $\operatorname{in}P\, X \leqslant n$ is true and $\operatorname{in}P\, X \leqslant n-1$ is false.

$\operatorname{in}P\, X = \infty$ if $\operatorname{in}P\, X \leqslant n$ is false for all integers $n \geqslant -1$.

Of course, inductive dimension is an example of an inductive invariant. In 1941, J. de Groot [1] used the family of compact spaces and gave a conjecture which is still unsettled to this day. (See [2] and [4] for discussions of this conjecture.) In 1964, A. Lelek [5] defined two more examples of inductive invariants. In fact, the name "inductive invariant" given above is due to A. Lelek. In the last mentioned paper, some interesting results concerning dimension and continuous mappings are proved.

In the present paper we address ourselves to a characterization problem posed by K. Menger in 1929. In [6], K. Menger discussed the problem of finding a characterization of the inductive dimension function. In carrying out his discussion, Menger introduced certain topologically closed families of spaces which arise naturally from the dimension function.

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With these topologically closed families in mind, we try to prove theorems analogous to those in dimension theory. That is, we wish to determine what part of dimension theory is due to the inductive nature of the definition and what part is due to the topologically closed family $P$. In this manner, we will isolate important necessary conditions needed for characterizing the dimension function. Employing these necessary conditions and others used by K. Menger, we will give a characterization of inductive dimension in section 8.

In section 1 we give some elementary properties of inductive invariant. Section 2 concerns the range of the inductive invariant functions. Sections 3 and 4 involve topologically closed families defined by Menger. In particular, section 3 deals with monotone properties and section 4 deals with sum and decomposition theorems. Sections 5 and 6 deal with separation and product. Some specific examples are discussed in section 7.

Throughout the paper, families $P$ will be assumed to be topologically closed.

**1. Some elementary properties.** In this section we give some elementary properties of $\operatorname{in}P\,X$. We do not prove the more obvious propositions.

1.1. PROPOSITION. *$\operatorname{in}P\,X$ is a topological invariant.*

1.2. PROPOSITION. *Suppose $n \geqslant -1$. Then $\operatorname{in}P\,X \leqslant n$ if and only if there is a countable basis of open sets whose boundaries have $\operatorname{in}P \leqslant n-1$, or $\operatorname{in}P\,X = -1$.*

1.3. THEOREM. *$\operatorname{in}P\,\emptyset \leqslant 0$ for every $P$.*

Proof. Suppose $\operatorname{in}P\,\emptyset \neq -1$. Then $\operatorname{in}P\,\emptyset \geqslant 0$. Clearly, each point of $\emptyset$ has arbitrarily small neighborhoods $U$ with boundary $B \in P$. Hence, $\operatorname{in}P\,\emptyset \leqslant 0$. The theorem is proved.

It should be remarked at this point that arbitrary closed subsets $X'$ of a space $X$ with $\operatorname{in}P\,X = -1$ need not have $\operatorname{in}P\,X' = -1$.

The next theorem follows easily by induction.

1.4. THEOREM. *Let $P$ and $Q$ be two families with $P \subset Q$. Then $\operatorname{in}Q\,X \leqslant \operatorname{in}P\,X$ for all $X$.*

1.5. LEMMA. *Suppose $X \neq \emptyset$ and $\operatorname{in}P\,X = 0$. Then there is a space $X'$ for which $\operatorname{in}P\,X' = -1$. I.e., $P \neq \emptyset$.*

1.6. EXAMPLE. We give here a useful example. Let $P_0$ be the empty family. Then $\operatorname{in}P_0\,X \geqslant \operatorname{in}P\,X$ for all $X$ and all $P$. Let us compute $\operatorname{in}P_0\,X$. Clearly $\operatorname{in}P_0\,X \geqslant 0$. By lemma 1.5, $X \neq \emptyset$ implies $\operatorname{in}P_0\,X \geqslant 1$. Hence, by theorem 1.3, $\operatorname{in}P_0\,X = 0$ if and only if $X = \emptyset$. Now it follows that $\operatorname{in}P_0\,X = \dim X + 1$ for all $X$. We summarize these facts into the following theorem.

1.7. THEOREM. *$\operatorname{in}P\,X \leqslant \operatorname{in}P_0\,X = \dim X + 1$ for all $X$ and all $P$.*



1.8. THEOREM. *$\emptyset \in P$ implies $\dim X \geqslant \operatorname{in}P\,X$ for all $X$.*

**2. The range problem.** Given a family $P$ and an integer $n \geqslant -1$, one can ask whether $\operatorname{in}P\,X = n$ for some $X$. This problem will be called the *existence problem for the family $P$*. Of course, the existence problem refers to a specific family $P$. In this section we deal with a problem related to the existence problem. Namely, we will determine exactly which subsets of the extended integers are the ranges of inductive invariant functions. We will call this problem the *range problem*. We proceed to its solution.

2.1. THEOREM. *Suppose $\operatorname{in}P\,X = n$ ($n$ finite). Then for each integer $m$ ($0 \leqslant m \leqslant n$) there is a closed subspace $X_m$ of $X$ such that $\operatorname{in}P\,X_m = m$.*

Proof. The theorem is obvious for $n = -1$ and $0$. We prove the theorem for $n \geqslant 0$ by induction. Suppose that the theorem is true for the integer $n$ ($n \geqslant 0$) and let $X$ be such that $\operatorname{in}P\,X = n+1$. Then, by theorem 1.3, $X \neq \emptyset$. Let $x$ be a point of $X$ which has a neighborhood $U$ whose boundary $B$ has $\operatorname{in}P\,B = n$. Such a neighborhood exists since $n+1 \neq -1$. The theorem now follows by induction.

2.2. COROLLARY. *Suppose $\emptyset \in P$ and $\operatorname{in}P\,X = n$ ($n$ finite). Then for every integer $m$ ($-1 \leqslant m \leqslant n$) there is a closed subspace $X_m$ of $X$ such that $\operatorname{in}P\,X_m = m$.*

It should be remarked that theorem 2.1 is best possible as $\operatorname{in}P_0$ of example 1.6 above shows.

For each extended integer $k$, $k \geqslant 0$, let $A_k = \{n \mid n \text{ is an integer}, -1 \leqslant n < k\}$, and $B_k = A_k \cup \{\infty\}$. Next, let $C$ be the set of non-negative extended integers. Then we have that the range of $\operatorname{in}P$ is $A_k$ or $B_k$ for some $k \geqslant 0$ or $C$. Namely, if $P$ is a nonempty family, then by theorem 2.1 the range of $\operatorname{in}P$ is $A_k$ or $B_k$, and otherwise by theorem 1.7 the range of $\operatorname{in}P_0$ is $C$. We show the converse to hold by examples.

2.3. EXAMPLE. For each extended integer $n$ ($-1 \leqslant n \leqslant \infty$) let $Q_n = \{X \mid \dim X \geqslant n\}$. Then

$$\operatorname{in}Q_n\,X = \begin{cases} -1 & \text{if and only if } \dim X \geqslant n; \\ j+1 & \text{if and only if } \dim X = j \ (-1 \leqslant j < n). \end{cases}$$

Proof. If $n = -1$, then the above statement is trivially true. Hence we will prove the above statement for all extended integers $n \geqslant 0$. The proof is by induction on $j$.

Let $j = -1$ and $n = 0$. Then, clearly, $\dim X = -1$ if and only if $\operatorname{in}Q_0\,X = 0$. Hence the statement is true for $j = -1$ and $n = 0$. Suppose $j = -1$ and $n > 0$. Since $\operatorname{in}Q_n\,\emptyset \neq -1$, we have $\operatorname{in}Q_n\,\emptyset = 0$ by theorem 1.3. Suppose $\dim X \geqslant 0$ and $\operatorname{in}Q_n\,X \geqslant 0$. Then $0 \leqslant \dim X < n$. Consequently, the boundary $B$ of every open set has $-1 \leqslant \dim B \leqslant \dim X < n$. That is, $\operatorname{in}Q_n B \geqslant 0$. Hence, $\operatorname{in}Q_n X \geqslant 1$. Thus we have shown that $j = -1$ and $n > j$ imply the above statement is true.

Assume the statement to be true for $\mu$ $(-1 \leqslant \mu \leqslant j)$. We prove the statement for extended integers $n > j+1$. Suppose $n > j+1$ and let $\dim X = j+1$. Then $\operatorname{in}Q_n X \neq -1$. We have $\operatorname{in}Q_n Y = i+1$ if and only if $\dim Y = i$ $(i = -1, 0, \ldots, j)$. Hence, $\operatorname{in}Q_n X \geqslant j+2$. Applying the definition of $\operatorname{in}Q_n X$, we have $\operatorname{in}Q_n X \leqslant j+2$. Thus, when $n > j+1$, we have that $\dim X = j+1$ implies $\operatorname{in}Q_n X = j+2$.

