On functions defined on Cartesian products

by

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It is generally known that, under certain assumptions about the spaces $\{X_s\}_{s\in S}$ and about the space Y, every continuous function $f\colon \underset{s\in S}{P} X_s \to Y$ depends only on a countable number of coordinates, i.e. that there exists a factorization $f = f'p_{S'}$, where $S' \subset S$ is a countable set and f' is a continuous function from $\underset{s\in S'}{P} X_s$ to Y [1].

The following table contains all theorems of this type which are known to the author:

	Assumptions on X_s's	Assumptions on Y	Authors
1	compact	metrizable	Y. Mibu [16]
2	T_2, countable base	T_2, diagonal of $Y\times Y$ is a G_δ-set	S. Mazur [15]
3	countable base	metrizable	H. H. Corson and J. R. Isbell [3], [4]
4	property (K)	metrizable	K. A. Ross and A. H. Stone [19]
5	separable	T_2, points are G_δ-sets	A. M. Gleason cited in [14]
6	caliber $\aleph_1$	T_2, points are G_δ-sets	A. Miščenko [18]

This table needs some explanations.

Ad 1. The theorem is an easy consequence of the Stone-Weierstrass theorem; it was rediscovered and cited as a well-known fact without the name of the author (see [2] and [7]).

Ad 2. The paper [15] is devoted to sequentially continuous mappings (see paragraph 3 below) and the theorem is not stated in it explicitly, but follows easily from theorem II. Let us remark that in theo-

(1) For any $S' \subset S$ the symbol $p_{S'}$ denotes the projection of the Cartesian product $\underset{s\in S}{P} X_s$ onto the partial product $\underset{s\in S'}{P} X_s$.

rem II of [15], in contradistinction to theorem III, no assumption on the power of the set of indices S is made (cf. theorem 3 below).

Ad 3. The common opinion (see [3], [4] or [19]) that the theorem formulated in the table under 2 follows from the results of S. Mazur only under certain assumptions on the power of S made H. H. Corson and J. R. Isbell re-prove it (in a completely different manner and without the assumption that X_s's are T_2-spaces; let us note that in the proof of S. Mazur the assumption that X_s's are T_1-spaces suffices).

Ad 4. A space has property (K) if every uncountable family of open sets contains an uncountable subfamily in which every two sets have a non-void intersection. Every separable space (i.e. every space containing a countable dense subset) has property (K).

Ad 6. A space is of caliber $\aleph_1$ if for every family $\{U_s\}_{s\in S}$, where $\overline{\overline{S}}=\aleph_1$ and $U_s\neq 0$ for $s\in S$, of open sets there exists a set $S_0\subset S$ such that $\overline{\overline{S_0}}=\aleph_1$ and $\bigcap_{s\in S_0} U_s\neq 0$. Every space which is of caliber $\aleph_1$ has property (K). A. Miščenko has also proved that if $\overline{\overline{S}}>\aleph_0$ and one of the spaces $\{X_s\}_{s\in S}$, which all have at least two points, is not of caliber $\aleph_1$, then there exists a Hausdorff space Y in which every point is a G_δ-set and a continuous function $f\colon \underset{s\in S}{P} X_s\to Y$ which depends on a uncountable number of coordinates.

Let us also remark that without any essential change in the proofs all the theorems from our table can be generalized, by—roughly speaking—replacing in the assumptions the number $\aleph_0$ by an arbitrary cardinal number $\mathfrak{m}\geqslant\aleph_0$; in this case the set S from the conclusion is of power $\leqslant\mathfrak{m}$.