We prove the converse implication next. Suppose $n > j+1$ and $\operatorname{in}Q_n X = j+2$. From theorem 1.7, $1+\dim X \geqslant \operatorname{in}Q_n X = j+2$. Hence, $\operatorname{in}Q_n X = j+2$ implies $\dim X \geqslant j+1$. Suppose $n = j+2$. Since $n = j+ +2 > 0$, we have $\operatorname{in}Q_n X = j+2$ implies $\dim X \leqslant j+1 = n-1 < n$. Hence, in the case $n = j+2$, we have $\operatorname{in}Q_n X = j+2$ implies $\dim X = j+1$. Suppose the extended integer $n > j+2$, $\dim X \geqslant j+2$ and $\operatorname{in}Q_n X \geqslant j+2$. Since $j+2 > 0$, $n > \dim X \geqslant j+2$. There is some point of $X$ such that sufficiently small open neighborhoods have boundaries $B$ with $j+1 \leqslant \dim B \leqslant \dim X < n$. That is, $\operatorname{in}Q_n B \geqslant j+2$. Hence, $\operatorname{in}Q_n X \geqslant j+3$. Consequently, for $n > j+2$, we have that $\operatorname{in}Q_n X = j+2$ implies $\dim X \leqslant j+1$. But we have already established above that $n > j+1$ and $\operatorname{in}Q_n X = j+2$ imply $\dim X \geqslant j+1$. Hence, $n > j+1$ and $\operatorname{in}Q_n X = j+2$ imply $\dim X = j+1$.

The statement is completely proved.

2.4. Example. For each extended integer $n$ $(-1 \leqslant n < \infty)$ let $R_n = \{X \mid X$ has transfinite dimension $\geqslant n\}$. Then

$\operatorname{in}R_n X = -1$ if and only if $X$ has transfinite dimension $\geqslant n$;
$\operatorname{in}R_n X = j+1$ if and only if $\dim X = j$ $(-1 \leqslant j < n)$;
$\operatorname{in}R_n X = \infty$ if and only if $X$ does not have transfinite dimension.

Proof. For each $n$ we have $R_n \subset Q_n$. Hence $\operatorname{in}P_0 X \geqslant \operatorname{in}R_n X \geqslant \operatorname{in}Q_n X$. Thus, if $\dim X$ is finite then, by 1.7 and 2.3, $\operatorname{in}R_n X = \operatorname{in}Q_n X$. If $\dim X$ is infinite, then either $X$ has transfinite dimension or not. The statement now follows. (We note that there are spaces which have no transfinite dimension.)

We now have the following theorem.

2.5. Theorem. *A subset $N$ of the extended integers is the range of some* $\operatorname{in}P$ *function if and only if $N$ is one of the following*:

(1) $A_k$ $(k = -1, 0, \ldots, \infty)$;
(2) $B_k$ $(k = -1, 0, \ldots, \infty)$;

*or*

(3) $C$.

The range problem is now completely solved.



**3. Monotone property.** This section concerns a property of families introduced by K. Menger. It is well known that the inductive dimension function is monotone. That is, if $X \subset Y$ then $\dim X \leqslant \dim Y$. Hence the family $P = \{X \mid \dim X \leqslant n\}$ has the property that $Y \in P$ and $X \subset Y$ imply $X \in P$. We will study families with properties similar to that above. Let us begin with a definition.

3.1. Definition. A family $P$ is said to be

$$\left\{\begin{array}{l} 1.\ \textit{monotone} \\ 2.\ F_\sigma\textit{-monotone} \\ 3.\ \textit{c-monotone} \end{array}\right\} \text{ if } X \text{ is a } \left\{\begin{array}{l} 1.\ \text{subset} \\ 2.\ F_\sigma \text{ subset} \\ 3.\ \text{closed subset} \end{array}\right\}$$

of a space $Y$ and $Y \in P$ imply $X \in P$.

An extended real-valued function $f$ on the collection of separable metrizable spaces is called

$$\left\{\begin{array}{l} 1.\ \textit{monotone} \\ 2.\ F_\sigma\textit{-monotone} \\ 3.\ \textit{c-monotone} \end{array}\right\} \text{ if } f(X) \leqslant f(Y)$$

$$\text{whenever } X \text{ is a } \left\{\begin{array}{l} 1.\ \text{subset} \\ 2.\ F_\sigma \text{ subset} \\ 3.\ \text{closed subset} \end{array}\right\} \text{ of } Y.$$

3.2. Proposition. *For families or functions, monotone implies $F_\sigma$-monotone, and $F_\sigma$-monotone implies c-monotone. For c-monotone functions, $f(X) \geqslant f(\emptyset)$ for every $X$.*

3.3. Theorem. *A family $P$ is*

$$\left\{\begin{array}{l} 1.\ \textit{monotone} \\ 2.\ F_\sigma\textit{-monotone} \\ 3.\ \textit{c-monotone} \end{array}\right\} \textit{ if and only if } \operatorname{in}P \textit{ is } \left\{\begin{array}{l} 1.\ \textit{monotone} \\ 2.\ F_\sigma\textit{-monotone} \\ 3.\ \textit{c-monotone} \end{array}\right\}.$$

Proof. We prove the c-monotone case. The other cases are proved in an analogous manner. Clearly, if $\operatorname{in}P$ is c-monotone then $P$ is also c-monotone. We prove the converse by induction. We prove the proposition:

*If $\operatorname{in}P X \leqslant n$ and $X'$ is closed in $X$, then $\operatorname{in}P X' \leqslant n$.*

The proposition is obvious for $n = -1$. Let $n$ be an integer $\geqslant -1$ and assume that the proposition is true for all integers $k$ $(-1 \leqslant k \leqslant n)$. Suppose that $X$ is such that $\operatorname{in}P X = n+1$ and $X'$ is closed in $X$. If $X' = \emptyset$, then $\operatorname{in}P \emptyset \leqslant 0 \leqslant n+1 = \operatorname{in}P X$. Suppose $X' \neq \emptyset$. Then each point of $X'$ has arbitrarily small open neighborhoods $U$ in $X$ such that the boundary $B$ of $U$ in $X$ has $\operatorname{in}P B \leqslant n$. Let $U' = U \cap X'$ and $B'$ be the boundary of $U'$ in $X'$. Then $B' \subset B \cap X'$ and $B'$ is closed in $B \cap X'$.

Also, $B \cap X'$ is closed in $B$. Hence $\operatorname{in}P\, B' \leqslant \operatorname{in}P\, B \cap X' \leqslant \operatorname{in}P\, B \leqslant n$. That is, $\operatorname{in}P\, X' \leqslant n+1$. The proposition now follows.

3.4. COROLLARY. *If $P$ is c-monotone then $\operatorname{in}P\, \emptyset \leqslant \operatorname{in}P\, X$ for all $X$.*

The next theorem is very useful in the succeeding section. The proof is the same as the analogous theorem for dimension. See [3], A), p. 27.

3.5. THEOREM. *Suppose that $P$ is c-monotone and $n \geqslant 0$. Then, a subspace $X'$ of a space $X$ has $\operatorname{in}P\, X' \leqslant n$ if and only if every point of $X'$ has arbitrarily small neighborhoods in $X$ whose boundaries have intersection with $X'$ of $\operatorname{in}P \leqslant n-1$.*

Proof. Suppose that $X'$ satisfies the conditions of the theorem. If $X' = \emptyset$ then $\operatorname{in}P\, \emptyset \leqslant n$. Hence we assume $X' \neq \emptyset$. Let $x \in X'$ and $U'$ be a neighborhood in $X'$ of $x$. Then there is a neighborhood $U$ in $X$ of $x$ such that $U' = U \cap X'$. Hence, there is a neighborhood $V$ in $X$ of $x$ such that $V \subset U$ and $\operatorname{in}P\, B \cap X' \leqslant n-1$, where $B$ is the boundary of $V$ in $X$. Let $V'$ be the intersection of $X'$ with the interior relative to $X$ of $V$ and let $B'$ be the boundary of $V'$ in $X'$. Then $V'$ is open in $X'$, $x \in V' \subset U'$, and $B'$ is closed in $B \cap X'$. Hence we have $\operatorname{in}P\, B' \leqslant \operatorname{in}P\, B \cap X' \leqslant n-1$. Consequently, $\operatorname{in}P\, X' \leqslant n$.

Conversely, suppose $\operatorname{in}P\, X' \leqslant n$. If $X' = \emptyset$ then the condition is trivially satisfied. Hence, suppose further that $X' \neq \emptyset$. Let $x \in X'$ and $U$ be a neighborhood in $X$ of $x$. Then there is an open neighborhood $V'$ in $X'$ of $x$ for which $V' \subset U$ and $\operatorname{in}P\, B' \leqslant n-1$, where $B'$ is the boundary in $X'$ of $V'$. Neither of the disjoint sets $V'$ and $X' \backslash \overline{V'}$ contains a cluster point of the other, where $\overline{M}$ means the closure of $M$ in $X$. So by the complete normality of $X$ there exists an open set $W$ satisfying $V' \subset W$ and $\overline{W} \cap (X' \backslash V') = \emptyset$. By replacing $W$ if necessary by $W \cap U$, we may assume $W \subset U$. The boundary $B$ of $W$ contains no points of $V'$. Hence $B \cap X' \subset B'$. Since $B \cap X'$ is closed in $B'$, we have $\operatorname{in}P\, B \cap X' \leqslant \operatorname{in}P\, B' \leqslant n-1$. Now the condition of the theorem is fulfilled by $W$. The theorem is now completely proved.