In [15] and [3] there were considered functions defined on Σ-products of spaces $\{X_s\}_{s\in S}$, i.e. on the sets of the form

$$\Sigma(a)=\Big\{\{x_s\}\in\underset{s\in S}{P} X_s\colon\ \overline{\overline{\{s\colon x_s\neq a_s\}}}\leqslant\aleph_0\Big\},$$

where $a=\{a_s\}\in\underset{s\in S}{P} X_s$, and the corresponding theorems were also proved for such functions. It is not difficult to see that this is also true for the situations listed in the table under 4 and 5. The fact that every continuous function from the Σ-product of compact spaces to a metrizable space depends only on a countable number of coordinates follows from the Mibu theorem and a theorem of I. Glicksberg [12] (see Corollary 1 below). The theorems for functions defined on Σ-products are evidently more general and give also some information on the Čech-Stone compactification or the Hewitt real-compactification of Σ-products (see [11] for the definitions).

In the first part of this paper we prove a theorem which completes our table and is a common generalization of the theorems of Mibu and Mazur. In the second part we examine the possibility of reinforcing the Gleason theorem by assuming that only points of a set Y_0 dense in Y are G_δ-sets. The last part is devoted to sequentially continuous mappings and contains variants of the results of S. Mazur [15]. The proofs of all theorems of this paper are obtained by some modifications of reasonings of A. Gleason and S. Mazur.

1. The following lemma was proved by N. Šanin ([20], p. 24) and S. Mazur ([15], lemma (vii)). The lemma follows also from Theorem I (ii) of [8], proved in an elegant manner by E. Michael in [17] (cf. lemma 1 in [6]).

LEMMA 1. *For every family $\{S_t\}_{t\in T}$ of finite sets, where $\overline{\overline{T}}>\aleph_0$, there exist a finite set Z and a subset T_0 of T such that $\overline{\overline{T_0}}>\aleph_0$ and $S_t\cap S_{t'}=Z$ for distinct $t,t'\in T_0$.*

From Lemma 1 we obtain the following variant of lemma (viii) from [15]; the set $\Sigma_0(a)$ occurring in its formulation is defined by the formula.

$$\Sigma_0(a)=\Big\{\{x_s\}\in\underset{s\in S}{P} X_s\colon\ \overline{\overline{\{s\colon x_s\neq a_s\}}}<\aleph_0\Big\},$$

where $a=\{a_s\}\in\underset{s\in S}{P} X_s$.

LEMMA 2. *If $\{X_s\}_{s\in S}$ is a family of T_1-spaces such that every finite product $X_{s_1}\times X_{s_2}\times\ldots\times X_{s_k}$, where $s_1,s_2,\ldots,s_k\in S$, has the Lindelöf property, then for any $a\in\underset{s\in S}{P} X_s$ and every family $\{x_t\}_{t\in T}$, where $\overline{\overline{T}}>\aleph_0$, of elements of $\Sigma_0(a)$, there exists a point $x_0\in\Sigma_0(a)$ whose every neighbourhood contains the point x_t for infinitely many $t\in T$.*

Proof. For every $t\in T$ thes et $S_t=\{s\in S\colon p_s(x_t)\neq a_s\}$ is finite. By Lemma 1, there exist a finite set $Z\subset S$ and a set $T_0\subset T$ of power $>\aleph_0$ such that $S_t\cap S_{t'}=Z$ for distinct $t,t'\in T_0$. If $Z=0$, then the point $x_0=a$ satisfies the conclusion of the lemma. Thus we can suppose that $Z=\{s_1,s_2,\ldots,s_k\}$. Since the space $X_Z=X_{s_1}\times X_{s_2}\times\ldots\times X_{s_k}$ has the Lindelöf property, there exists a point $x_0'\in X_Z$ whose every neighbourhood contains the point $p_Z(x_t)$ for infinitely many $t\in T_0$. It is easy to prove, using the fact that $S_t\cap S_{t'}=Z$ for distinct $t,t'\in T_0$, that the point $x_0\in\Sigma_0(a)$, defined by the conditions $p_{S\setminus Z}(x_0)=p_{S\setminus Z}(a)$ and $p_Z(x_0)=x_0'$, satisfies the conclusion of the lemma.