The following analogue of proposition 1.2 is easily proved.

3.6. PROPOSITION. *Suppose that $P$ is c-monotone and $n \geqslant 0$. Then $\operatorname{in}P\, X \leqslant n$ if and only if there is a countable basis of open sets such that the $\operatorname{in}P$ of the boundaries are $\leqslant n-1$.*

We conclude this section with a characterization of inductive dimension.

3.7. THEOREM. *Let $P$ be a family of spaces. Then $\operatorname{in}P\, X = \dim X$ if and only if $P$ is c-monotone and $\operatorname{in}P\, \{\text{point}\} = 0$.*

Proof. If $\operatorname{in}P\, X = \dim X$ then $P$ is c-monotone and $\operatorname{in}P\, \{\text{point}\} = 0$. We prove the converse. Obviously, $\operatorname{in}P\, \emptyset = -1$, for otherwise $\operatorname{in}P\, X = \dim X + 1$ and we would contradict the hypothesis that $\operatorname{in}P\, \{\text{point}\} = 0$.



Since $P$ is c-monotone, $X \neq \emptyset$ implies $\operatorname{in}P\, X \geqslant \operatorname{in}P\, \{\text{point}\} = 0$. Consequently, $\operatorname{in}P\, X = -1$ if and only if $X = \emptyset$. Therefore, $\operatorname{in}P\, X = \dim X$.

**4. Sum and decomposition theorems.** In this section we investigate to what extent the sum and decomposition theorems of dimension are valid. (See [3] for the dimension theorems.)

4.1. THEOREM. *If $P$ is c-monotone then $\operatorname{in}P\, X \cup Y \leqslant \operatorname{in}P\, X + \dim Y + 1$.*

Proof. If $\dim Y = -1$ then the proposition holds for all $X$. Assume that the proposition holds for all spaces $Y$ with $\dim Y \leqslant n$ ($n \geqslant -1$) and all $X$. Let $\dim Y = n+1$. Then by [3] B), page 34, each point of $X \cup Y$ has arbitrarily small open neighborhoods $U$ whose boundaries $B$ meet $Y$ in a set of dimension $\leqslant n$. Hence $\operatorname{in}P\, B \leqslant \operatorname{in}P\, B \cap X + \dim B \cap Y + 1 \leqslant \operatorname{in}P\, X + \dim B \cap Y + 1$. Therefore, $\operatorname{in}P\, X \cup Y \leqslant \operatorname{in}P\, X + \dim Y + 1$. The theorem now follows.

4.2. THEOREM. *If $P$ is c-monotone and $X \supset Y$, then $\operatorname{in}P\, X \backslash Y \leqslant \operatorname{in}P\, X + \dim Y + 1$.*

Proof. The proposition is true for $\dim Y = -1$ and all $X$. Assume the proposition holds for all spaces $Y$ with $\dim Y \leqslant n$ ($n \geqslant -1$) and all $X \supset Y$. Let $\dim Y = n+1$ and $X \supset Y$. Each point of $X \backslash Y$ has arbitrarily small open neighborhoods $U$ whose boundaries $B$ meet $Y$ in a set of dimension $\leqslant n$. Hence,

$$\operatorname{in}P(B \backslash B \cap Y) \leqslant \operatorname{in}P\, B + \dim B \cap Y + 1 \leqslant \operatorname{in}P\, X + \dim B \cap Y + 1 .$$

Therefore, by theorem 3.5, $\operatorname{in}P\, X \backslash Y \leqslant \operatorname{in}P\, X + \dim Y + 1$. The theorem now follows.

We now proceed to investigate the analogues of the sum theorem of dimension, [3], Theorem III 2. The next definition is motivated by the sum theorem. Similar definitions were given by K. Menger in [6].

4.3. DEFINITION. A family $P$ is called *$F_\sigma$-constant* if each space $X$ which is the countable union of closed subsets each a member of $P$ is also a member of $P$.

An extended real-valued function $f$ on the collection of separable metrizable spaces is called *$F_\sigma$-constant* if for each space $X$ which is the countable union of closed subsets $X_i$ we have $f(X) \leqslant \operatorname{Sup} f(X_i)$.

4.4. PROPOSITION. *A family which is c-monotones and $F_\sigma$-constant is also $F_\sigma$-monotone. The family of finite dimensional spaces is monotone but not $F_\sigma$-constant.*

4.5. PROPOSITION. *If $\operatorname{in}P$ is $F_\sigma$-constant then $P$ is $F_\sigma$-constant. There exists an $F_\sigma$-constant family $P$ which is not c-monotone such that $\operatorname{in}P$ is not $F_\sigma$-constant.*

Proof. The first statement is obvious. To show the second statement, we consider the family $\overline{P}$ of $\sigma$-compact zero-dimensional spaces. Clearly, $\overline{P}$ is closed under countable unions and is not c-monotone. Since $\operatorname{in}\overline{P}\,\emptyset = 0$, we have that the subspace $X = [0,1] \cup \{2\}$ of the real line has $\operatorname{in}\overline{P}\,X = 1$. Now, $\operatorname{in}\overline{P}\,[0,1] = 0$ and $\operatorname{in}\overline{P}\,\{2\} = -1$. Hence, $\operatorname{in}\overline{P}$ is not $F_\sigma$-constant.

With proposition 4.5 in mind, we now prove a sum theorem for inductive invariants. The proof is modeled after the proof of the sum theorem of dimension given in [3]. The major difference is the use of theorem 4.1 above.

4.6. THEOREM. *Suppose that $P$ is c-monotone. Then $P$ is $F_\sigma$-constant if and only if $\operatorname{in}P$ is $F_\sigma$-constant.*

Proof. Due to proposition 4.5, we need only prove one implication. Suppose that $P$ is c-monotone and $F_\sigma$-constant. We assume $\operatorname{in}P\,\emptyset = -1$ because $\operatorname{in}P\,\emptyset = 0$ implies $\operatorname{in}P\,X = \operatorname{in}P_0\,X = \dim X + 1$ and the theorem is true for dimension.

We prove by induction the following proposition:

$\Sigma_n$. *If $X$ is the countable union of $F_\sigma$ subsets $X_i$, where $\operatorname{in}P\,X_i \leqslant n$, then $\operatorname{in}P\,X \leqslant n$.*

$\Sigma_{-1}$ is trivial. We deduce $\Sigma_n$ from $\Sigma_{n-1}$, making use of theorem 4.1.

First, we prove for $n \geqslant 0$ that $\Sigma_{n-1}$ implies the following proposition:

$\Delta_n$. *Any space of $\operatorname{in}P \leqslant n$ is the union of a subspace of $\operatorname{in}P \leqslant n-1$ and a subspace of dimension $\leqslant 0$.*

Proof of $\Delta_n$. Let $X$ be a space of $\operatorname{in}P \leqslant n$. Then there is a countable basis $\{U_i\}$ of open sets of $X$ made up of sets whose boundaries $B_i$ have $\operatorname{in}P \leqslant n-1$. (See proposition 3.6) From $\Sigma_{n-1}$ it follows that $B = \bigcup_i B_i$ has $\operatorname{in}P \leqslant n-1$. Now $\dim X \backslash B \leqslant 0$. Hence we have shown that $X$ is the union of a subspace of $\operatorname{in}P \leqslant n-1$ and a subspace of dimension $\leqslant 0$.

We now combine $\Sigma_{n-1}$ and $\Delta_n$ to prove $\Sigma_n$. Suppose that $X$ is the countable union of closed sets $C_i$ with $\operatorname{in}P\,C_i \leqslant n$. We want to show $\operatorname{in}P\,X \leqslant n$. Let $K_1 = C_1$ and $K_i = C_i \backslash \bigcup_{j=1}^{i-1} C_j$ $(i = 2, 3, \ldots)$. Then $X = \bigcup_i K_i$, $K_i \cap K_j = \emptyset$ if $i \neq j$, $K_i$ is an $F_\sigma$ set in $X$ and $\operatorname{in}P\,K_i \leqslant n$ $(i = 1, 2, \ldots)$. The last fact follows from proposition 4.4 and theorem 3.3.

Applying $\Delta_n$ to each $K_i$, we have $K_i = M_i \cup N_i$, where $\operatorname{in}P\,M_i \leqslant n-1$ and $\dim N_i \leqslant 0$. Let $M$ be the union of the $M_i$ and $N$ be the union of the $N_i$. Then $X = M \cup N$. Each $M_i$ is an $F_\sigma$ subset of $M$ and each $N_i$ is an $F_\sigma$ subset of $N$. Hence by $\Sigma_{n-1}$, $\operatorname{in}P\,M \leqslant n-1$. Also, $\dim N \leqslant 0$. By theorem 4.1, we have $\operatorname{in}P\,X \leqslant \operatorname{in}P\,M + \dim N + 1 \leqslant n$.

The theorem 4.6 is now completely proved.

The following corollaries are easily proved.