Remark 1. Let us note that if X_s's satisfy the first axiom of countability, then there exists a sequence $t_1,t_2,\ldots$ of distinct elements of T such that the sequence $x_{t_1},x_{t_2},\ldots$ converges to x_0 (see paragraph 3); in fact it is an arbitrary sequence $t_1,t_2,\ldots$ of distinct elements of T such that the sequence $p_Z(x_{t_1}),p_Z(x_{t_2}),\ldots$ converges to x_0'. If X_s's have countable bases, then one can suppose moreover that $x_0'=p_Z(x_{t_0})$ for some $t_0\in T$.

Remark 2. It is well known that the assumption that the spaces $\{X_s\}_{s\in S}$ have the Lindelöf property does not imply that finite products $X_{s_1}\times X_{s_2}\times\ldots\times X_{s_k}$ also have this property. Let us note that finite products have the Lindelöf property if for every $s\in S$ the space X_s is compact or has a countable base and if X_s has, for every $s\in S$, the Lindelöf property and is complete in the sense of Čech (i.e. is a Tychonoff space which is a G_δ-set in some of its compactifications). The last fact follows from paper [10] of Z. Frolík. Indeed, it is enough to notice that if the space P in the proof of theorem 3 from [10] has the Lindelöf property, then one can suppose that the coverings $\mathfrak{B}_n$ are countable and it easily follows that the space Q contains a countable dense subset. Using the fact that the Lindelöf property is invariant (in both directions) under perfect mapping (see [13], theorem 2.2) we conclude that a space has the Lindelöf property and is complete in the sense of Čech if and only if it can be transformed by a perfect mapping onto a separable and complete metric space (cf. theorem 3 in [10]). Lastly, the invariance of the class of spaces which have the Lindelöf property and are complete in the sense of Čech under countable Cartesian multiplication follows from lemma 4 of [10].

THEOREM 1. *If $\{X_s\}_{s\in S}$ is a family of T_1-spaces such that every finite product $X_{s_1}\times X_{s_2}\times\ldots\times X_{s_k}$, where $s_1, s_2, \ldots, s_k\in S$, has the Lindelöf property and Y is a Hausdorff space such that the diagonal of $Y\times Y$ is a G_δ-set, then for any $a\in\underset{s\in S}{P}X_s$ and a continuous function $f\colon \Sigma(a)\to Y$ there exists a factorization $f=f'(p_{S'}|\Sigma(a))$, where $S'\subset S$ is a countable set and f' is a continuous function from $\underset{s\in S'}{P}X_s$ to Y. In particular f is in this case extendable to the whole product $\underset{s\in S}{P}X_s$.*

Proof. We shall show that the set S' of those $s\in S$ for which there exist points $x, x'\in\Sigma_0(a)$ such that

$$f(x)\neq f(x')\quad\text{and}\quad p_{S\setminus\{s\}}(x)=p_{S\setminus\{s\}}(x'),$$

is countable. Let us suppose that $\overline{\overline{S'}}>\aleph_0$ and let points $x(s), x'(s)\in\Sigma_0(a)$ satisfy the conditions

$$(1)\qquad f\big(x(s)\big)\neq f\big(x'(s)\big)\quad\text{and}\quad p_{S\setminus\{s\}}\big(x(s)\big)=p_{S\setminus\{s\}}\big(x'(s)\big)$$

for $s\in S'$. Since the diagonal $\Delta\subset Y\times Y$ is a G_δ-set, there exist an uncountable set $S''\subset S'$ and a closed set $F\subset Y\times Y\setminus\Delta$ such that $\big(f(x(s)), f(x'(s))\big)\in F$ for $s\in S''$. From Lemma 2 if follows that there exists a point $x_0\in\Sigma_0(a)$ whose any neighbourhood W contains the point $x(s)$ for all s in an infinite subset $S''(W)$ of S''. From the second part of (1) we conclude that W contains also the point $x'(s)$ for infinitely many $s\in S''(W)$.