4.7. COROLLARY. *Suppose that $P$ is c-monotone and $F_\sigma$-constant. Let $X = A \cup B$, where $B$ is closed, $\operatorname{in}P\,A \leqslant n$ and $\operatorname{in}P\,B \leqslant n$. Then $\operatorname{in}P\,X \leqslant n$.*

4.8. COROLLARY. *Suppose that $P$ is c-monotone and $F_\sigma$-constant. If $X \neq \emptyset$ then $\operatorname{in}P\,(X \cup \{\text{point}\}) = \operatorname{in}P\,X$.*

4.9. COROLLARY. *Suppose that $P$ is c-monotone and $F_\sigma$-constant, and $n > \operatorname{in}P\,\emptyset$. If $\operatorname{in}P\,X' \leqslant n$ and $X' \subset X$ then each point of $X$ has arbitrarily small neighborhoods in $X$ whose boundaries $B$ have $\operatorname{in}P\,B \cap X' \leqslant n-1$.*

4.10. COROLLARY. *Suppose that $P$ is c-monotone and $F_\sigma$-constant, and $n > \operatorname{in}P\,\emptyset$. If $\operatorname{in}P\,X \leqslant n$ then $X = X_0 \cup X_1$ where $\operatorname{in}P\,X_0 \leqslant n-1$ and $\dim X_1 \leqslant 0$.*

Next, we prove a decomposition theorem which involves both $\operatorname{in}P$ and dimension.

4.11. THEOREM. *Suppose that $P$ is c-monotone and $F_\sigma$-constant. Let $n$ be such that $\infty > n \geqslant 0$. Then $\operatorname{in}P\,X \leqslant n$ if and only if $X$ is the union of $n+1$ subsets $X_i$ $(i = 0, 1, 2, \ldots, n)$ such that $\operatorname{in}P\,X_0 \leqslant 0$ and $\dim X_i \leqslant 0$ $(i = 1, 2, \ldots, n)$.*

Proof. If $X = \bigcup_{i=0}^{n} X_i$ where $\operatorname{in}P\,X_0 \leqslant 0$ and $\dim X_i \leqslant 0$ $(i = 1, 2, \ldots, n)$, then by theorem 4.1, we have $\operatorname{in}P\,X \leqslant \operatorname{in}P\,X_0 + \dim \bigcup_{i=1}^{n} X_i + 1 \leqslant n$.

Suppose $\operatorname{in}P\,X \leqslant n$. Then by repeated application of corollary 4.10, we have $X = \bigcup_{i=0}^{n} X_i$ where $\operatorname{in}P\,X_0 \leqslant 0$ and $\dim X_i \leqslant 0$ $(i = 1, 2, \ldots, n)$.

Later, we will prove another decomposition theorem (theorem 4.21) which involves the inductive invariant only.

4.12. THEOREM. *Suppose that $P$ is c-monotone and $F_\sigma$-constant, and $\operatorname{in}P\,\emptyset = -1$. Let $\operatorname{in}P\,X = n < \infty$. If $\alpha, \beta \geqslant -1$ and $\alpha + \beta + 1 = n$ then there exist two subsets $A$ and $B$ of $X$ such that $X = A \cup B$, $\operatorname{in}P\,A = \alpha$ and $\operatorname{in}P\,B = \beta$.*

Proof. If $\alpha = -1$ or $\beta = -1$ then we let $A = \emptyset$ and $B = X$ or $A = X$ and $B = \emptyset$. Hence, we assume $\alpha \neq -1 \neq \beta$. With the aid of theorem 4.11, we can find two sets $A'$ and $B'$ such that $X = A' \cup B'$, $\operatorname{in}P\,A' \leqslant \alpha$ and $\dim B' \leqslant \beta$. By theorem 4.1, we have $n = \operatorname{in}P\,X \leqslant \operatorname{in}P\,A' + \dim B' + 1 \leqslant \alpha + \beta + 1 = n$. Hence $\operatorname{in}P\,A' = \alpha$ and $\dim B' = \beta$. Since $n$ is finite, there is a closed subset $C$ of $X$ such that $\operatorname{in}P\,C = \beta$. (See corollary 2.2.) Consequently, c-monotone and corollary 4.7 imply $\operatorname{in}P\,C \cup B' = \beta$ since Theorem 1.8 gives $\operatorname{in}P\,B' \leqslant \dim B'$. Hence we let $A = A'$ and $B = C \cup B'$. The theorem is proved.

We next investigate the following sum theorem for dimension:

$$\dim A \cup B \leqslant \dim A + \dim B + 1 .$$

The proof of this theorem is a straight forward induction and begins with the fact that $\emptyset \cup \emptyset = \emptyset$. That is, the family $\{\emptyset\}$ is additive. Thus, we give the following definition.

4.13. DEFINITION. A family $P$ is called *additive* (c-*additive*) if $X = A \cup B$ ($A$ and $B$ closed in $X$) and $A, B \in P$ imply $X \in P$.

The inequality $\dim A \cup B \leqslant \dim A + \dim B + 1$ is much akin to the subadditive condition $\mu(A \cup B) \leqslant \mu(A) + \mu(B)$ for outer measures $\mu$. The extra term of the first inequality reflects the fact that $\dim X \geqslant -1$ instead of $\mu(X) \geqslant 0$ for outer measures $\mu$. Consequently, we define the following:

4.14. DEFINITION. An extended real-valued function $f$ on the collection of separable metrizable spaces is called *inductively subadditive* (c-*subadditive*) if $X = A \cup B$ ($A$ and $B$ closed in $X$) implies $f(X) \leqslant f(A) + f(B) + 1$.

4.15. Remarks. A family which is additive is c-additive. A family which is $F_\sigma$-constant is c-additive. An inductively subadditive function is also inductively c-subadditive. Of course, the converses of the above statements are false. If $\operatorname{in}P$ is inductively subadditive then $P$ is additive. The corresponding statement is true for inductively c-subadditive and c-additive. The converses are discussed below.

4.16. THEOREM. *Suppose that $P$ is* c-*monotone. Then $P$ is* c-*additive if and only if* $\operatorname{in}P$ *is inductively* c-*subadditive.*

Proof. The theorem follows immediately from the next theorem.

4.17. THEOREM. *Suppose that $P$ ic* c-*monotone and* c-*additive. If*

(1) *$A$ and $B$ are closed in $A \cup B$,*

(2) $\operatorname{in}P\ A \leqslant n$ *and* $\operatorname{in}P\ B \leqslant n$,

(3) $\operatorname{in}P\ A \cap B \leqslant m$,

*then*

$$\operatorname{in}P\ A \cup B \leqslant n+m+1\,.$$

Proof. If $\operatorname{in}P\emptyset = 0$ then $\operatorname{in}P\ X = \dim X + 1$. Hence, the theorem follows for this case since the theorem holds for $\dim X$. Thus we assume $\operatorname{in}P\emptyset = -1$. The proof is by induction on $n$ and $m$. The case $n = \infty$ or $m = \infty$ is trivial.

The proposition is true for $n = m = -1$. Suppose that the proposition is true for $m = -1$ and $n$ ($n \geqslant -1$). Let $\operatorname{in}P\ A \leqslant n+1$, $\operatorname{in}P\ B \leqslant n+1$ and $\operatorname{in}P\ A \cap B = -1$. Since $A$ and $B$ are closed in $A \cup B$, we have $(A \backslash B) \cup (B \backslash A)$ to be open. Hence each point of $(A \backslash B) \cup (B \backslash A)$ has arbitrarily small neighborhoods whose boundaries have $\operatorname{in}P \leqslant n$. (See proposition 3.6.) Consider next a point $x$ of $A \cap B$. By theorem 3.5, we can find an arbitrarily small neighborhood $U$ of $x$ such that the boundary $C$ of $U$ has $\operatorname{in}P\ C \cap A \leqslant n$. Also, we can find a neighborhood $V$ of $x$ such that $V \subset U$ and the boundary $C'$ of $V$ has $\operatorname{in}P\ C' \cap B \leqslant n$. Let $W = V \cup (U \backslash B)$. Then $W$ is a neighborhood of $x$, $W \subset U$ and the boundary $C''$ of $W$ is a closed subset of $M = (C \cap A) \cup (A \cap B) \cup (C' \cap B)$. Now, $\operatorname{in}P\ M \leqslant n$. Since $\operatorname{in}P$ is c-monotone, $\operatorname{in}P\ C'' \leqslant n$. Thus we have shown that each point of $A \cup B$ has arbitrarily small neighbourhoods whose boundaries have $\operatorname{in}P \leqslant n$. That is, $\operatorname{in}P\ A \cup B \leqslant n+1$.



Next, suppose the proposition is true for $m$ ($m \geqslant -1$) and all $n \geqslant m$. Let $\operatorname{in}P\ A \leqslant n$, $\operatorname{in}P\ B \leqslant n$ and $\operatorname{in}P\ A \cap B = m+1$. Again, $(A \backslash B) \cup (B \backslash A)$ is open in $A \cup B$ and hence each point of $(A \backslash B) \cup (B \backslash A)$ has arbitrarily small neighborhoods whose boundaries have $\operatorname{in}P \leqslant n-1 \leqslant n+m+1$. (Note: $n \geqslant m+1$ and $m \geqslant -1$. Consequently, $n-1 \geqslant -1$.) Let $x$ be a point of $A \cap B$. Since $m+1 > -1 = \operatorname{in}P\emptyset$, by theorem 3.5, there are arbitrarily small neighborhoods $U$ of $x$ whose boundaries $C$ have $\operatorname{in}P\ C \cap A \cap B \leqslant m$. Also, $C \cap A$ and $C \cap B$ are closed in $C$ and $\operatorname{in}P\ C \cap A \leqslant n$ and $\operatorname{in}P\ C \cap B \leqslant n$. Hence we have $\operatorname{in}P\ C \leqslant n+m+1$. Thus we have that $\operatorname{in}P\ A \cup B \leqslant n+m+2$. The induction is completed and the theorem follows.

4.18. PROPOSITION. *There exists a* c-*additive family $P$ which is not* c-*monotone such that* $\operatorname{in}P$ *is not inductively* c-*subadditive.*

Proof. See example $\overline{P}$ of proposition 4.5.

4.19. THEOREM. *Suppose that $P$ is* c-*monotone and $F_\sigma$-constant. Then $P$ is additive if and only if* $\operatorname{in}P$ *is inductively subadditive.*

Proof. Only one implication must be proved due to remark 4.15. We need only consider the case where $\operatorname{in}P\emptyset = -1$ since $\operatorname{in}P\emptyset = 0$ implies $\operatorname{in}P\ X = \dim X + 1$. The proof is by induction. We prove the proposition $\operatorname{in}P\ A \leqslant n$ and $\operatorname{in}P\ B \leqslant m$ imply $\operatorname{in}P\ A \cup B \leqslant n+m+1$.

The proposition is trivial if $n = -1 = m$. Hence, assume that the proposition is true for $n = -1$ and $m$ ($m \geqslant -1$). Let $\operatorname{in}P\ A = -1$ and $\operatorname{in}P\ B \leqslant m+1$. By corollary 4.10, $B = B_0 \cup B_1$ where $\operatorname{in}P\ B_0 \leqslant m$ and $\dim B_1 \leqslant 0$. Hence

$$\operatorname{in}P\ A \cup B = \operatorname{in}P\,[(A \cup B_0) \cup B_1] \leqslant m+1\,,$$

by theorem 4.1. Thus we have shown the proposition holds when $n = -1$ or $m = -1$.

Next, suppose the proposition holds for $\operatorname{in}P\ A \leqslant n$ and $\operatorname{in}P\ B \leqslant m-1$ or $\operatorname{in}P\ A \leqslant n-1$ and $\operatorname{in}P\ B \leqslant m$ ($m \geqslant 0$, $n \geqslant 0$). Let $\operatorname{in}P\ A \leqslant n$ and $\operatorname{in}P\ B \leqslant m$. Each point of $A$ has arbitrarily small neighborhoods $U$ whose boundaries $C$ have $\operatorname{in}P\ C \cap A \leqslant n-1$. Also $\operatorname{in}P\ C \cap B \leqslant m$. Hence, $\operatorname{in}P\ C \leqslant n+m$. By a symmetrical argument, each point of $B$ has arbitrarily small neighborhoods whose boundaries have $\operatorname{in}P \leqslant n+m$. Hence $\operatorname{in}P\ A \cup B \leqslant n+m+1$. The induction is now complete.

The theorem follows easily.

4.20. PROPOSITION.

(1) *There exists an additive family $P$ which is not* c*-monotone but $F_\sigma$-constant such that* in$P$ *not inductively subadditive.*

(2) *There exists an additive family $P$ which is* c*-monotone but not $F_\sigma$-constant such that* in$P$ *is not inductively subadditive.*

(3) *There exists an additive family $P$ which is neither* c*-monotone nor $F_\sigma$-constant such that* in$P$ *is not inductively subadditive.*

Proof. (1) The family $\overline{P}$ of proposition 4.5 is an example.

(2) Example $P_1$ of section 7.3 below is an example. For consider the subspace $X = \{x \mid \|x\| < 1\} \cup \{(0,1)\}$ of the cartesian product $R^2$ with the usual norm. It is not difficult to show $\operatorname{in}P_1\, X = 1$, $\operatorname{in}P_1\, \{x \mid \|x\| < 1\} = 0$ and $\operatorname{in}P_1\, \{(0,1)\} = -1$. Hence $\operatorname{in}P_1$ is not inductively subadditive.

(3) Let $P$ be the family of nonempty finite spaces. Then, clearly, $P$ is not c-monotone nor $F_\sigma$-constant. $P$ is additive. The subspace $X = [0,1] \cup \{2\}$ of the real line has $\operatorname{in}P\, X = 1$ since $\operatorname{in}P\, \emptyset = 0$. Also, $\operatorname{in}P\, [0,1] = 0$ and $\operatorname{in}P\, \{2\} = -1$. Hence, in$P$ is not inductively subadditive.

We now establish a decomposition theorem in terms of inductive invariants alone.

4.21. THEOREM. *Suppose that $P$ is* c*-monotone, $F_\sigma$-constant and additive. Furthermore, suppose that $\operatorname{in}P\, \emptyset = -1$ and $\infty > n \geqslant 0$. Then $\operatorname{in}P\, X \leqslant n$ if and only if $X$ is the union of $n+1$ subspaces of $\operatorname{in}P \leqslant 0$.*

Proof. The sufficiency follows from theorem 4.19. The necessity follows from theorems 4.11 and 1.8.

4.22. Remark. In theorem 4.21, it is not possible to let $-1 \leqslant n < \infty$ unless $\operatorname{in}P\, X = \dim X$. Also, we remark that there are families other than $P = \{\emptyset\}$ which satisfy the hypotheses of theorem 4.21. (See section 7.)

In summary, we have the following theorem which isolates some properties found in dimension theory that are due to the inductive nature of the definition and not the particular family $P = \{\emptyset\}$.

4.23. THEOREM. *A family $P$ is $F_\sigma$-constant, additive and*

$$\left\{\begin{array}{l} 1.\ \textit{monotone} \\ 2.\ F_\sigma\textit{-monotone} \\ 3.\ \text{c-}\textit{monotone} \end{array}\right\} \textit{ if and only if } \operatorname{in}P \textit{ is } F_\sigma\textit{-constant},$$

$$\textit{inductively subadditive and } \left\{\begin{array}{l} 1.\ \textit{monotone} \\ 2.\ F_\sigma\textit{-monotone} \\ 3.\ \text{c-}\textit{monotone} \end{array}\right\}.$$

**5. Separation theorems.** For separable metrizable spaces, there are several equivalent definitions for dimension. (See [3], introduction and appendix for a discussion.) These definitions are interrelated by certain separation theorems. Of course, such separation theorems need not be valid for arbitrary families $P$. Hence, it would be of interest to investigate the analogues in the present setting (if any exist) of the various definitions of dimension.



In this section we prove two theorems on separation. (See [3] B) and C), pages 34-35.)

5.1. THEOREM. *Suppose that $P$ is* c*-monotone and $F_\sigma$-constant, and $n > \operatorname{in}P\, \emptyset$. Let $C_1$ and $C_2$ be two disjoint closed subsets of $X$, $A \subset X$ and $\operatorname{in}P\, A \leqslant n$. Then there exists a closed subset $B$ of $X$ which separates $C_1$ and $C_2$ in $X$ and $\operatorname{in}P\, A \cap B \leqslant n-1$.*

Proof. By corollary 4.10, we can find $A_0$ and $A_1$ such that $A = A_0 \cup A_1$, $\operatorname{in}P\, A_0 \leqslant n-1$ and $\dim A_1 \leqslant 0$. By [3] F), page 16, there is a closed subset $B$ of $X$ which separates $C_1$ and $C_2$ in $X$ and $B \cap A_1 = \emptyset$. Clearly, $\operatorname{in}P\, A \cap B = \operatorname{in}P\, A_0 \cap B \leqslant n-1$. Thus the theorem is proved

5.2. THEOREM. *Suppose that $P$ is* c*-monotone and $F_\sigma$-constant, and $\operatorname{in}P\, \emptyset = -1$. Let $\operatorname{in}P\, X \leqslant n-1$ and $C_i$, $C_i'$ be a pair of disjoint closed subsets of $X$ $(i = 1, 2, \ldots, n)$. Then there exist closed sets $B_i$ which separate $C_i$ and $C_i'$ in $X$ $(i = 1, 2, \ldots, n)$ such that $\operatorname{in}P \bigcap_{i=1}^{n} B_i = -1$.*

Proof. This follows from theorem 5.1.

**6. Product theorems.** We next discuss product theorems. The main product theorem in dimension theory is the logarithmic inequality: $\dim A \times B \leqslant \dim A + \dim B$ where $A \neq \emptyset$. Of course, one cannot hope for such an inequality for arbitrary families $P$. But the difficulty lies even deeper, for the logarithmic inequality is not valid when both factors $A$ and $B$ are empty. Hence the fact that a family is closed under product (i.e., $A, B \in P$ implies $A \times B \in P$) does not lead to a logarithmic inequality. We will give two positive results on products in this section which will be useful in section 7 below.