But the counterimage $f^{-1}(V)$, where V is an arbitrary neighbourhood of $f(x_0)\in Y$ which satisfies $V\times V\subset Y\times Y\setminus F$, does not contain the points $x(s)$ and $x'(s)$ simultaneously for any $s\in S''$. Hence $\overline{\overline{S'}}\leqslant\aleph_0$.

Applying induction we conclude that if for $x, x'\in\Sigma_0(a)$ there exists a finite set $S''\subset S\setminus S'$ such that $p_{S\setminus S''}(x)=p_{S\setminus S''}(x')$, then $f(x)=f(x')$. From the continuity of f and the fact that $\Sigma_0(a)$ is dense in $\Sigma(a)$ it follows that if for $x, x'\in\Sigma(a)$ we have $p_{S'}(x)=p_{S'}(x')$, then $f(x)=f(x')$. This implies that there exists a function $f'\colon\underset{s\in S'}{P}X_s\to Y$ which satisfies $f=f'(p_{S'}|\Sigma(a))$. It is easily seen that f' is continuous.

From Theorem 1 we obtain, as corollaries, theorem 2 of I. Glicksberg [12] and a reinforcement of theorem 2 of H. H. Corson [3]:

COROLLARY 1. *The Cartesian product $\underset{s\in S}{P}X_s$ of a family of compact spaces is the Čech-Stone compactification of $\Sigma(a)$ for every $a\in\underset{s\in S}{P}X_s$.*

COROLLARY 2. *The Cartesian product $\underset{s\in S}{P}X_s$ of a family of Tychonoff spaces $\{X_s\}_{s\in S}$ such that every finite product $X_{s_1}\times X_{s_2}\times\ldots\times X_{s_k}$, where $s_1, s_2, \ldots, s_k\in S$, has the Lindelöf property, is the Hewitt real-compactification of $\Sigma(a)$ for every $a\in\underset{s\in S}{P}X_s$.*

COROLLARY 3. *Every metric space which is a continuous image of a Σ-product of a family $\{X_s\}_{s\in S}$ of T_1-spaces, such that every finite product $X_{s_1}\times X_{s_2}\times\ldots\times X_{s_k}$, where $s_1, s_2, \ldots, s_k\in S$, has the Lindelöf property, is separable.*

Indeed, such a space is a continuous image of a countable Cartesian product $X_{s_1}\times X_{s_2}\times\ldots\times X_{s_k}\ldots$ which contains a dense subspace with the Lindelöf property.

2. The Gleason theorem implies the well-known theorem (see [9]) on continuous images of Cartesian products of compact spaces:

ESENIN-VOLPIN THEOREM. *Let $\{X_s\}_{s\in S}$ be a family of compact spaces with countable bases, and let $f\colon\underset{s\in S}{P}X_s\to Y$ be a continuous function onto a Hausdorff space Y. If every point of Y is a G_δ-set, then Y has a countable base.*

The last theorem can be reinforced (see [5]) by assuming that only points of a set Y_0 dense in Y are G_δ-sets.

We shall show that the Gleason theorem cannot be reinforced in this manner (which answers a question raised by A. Miščenko).

We begin with a generalization of the "double circumference" of P. Alexandroff (example A_2 in [1], p. 13). Let X be an arbitrary topological space and for $x\in X$ let $V(x)$ denote the family of all open sets containing x. Consider the product $A(X) = D\times X$, where $D = \{0, 1\}$, and put for $A\subset X$ and $x\in X$

$$A^{(i)} = \{i\}\times A, \quad x^{(i)} = (i, x), \quad \text{where} \quad i = 0, 1.$$

Assuming the class $\{\mathfrak{B}(z)\}_{z\in A(X)}$, where $\mathfrak{B}(x^{(0)}) = \{\{x^{(0)}\}\}$ and $\mathfrak{B}(x^{(1)}) = \{(V^{(0)}\setminus\{x^{(0)}\})\cup V^{(1)}\}_{V\in\mathcal{V}(x)}$, to be a system of neighbourhoods, we introduce a topology in $A(X)$. It is easy to see that $X^{(1)}$ is homeomorphic to X and that $X^{(0)}$ is composed of isolated points. A straighforward argument shows that $A(X)$ is compact for compact X. In this case $A^{(0)}\cup X^{(1)}$ is compact for any $A\subset X$. If A is dense in X and X is dense in itself, the set $A^{(0)}$ is dense in $A^{(0)}\cup X^{(1)}$.