6.1. DEFINITION. Let $Y$ be a closed subset of $X$ and $f$ be a continuous real-valued function on $X$ such that $f \geqslant 0$ and $f^{-1}(0) = Y$. By the triple $[X, Y, f]$ we mean the subset

$$\{(x,t) \mid t = f(x),\ x \in X\} \cup \{(x,t) \mid t = -f(x),\ x \in X\} \quad \text{of } X \times R .$$

It is clear that $[X, Y, f]$ and $[X, Y, g]$ are homeomorphic. Hence we write $[X, Y, f]$ as $[X, Y]$. We call the pair $[X, Y]$ the *double of $X$ modulo $Y$*.

6.2. LEMMA. *Suppose that $P$ is* c*-monotone and* c*-additive. If $Y$ is a closed subset of $X$, then $\operatorname{in}P\, [X, Y] = \operatorname{in}P\, X$.*

Proof. Consider the triple $[X, Y, f]$. Suppose $\operatorname{in}P\, X = -1$. Then $\operatorname{in}P\, \{(x,t) \mid t = f(x),\ x \in X\} = -1$. Hence $\operatorname{in}P\, [X, Y, f] = -1$ since $P$ is c-additive. Suppose that the equality holds whenever $\operatorname{in}P\, X \leqslant n$ $(n \geqslant -1)$ and let $\operatorname{in}P\, X = n+1$. Since $\{(x,t) \mid t = f(x),\ t > 0,\ x \in X\}$ is homeomorphic to $X\setminus Y$, each point $(x, t) \in [X, Y, f]$ with $t \neq 0$ has arbitrarily small neighborhoods in $[X, Y, f]$ whose boundaries have $\operatorname{in} P \leqslant n$. Let $(x, 0) \in [X, Y, f]$. In $X$, $x$ has arbitrarily small neighborhoods $U$ whose boundaries $B$ have $\operatorname{in} P\, B \leqslant n$. Then $[B, B \cap Y, f]$ has $\operatorname{in} P \leqslant n$. Now $[U, U \cap Y, f]$ is a neighborhood of $(x, 0)$ in $[X, Y, f]$ and its boundary is $[B, B \cap Y, f]$. Hence each point of $[X, Y, f]$ has arbitrarily small neighborhoods whose boundaries have $\operatorname{in} P \leqslant n$. That is, $\operatorname{in} P\,[X, Y, f] \leqslant n+1$. The lemma now follows, since $n+1 \leqslant \operatorname{in} P\, X \leqslant \operatorname{in} P\,[X, Y, f]$.

6.3. THEOREM. *Suppose that $P$ is c-monotone and c-additive. Then* $\operatorname{in} P\, A \times R^n \leqslant \operatorname{in} P\, A + n$.

Proof. We consider $A \times R$. By lemma 6.2, each point $(x, t) \in A \times R$ has arbitrarily small neighborhoods whose boundaries have $\operatorname{in} P \leqslant \operatorname{in} P\, A$. Hence $\operatorname{in} P\, A \times R \leqslant \operatorname{in} P\, A + 1$. The theorem now follows from induction.

Suppose that $P$ is a family with $\operatorname{in} P\, \emptyset = -1$. We define $p$ to be an extended real-valued function on the extended integers $n$ $(n \geqslant 0)$ such that

(1) $p(n+1) \geqslant p(n)+1$;

(2) $\operatorname{in} P\, A = -1$ and $\operatorname{in} P\, B \leqslant n$ imply $\operatorname{in} P\, A \times B \leqslant p(n)-1$.

6.4. THEOREM. *Suppose that $P$ is c-monotone and $F_\sigma$-constant, and $\operatorname{in} P\, \emptyset = -1$. Then $\operatorname{in} P\, A \leqslant n$ and $\operatorname{in} P\, B \leqslant m$ $(n \geqslant 0,\ m \geqslant 0)$ imply $\operatorname{in} P\, A \times B \leqslant p(n+m)$.*

Proof. Suppose $n = 0 = m$. If $\operatorname{in} P\, A = -1$ and $\operatorname{in} P\, B \leqslant 0$, then $\operatorname{in} P\, A \times B \leqslant p(0)-1 \leqslant p(0)$. Suppose $\operatorname{in} P\, A = 0$ and $\operatorname{in} P\, B = 0$. Then $A \times B \neq \emptyset$. Let $(a, b) \in A \times B$. Now $a$ has arbitrarily small neighborhoods $U$ whose boundaries $C$ have $\operatorname{in} P\, C = -1$, and $b$ has arbitrarily small neighborhoods $V$ whose boundaries $D$ have $\operatorname{in} P\, D = -1$. The boundary of $U \times V$ is $(\overline{U} \times D) \cup (C \times \overline{V})$ where $\overline{U}$ and $\overline{V}$ are the closures of $U$ and $V$ respectively. Now, $\operatorname{in} P\, \overline{U} \leqslant 0$ and $\operatorname{in} P\, \overline{V} \leqslant 0$. Hence by theorem 4.6, $\operatorname{in} P\,[(\overline{U} \times D) \cup (C \times \overline{V})] \leqslant p(0)-1$. Consequently, each point of $A \times B$ has arbitrarily small neighborhoods whose boundaries have $\operatorname{in} P \leqslant p(0)-1$. This is, $\operatorname{in} P\, A \times B \leqslant p(0)$.

Suppose that the proposition is true for $n = 0$ and $m$ $(m \geqslant 0)$. If $\operatorname{in} P\, A = -1$ and $\operatorname{in} P\, B = m+1$ then clearly $\operatorname{in} P\, A \times B \leqslant p(m+1)-1 \leqslant p(m+1)$. Suppose $\operatorname{in} P\, A = 0$ and $\operatorname{in} P\, B = m+1$. Then $A \times B \neq \emptyset$. Let $(a, b) \in A \times B$. We can find arbitrarily small neighborhoods $U \times V$ of $(a, b)$ such that the boundary $C$ of $U$ has $\operatorname{in} P\, C = -1$ and the boundary $D$ of $V$ has $\operatorname{in} P\, D \leqslant m$. Hence, by theorem 4.6, we have

$$\begin{aligned}\operatorname{in} P\,[(\overline{U} \times D) \cup (C \times \overline{V})] &\leqslant \max\{\operatorname{in} P\, \overline{U} \times D, \operatorname{in} P\, C \times \overline{V}\} \\ &\leqslant \max\{p(m), p(m+1)-1\} = p(m+1)-1\,.\end{aligned}$$

Hence, $\operatorname{in} P\, A \times B \leqslant p(m+1)$.



Next, assume that the proposition holds for $\operatorname{in} P\, A \leqslant n$ and $\operatorname{in} P\, B < m$ or $\operatorname{in} P\, A < n$ and $\operatorname{in} P\, B \leqslant m$ $(n \geqslant 1, m \geqslant 1)$. Let $\operatorname{in} P\, A = n$ and $\operatorname{in} P\, B = m$. Then $A \times B \neq \emptyset$. As in the calculations above, we can show that each point of $A \times B$ has arbitrarily small neighborhoods whose boundaries have $\operatorname{in} P \leqslant p(n+m-1)$. But, $p(m+n-1) \leqslant p(n+m)-1$. Hence we have $\operatorname{in} P\, A \times B \leqslant p(n+m)$. The induction is complete.

The case where $\operatorname{in} P\, A = \infty$ or $\operatorname{in} P\, B = \infty$ is obvious since $p(\infty) = \infty$. Thus the theorem is proved.

**7. Examples.** In this section we give various examples which serve to emphasise the distinction between the various types of families.

7.1. The families $S_n$ $(n = -1, 0, \ldots, \infty)$. Let $S_n$ be the family of spaces $X$ for which $\dim X \leqslant n$.

7.1.1. *If $n < \infty$ then*

$\operatorname{in} S_n\, X = -1$ *if and only if* $\dim X \leqslant n$;

$\operatorname{in} S_n\, X = k$ *if and only if* $\dim X = k+n+1$ $(k \geqslant 0)$;

$\operatorname{in} S_\infty\, X = -1$ *for all* $X$.

7.1.2. *$S_n$ is monotone and $F_\sigma$-constant.*

7.1.3. *$S_n$ is not additive when $-1 < n < \infty$.*

7.1.4. $\operatorname{in} S_n\, A \times B \leqslant \operatorname{in} S_n\, A + \operatorname{in} S_n\, B + n + 1$ $(-1 < n < \infty)$.

Proof. The proof is an immediate consequence of the logarithmic inequality for dimension.

7.1.5. $\operatorname{in} S_n\, R^m = \max(m-n-1, -1)$ $(n < \infty)$. $\operatorname{in} S_\infty\, R^m = -1$.

7.2. The families $\widetilde{T}, \overline{T}, T$ and $T_n$ $(n = 0, 1, \ldots)$. Let $T_n$ be the family of spaces with at most $n$ points.

Let $T$ be the family of finite spaces.

Let $\overline{T}$ be the family of spaces which are countable.

Let $\widetilde{T}$ be the family of spaces which are at most zero dimensional and $\sigma$-compact.