Now let $X = \underset{t\in T}{P} D_t$, where $\overline{\overline{T}} = 2^{\aleph_0}$ and $D_t = \{0, 1\}$ with discrete topology, be the Cantor cube of the weight $2^{\aleph_0}$, and let $Q = \{x_1, x_2, \ldots\}$ be a countable dense subset of X (see [19]). Consider the space $Y = Q^{(0)}\cup X^{(1)}\subset A(X)$, the Cartesian product $N\times X$, where N is the set of all non-negative integers with discrete topology, and the function $f\colon N\times X\to Y$ defined by

$$f(0, x) = x^{(1)}, \quad f(i, x) = x_i^{(0)} \quad \text{for} \quad x\in X \text{ and } i = 1, 2, \ldots$$

It is easy to verify that f is continuous and there exists no factorization $f = pf'$, where p is the projection onto some proper partial product of $N\times\underset{t\in T}{P} D_t$. Since factors of the product $N\times\underset{t\in T}{P} D_t$ are separable (what is more, they have countable bases) and points of the set $Q^{(0)}$ dense in Y are open sets, this example shows that the Gleason theorem cannot be reinforced in the manner mentioned above. We shall now prove that such a reinforcement is possible if Y_0 is a dense set of a special kind and factors of the product have countable bases.

THEOREM 2. *Let f be a continuous function from the Cartesian product $\underset{s\in S}{P} X_s$, of spaces with countable bases to a Hausdorff space Y. If, for some Q dense in $\underset{s\in S}{P} X_s$, every point of $f(Q)$ is a G_δ-set in Y then there exists a factorization $f = f'p_{S'}$, where $S'\subset S$ is countable and f' is a continuous function from $\underset{s\in S'}{P} X_s$ to Y.*

Proof. For every $q\in Q\subset X = \underset{s\in S}{P} X_s$ there exists a countable set $S(q)\subset S$ such that

(2) $\quad$ if $p_{S(q)}(x) = p_{S(q)}(q)$ for $x\in X$, then $f(x) = f(q)$.

We shall define, by means of induction, a sequence $Q_0, Q_1, \ldots$ of countable subsets of Q, which satisfy the condition

(3) $\quad$ the set $p_{S_i}(Q_{i+1})$ is dense in $p_{S_i}(Q)$,

where

(4) $$S_i = \bigcup_{j=0}^{i}\bigcup_{q\in Q_j} S(q).$$

Let $Q_0 = \{q_0\}$ where q_0 is an arbitrary point of Q (the case $Q = 0$ being trivial) and let us suppose that $Q_0, Q_1, \ldots, Q_k$ are defined and satisfy (3). The set S_k defined by (4) is countable and hence the product $\underset{s\in S_k}{P} X_s = p_{S_k}(X)$ and also its subset $p_{S_k}(Q)$ have countable bases; it follows that there exists a countable set $Q_{k+1}\subset Q$ such that (3) holds for $i = k$.

We shall show that the countable set $S' = \bigcup_{i=0}^{\infty} S_i$ satisfies the conclusion of the theorem. From the openness of $p_{S'}$ it follows that every f' satisfying $f = f'p_{S'}$ is continuous, whence it suffices to show that for any $x_1, x_2\in X$ if $p_{S'}(x_1) = p_{S'}(x_2)$, then $f(x_1) = f(x_2)$. Let us suppose that $f(x_1)\neq f(x_2)$ and let $U_1, U_2\subset Y$ be disjoint neighbourhood of $f(x_1)$ and $f(x_2)$. The set $f^{-1}(U_i)$ is a neighbourhood of x_i and thus contains its neighbourhood of the form $\underset{s\in S}{P} U_s^i$, where U_s^i is open in X_s and $U_s^i = X_s$ for all but a finite number of $s\in S$. Moreover, we can suppose that $U_s^1 = U_s^2$ for $s\in S'$. By (3) there exists a point $q = \{q_s\}\in\bigcup_{i=0}^{\infty} Q_i$ such that $q_s\in U_s^1 = U_s^2$ for $s\in S'$. By (2), (4) and from the definition of S' it follows that for $z_1, z_2\in X$, defined by the conditions $p_s(z_i) = q_s$ for $s\in S'$, $p_s(z_i) = p_s(x_i)$ for $s\in S\setminus S'$ and $i = 1, 2$, we have $f(z_1) = f(q) = f(z_2)$. Since $z_i\in\underset{s\in S}{P} U_s^i$, we infer that $U_1\cap U_2\neq 0$, which is impossible.

Let us remark that an analogous theorem for functions defined on open subsets of products also holds (the proof is similar to the proof of the corresponding theorem of Gleason ([14], p. 132) and that Theorem 2 can easily be generalized by supposing that X_s's have bases of power $\leqslant\mathfrak{m}\geqslant\aleph_0$ and that points of $f(Q)$ are intersections of $\leqslant\mathfrak{m}$ open sets; of course in this case $\overline{\overline{S}}\leqslant\mathfrak{m}$.

It seems to the author that Theorem 2 does not hold under the assumption of separability of X_s's, but he has not been able to construct a corresponding example. M. Karłowicz has remarked in this context that Theorem 2 is still valid if we suppose that X_s's have countable dense subsets at the points of which there exist countable bases. Indeed, in this case the product $p_{S_k}(X)$ has a countable dense subset $\{x_1, x_2, \ldots\}$ such that every x_i has a countable base $\mathfrak{B}_i$ of neighbourhoods. Choosing a point of $p_{S_k}(Q)$ in every element of $\mathfrak{B}_i$ for $i = 1, 2, \ldots$ we obtain a countable subset of $p_{S_k}(Q)$ dense in $p_{S_k}(X)$, and the existence of Q_{k+1} follows.

3. By applying Gleason's reasoning in the proof of Mazur's theorems II and III in [15] we obtain some interesting modifications of those theorems. Before formulating them we give some definitions.

For $x, x' \in \underset{s\in S}{P} X_s$ and $T \subset S$ we denote by $x(T, x')$ the point of $\underset{s\in S}{P} X_s$ defined by the conditions

$$p_T\big(x(T, x')\big) = p_T(x), \qquad p_{S\setminus T}\big(x(T, x')\big) = p_{S\setminus T}(x').$$

The set $A \subset \underset{s\in S}{P} X_s$ will be called *invariant under composition* if for any $x, x' \in A$ and $T \subset S$ we have $x(T, x') \in A$. For every $a = \{a_s\} \in \underset{s\in S}{P} X_s$ an example of a set invariant under composition is given by the sets $\Sigma(a)$ and $\Sigma_0(a)$.

A sequence $\{x_n\}$ of points of a topological space X is called *convergent* to $x \in X$ if every neighbourhood of x contains all points x_n perhaps with the exception of a finite number; in this case we write $x \in \lim x_n$. The set $F \subset X$ is called *sequentially closed* if $x_n \in F$ and $x \in \lim x_n$ implies $x \in F$. Every countable union of sequentially closed sets is called a *sequentially* F_σ*-set*, and its complement is called a *sequentially* G_δ*-set*. It is easy to see that every closed (F_σ or G_δ) set in X is sequentially closed (F_σ or G_δ). The function f from a topological space X to a topological space Y is called *sequentially continuous* if $x \in \lim x_n$ in X implies $f(x) \in \lim f(x_n)$ in Y; in a space X which satisfies the first axiom of countability sequential continuity implies continuity. In a Hausdorff space any sequence converges to at most one point; we write in this case $x = \lim x_n$ instead of $x \in \lim x_n$.