7.2.1. $T_0 \subset T_n \subset T_{n+1} \subset T \subset \overline{T} \subset \widetilde{T} \subset S_0$. Hence,

$$\begin{aligned}\dim X = \operatorname{in} T_0\, X \geqslant \operatorname{in} T_n\, X \geqslant \operatorname{in} T_{n+1}\, X \geqslant \operatorname{in} T\, X \geqslant \operatorname{in} \overline{T}\, X \geqslant \operatorname{in} \widetilde{T}\, X \\ \geqslant \operatorname{in} S_0 X = \max\{\dim X - 1, -1\}\,.\end{aligned}$$

Thus the range of each of the above functions is $\{-1, 0, 1, \ldots, \infty\}$.

7.2.2. *$\overline{T}$, $T$ and $T_n$ are monotone. $\widetilde{T}$ is c-monotone but not monotone.*

7.2.3. *$\widetilde{T}$, $\overline{T}$ and $T_0$ are $F_\sigma$-constant and additive. $T_n$ $(n = 1, 2, \ldots)$ are not additive. $T$ is additive but not $F_\sigma$-constant.*

7.2.4. $\operatorname{in} \overline{T}\, A \times B \leqslant \operatorname{in} \overline{T}\, A + \operatorname{in} \overline{T}\, B + 1$.

Proof. We prove by induction the proposition: $\operatorname{in} \overline{T}\, A = -1$ and $\operatorname{in} \overline{T}\, B \leqslant m$ imply $\operatorname{in} \overline{T}\, A \times B \leqslant p(m)-1$ where $p(m) = m+1$.

Suppose $m = -1$. Then clearly $A\times B$ is countable, and hence $\operatorname{in}\overline{T}\, A\times B = -1$. Assume that the proposition is true for all integers $< m$ $(m \geqslant 0)$. Let $\operatorname{in}\overline{T}\, B = m$. Then each point of $A\times B$ has arbitrarily small neighborhoods $U\times V$ such that its boundary is of the form $(\overline{U}\times D)\cup(\emptyset\times\overline{V})$ where $\operatorname{in}\overline{T}\, D < m$. Hence $\operatorname{in}\overline{T}\, \overline{U}\times D < p(m)-1$. That is $\operatorname{in}\overline{T}\, A\times B \leqslant p(m)-1$. The induction is completed.

By theorem 6.4, $\operatorname{in}\overline{T}\, A \geqslant 0$, $\operatorname{in}\overline{T}\, B\geqslant 0$ imply $\operatorname{in}\overline{T}\, A\times B \leqslant \operatorname{in}\overline{T}\, A + \operatorname{in}\overline{T}\, B + 1$. Checking the formula above for $\operatorname{in}\overline{T}\, A = -1$, we find the inequality true for all $A$ and $B$.

7.2.5. $\operatorname{in}\widetilde{T}\, A\times B \leqslant \operatorname{in}\widetilde{T}\, A + \operatorname{in}\widetilde{T}\, B + 1$.

The proof is similar to that of 7.2.4.

7.2.6. $m-1 \geqslant \operatorname{in}T_n\, R^m \geqslant \operatorname{in}T\, R^m \geqslant \operatorname{in}\overline{T}\, R^m \geqslant \operatorname{in}\widetilde{T}\, R^m \geqslant \operatorname{in}S_0\, R^m = m-1$ $(n\geqslant 2)$.

Proof. We need only prove the first inequality. It is clear that $\operatorname{in}T_n\, R = 0$ for $n\geqslant 2$. Hence the inequality is valid when $m = 1$. Now for $m>1$, $R^m = R\times R^{m-1}$. Hence by theorem 6.3, we have $\operatorname{in}T_n\, R^m \leqslant 0+(m-1) = m-1$.

7.3. The families $P_1$ and $P_2$. Let $P_1$ be the family of compact spaces. Let $P_2$ be the family of $\sigma$-compact spaces. The family $P_1$ has been investigated to some extent in [1] and [2].

7.3.1. $T_0\subset P_1\subset P_2$. If we show the range of $\operatorname{in}P_2\, X$ is $\{-1, 0, 1, \ldots, \infty\}$, then we have that the range of $\operatorname{in}P_1$ is also the same set. We will use totally imperfect spaces to show the existence of an $X_n$ with $\operatorname{in}P_2\, X_n = n$ for each $n$. A space $X$ is *totally imperfect* if every compact subspace $M$ of $X$ is countable.

We prove

THEOREM. *Suppose that*

(i) *$X$ is a cantor manifold,*

(ii) $\dim X\geqslant n$,

(iii) *$X = X_1\cup X_2$ where $X_1$ and $X_2$ are disjoint totally imperfect sets.*

*Then* $\operatorname{in}P_2\, X_i \geqslant n-2$ $(i = 1, 2)$.

Proof. If $n = 1$, then the proposition is obvious. Assume that the proposition is true for all integers $< n$ and let $X$ be a cantor manifold with $\dim X\geqslant n$ and $X_1, X_2$ be disjoint totally imperfect subsets of $X$ whose union is $X$. Let $x\in X_1$ and $U$ be any neighborhood of $x$ whose boundary $B$ disconnects $X$. Then $\dim B\geqslant n-1$ and $B$ is compact. Now, $B$ contains a cantor manifold $X'$ with $\dim X' = \dim B \geqslant n-1$. Since $X'$ is uncountable and compact, we have $X'\cap X_i\neq\emptyset$ $(i = 1, 2)$. Clearly, $X'\cap X_1$ and $X'\cap X_2$ are disjoint totally imperfect sets. Hence $\operatorname{in}P_2\, X'\cap X_1 \geqslant n-3$. Consequently, $\operatorname{in}P_2\, B\cap X_1\geqslant \operatorname{in}P_2\, X'\cap X_1\geqslant n-3$. That is, $\operatorname{in}P_2\, X_1\geqslant n-2$. By symmetry, $\operatorname{in}P_2\, X_2\geqslant n-2$. The theorem is proved.

COROLLARY. *Suppose that $\dim X = n < \infty$ and $X$ is a cantor manifold. If $X = X_1\cup X_2$, where $X_1$ and $X_2$ are disjoint totally imperfect sets, then $n\geqslant \operatorname{in}P_1\, X_1 \geqslant \operatorname{in}P_2\, X_1\geqslant n-2$. Thus the existence problem is solved for $P_1$ and $P_2$.*

We remark that each cantor manifold has a decomposition satisfying condition (iii) of the above theorem (see [7], Bernstein's Theorem, p. 422).

7.3.2. *$P_1$ is c-monotone but not $F_\sigma$-monotone. $P_2$ is $F_\sigma$-monotone but not monotone.*

7.3.3. *$P_1$ is additive but not $F_\sigma$-constant. $P_2$ is additive and $F_\sigma$-constant.*

7.3.4. $\operatorname{in}P_1\, R^n = 0$, $\operatorname{in}P_2\, R^n = -1$.

7.4. Remarks. All the example 7.1-7.3 have some sort of monotone property. The examples $Q_n$ and $R_n$ of sections 2.3 and 2.4 are not monotone in any sense when $n\geqslant 0$. $\operatorname{in}Q_n\, R^m$ and $\operatorname{in}R_n\, R^m$ are easily computed. $Q_n$ is $F_\sigma$-constant for all $n$, whereas $R_n$ is not for all n.

Of course, there are many more examples. We refer the reader to the references for other examples and their applications.

**8. A characterization theorem.** We have already given a characterization of dimension in section 3 in terms of inductive invariants. Now, we will give an axiomatic characterization of the dimension function in the spirit of inductive invariants.

Let us begin with a definition.

8.1. DEFINITION. An extended real-valued function $f$ on the collection of separable metrizable spaces is called *pseudo-inductive* if for each space $X$ and $x\in X$ there are arbitrarily small open neighborhoods $U$ of $x$ such that the boundary $B$ of $U$ has $f(B)\leqslant f(X)-1$. (We agree that $\infty-1 = \infty$.)

An extended real-valued function $f$ on the collection of separable metrizable spaces is called *topological* if $X$ homeomorphic to $Y$ implies $f(X) = f(Y)$.

Clearly, inductive dimension is pseudo-inductive and topological.

Returning to theorem 4.23, we find that monotone, $F_\sigma$-constant and inductively subadditive are desirable conditions in an axiomatic characterization of the inductive dimension function. Finally, from theorem 3.7, we find that $f(\{\emptyset\}) = 0$ is also desirable.

Now, it would be pleasant if the six conditions mentioned above would characterize dimension. But, unfortunately, this is not the case as the following example shows: $f(\emptyset) = -1$, $f(X) = \operatorname{in}\overline{T}\, X+1$, $X\neq\emptyset$, where $\overline{T}$ is as in example 7.2.

To find our last condition, we go to a characterization of dimension given by K. Menger [6] for subspaces of the plane.

8.2. THEOREM (K. Menger). *Let $f$ be a real-valued function on the collection of subspaces of the plane. Then $f$ is the dimension function if and only if $f$ satisfies the following five conditions*:

(a) *$f$ is monotone.*

(b) *$f$ is $F_\sigma$-constant.*

(c) *$f$ is topological.*

(d) *$f$ is compactifiable; that is, every space $X$ is homeomorphic to a subspace of a compact space $Y$ for which $f(X) = f(Y)$.*

(e) *$f$ is normed; that is,* $f(\emptyset) = -1$, $f(\{\text{point}\}) = 0$, $f(\text{line}) = 1$ $f(\text{plane}) = 2$.

*The five conditions are independent.*

In our discussion of inductive invariants, the condition (d) was not considered. It seems that this compactifiability condition is not so much a property of inductive dimension but more a property of dimension defined in terms of finite open covers. The conditions (a), (b), and (c) are definitely related to inductive invariants as we have discovered. The remaining condition (e) can be derived by elementary means from inductive dimension without the use of the covering theorems of dimension. Analysis of inductive invariants shows we only need the condition $f(\{\text{point}\}) = 0$. Consequently, we will take condition (d) as our last condition.

Now we can state our characterization.

8.3. THEOREM. *Suppose that $f$ is an extended real-valued function on the collection of separable metrizable spaces. Then $f$ is the dimension function if and only if $f$ satisfies the following seven conditions*:

(1) *$f$ is topological.*

(2) *$f$ is monotone.*

(3) *$f$ is $F_\sigma$-constant.*

(4) *$f$ is inductively subadditive.*

(5) *$f$ is compactifiable.*

(6) *$f$ is pseudo-inductive.*

(7) $f(\{\emptyset\}) = 0$.

*Furthermore, the seven conditions are independent.*

Proof. Clearly, the dimension function satisfies the conditions (1)-(7). We prove the converse in five parts. Suppose that $f$ satisfies the seven conditions of the theorem.

Part I. $f(X) = -1$ *if and only if* $X = \emptyset$.

Proof. Conditions (6) and (7) imply $f(\emptyset) \leqslant -1$. Using conditions (4) and (7), we have

$$0 = f(\{\emptyset\}) = f(\{\emptyset\} \cup \emptyset) \leqslant f(\{\emptyset\}) + f(\emptyset) + 1 = f(\emptyset) + 1\,.$$



Hence, $f(\emptyset) = -1$. Now, suppose $X \neq \emptyset$. Then by (1), (2), and (7), we have $f(X) \geqslant f(\{\emptyset\}) = 0 > -1$. Thereby, part I is proved.

Part II. $\dim X = 0$ *implies* $f(X) = 0$.

Proof. By conditions (1), (3), and (7) we have that the set of rational numbers $Q$ has $f(Q) = 0$. Conditions (5) and (2) then imply that there is a nonempty, compact dense-in-itself space $X'$ with $f(X') = 0$. Let $X$ be a zero-dimensional space. Then $X$ can be embedded in $X'$. From (1), (2), and part I, we have $0 = f(X') \geqslant f(X) \geqslant 0$. Thus, part II is proved.

Part III. *For each extended integer $n$ ($n \geqslant -1$), we have*

$$n \geqslant \dim X \quad \textit{implies} \quad n \geqslant f(X)\,.$$

Proof. Suppose $\dim X \leqslant n < \infty$. Then there is a decomposition $X = \bigcup_{i=0}^{n} X_i$, where $\dim X_i \leqslant 0$ $(i = 0, 1, \ldots, n)$ ([3], Theorem III3). By (4) and part II, we have $f(X) \leqslant \sum_{i=0}^{n} f(X_i) + n = n$. Part III is now proved.

Part IV. *For each extended integer $n$ ($n \geqslant -1$), we have*

$$n \geqslant f(X) \quad \textit{implies} \quad n \geqslant \dim X\,.$$

Proof. The proposition is true for $n = -1$ by part I and condition (2). Suppose that the proposition is true for $n$ ($n < \infty$) and let $f(X) \leqslant n+1$. By (6), each point of $X$ has arbitrarily small open neighborhoods $U$ whose boundaries $B$ have $f(B) \leqslant f(X) - 1 \leqslant n$. Hence $\dim B \leqslant n$. Thus we have shown that $\dim X \leqslant n+1$. The induction is completed and part IV now follows.

Part V. $f(X) = \dim X$ *for all* $X$.

Proof. This follows from parts III and IV.

The proof of the converse is now completed. We prove the independence of the seven conditions in the next subsection.

8.4. Independence of conditions (1)-(7). In each of the following examples, we negate exactly one of the seven conditions. The verification in each case is straight forward.

8.4.1. Negation of condition (1). Let $f$ be defined as follows: $f(\emptyset) = -1$; $f(\{\emptyset\}) = 0$; $f(X) = \operatorname{in} P_0 X$ if and only if $X \neq \emptyset$ or $X \neq \{\emptyset\}$.

8.4.2. Negation of condition (2). Let $f$ be defined as follows: $f(\emptyset) = -1$; $f(X) = \operatorname{in} \tilde{T}\, X + 1$ if and only if $X \neq \emptyset$.

8.4.3. Negation of condition (3). Let $f$ be defined as follows: $f(X) = \dim X$ if and only if $X$ is finite; $f(X) = \operatorname{in} P_0 X$ if and only if $X$ is infinite.

8.4.4. Negation of condition (4). Let $f$ be defined as follows: $f(X) = \dim X$ if and only if $\dim X \leqslant 0$; $f(X) = \mathrm{in} P_0 X$ if and only if $\dim X > 0$.

8.4.5. Negation of condition (5). Let $f$ be defined as follows: $f(\emptyset) = -1$; $f(X) = \mathrm{in} \overline{T}\, X + 1$ if and only if $X \neq \emptyset$.

8.4.6. Negation of condition (6). Let $f$ be defined as follows: $f(\emptyset) = -1$; $f(X) = \dim X / \mathrm{in} P_0 X$ if and only if $-1 < \dim X < \infty$; $f(X) = 1$ if and only if $\dim X = \infty$.

8.4.7. Negation of condition (7). Let $f(X) = \mathrm{in} P_0 X$ for all $X$.

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# On some numerical constants associated with abstract algebras


by

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**1. Introduction.** For the terminology and notation used here, see [5]. In particular, for a given abstract algebra $\mathfrak{A} = (A; \mathbf{F})$, where $A$ is a non-void set and $\mathbf{F}$ is a class of fundamental operations, by $\mathbf{A}(\mathfrak{A})$ or $\mathbf{A}(\mathbf{F})$ we shall denote the class of all algebraic operations, i.e. the smallest class, closed under the composition, containing all fundamental operations and all trivial operations $e_k^{(n)}$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ defined by the formula

$$e_k^{(n)}(x_1, x_2, \ldots, x_n) = x_k .$$

The subclass of all $n$-ary algebraic operations in $\mathfrak{A}$ will be denoted by $\mathbf{A}^{(n)}(\mathfrak{A})$ or $\mathbf{A}^{(n)}(\mathbf{F})$ $(n \geqslant 0)$. Two algebras $(A; \mathbf{F}_1)$ and $(A; \mathbf{F}_2)$ having the same class of algebraic operations will be treated here as identical. If a non-void subset $B$ of $A$ is closed with respect to $\mathbf{F}$, then the algebra $(B; \mathbf{F})$ is called a *subalgebra* of the algebra $(A; \mathbf{F})$. An algebra $(A; \mathbf{G})$ is called a *reduct* of the algebra $(A; \mathbf{F})$ if $\mathbf{A}(\mathbf{G}) \subset \mathbf{A}(\mathbf{F})$. Further, by $\mathfrak{A}^{(n)}$ we shall denote the algebra of all $n$-ary algebraic operations in the algebra $\mathfrak{A}$.

In his study of certain numerical constants associated with abstract algebras, E. Marczewski introduced the order of enlargeability (called by him the *degree of extendability*) of abstract algebras (see [7], p. 182). We recall his definition of this concept. Let $\mathfrak{A} = (A; \mathbf{F})$. We say that a non-negative integer $n$ belongs to the set $N(\mathfrak{A})$ if for every family $\mathbf{G}$ of operations in the set $A$ the equation $\mathbf{A}^{(n)}(\mathbf{F}) = \mathbf{A}^{(n)}(\mathbf{G})$ implies the inclusion $\mathbf{A}(\mathbf{F}) \supset \mathbf{A}(\mathbf{G})$. In other words, $n \in N(\mathfrak{A})$ if and only if for every family $\mathbf{G}$ satisfying the condition $\mathbf{A}^{(n)}(\mathbf{F}) = \mathbf{A}^{(n)}(\mathbf{G})$ the algebra $(A; \mathbf{G})$ is a reduct of the algebra $(A; \mathbf{F})$. Further, let $\varepsilon(\mathfrak{A})$ be the smallest integer belonging to $N(\mathfrak{A})$ if the set $N(\mathfrak{A})$ is non-void and let $\varepsilon(\mathfrak{A}) = \infty$ in the opposite case. The quantity $\varepsilon(\mathfrak{A})$ is called the *order of enlargeability* of the algebra $\mathfrak{A}$. It is evident that

(i) *For an algebra* $\mathfrak{A} = (A; \mathbf{F})$ *the inequality* $\varepsilon(\mathfrak{A}) > k$ *holds if and only if there exists an operation* $f$ *in* $A$ *such that* $\mathbf{A}^{(k)}(\mathbf{F}) = \mathbf{A}^{(k)}(\mathbf{F} \cup \{f\})$ *and* $f \notin \mathbf{A}(\mathbf{F})$.