THEOREM 3. *If A is a subset of the Cartesian product $\underset{s\in S}{P} X_s$ of T_1-spaces with countable bases, invariant under composition and such that $A \subset \Sigma(a)$ for some $a \in A$, and $f\colon A \to Y$ is a sequentially continuous function from A to a Hausdorff space Y in which every point is a sequentially G_δ-set, then there exists a factorization $f = f'(p_{S'}|A)$ where $S' \subset S$ is countable and f' is a continuous function from $p_{S'}(A)$ to Y.*

Moreover, *if $A = \Sigma(a)$ for some $a \in A$, then it suffices to suppose that X_s's are T_1, separable, satisfy the first axiom of countability, and every finite product $X_{s_1} \times X_{s_2} \times \ldots \times X_{s_k}$, where $s_1, s_2, \ldots, s_k \in S$, has the Lindelöf property* (2).

(2) Theorems II and III from [15] differ from the first parts of Theorems 3 and 4 only in the fact that the diagonal of $Y \times Y$ is supposed to be a sequentially G_δ-set and A is supposed to be invariant under composition of any $x \in A$ with $a \in A$. Let us also remark that if in theorem II we have $A = \Sigma(a)$ for some $a \in A$ and in theorem III we have $\Sigma(a) \subset A$ for some $a \in A$, then it suffices to suppose that X_s's satisfy the first axiom of countability and every finite product $X_{s_1} \times X_{s_2} \times \ldots \times X_{s_k}$ has the Lindelöf property.

Proof. First, we prove that for every $y \in Y$ there exists a countable set $S_0(y) \subset S$ such that if $x, x' \in Q = A \cap \Sigma_0(a)$, $f(x) = y$ and the set $\{s\colon p_s(x) \neq p_s(x')\}$ has at most one element and is disjoint with $S_0(y)$ then $f(x') = y$.

Suppose the contrary. Thus there exists an uncountable set $S_1 \subset S$ and for every $s \in S_1$ we have points $x(s), x'(s) \in Q$ such that

(5) $$f\big(x(s)\big) = y \neq f\big(x'(s)\big) \quad \text{and} \quad p_{S\setminus\{s\}}\big(x(s)\big) = p_{S\setminus\{s\}}\big(x'(s)\big).$$

Since $Y\setminus\{y\}$ is a sequentially F_σ-set, there exist a sequentially closed set $F \subset Y\setminus\{y\}$ and an uncountable set $S_2 \subset S_1$ such that $f\big(x'(s)\big) \in F$ for $s \in S_2$. By remark 1, there exists a sequence $s_1, s_2, \ldots$ of distinct elements of S_2 such that $x_0 \in A \cap \lim x(s_n)$. Since $s_n \neq s_m$ for $n \neq m$, from the second part of (5) it follows that $x_0 \in \lim x'(s_n)$ and we have $\lim f\big(x'(s_n)\big) = f(x_0) = \lim f\big(x(s_n)\big) = y$, which is impossible.

From the fact just proved it follows that for every $x \in Q$ there exists a countable set $S(x)$ such that

(6) $$f\big(x(S\setminus T, x')\big) = f(x) \quad \text{for every finite } T \subset S\setminus S(x) \text{ and } x' \in Q.$$

Now, exactly as in the proof of theorem 2, we define a sequence $Q_0, Q_1, \ldots$ of countable subsets of Q satisfying (3). We can evidently suppose that $p_{S\setminus S_k}(Q_{k+1}) = p_{S\setminus S_k}(a)$. We shall show that the countable set $S' = \bigcup_{i=0}^{\infty} S_i$, where S_i is defined by (4), satisfies the conclusion of the theorem. It suffices to show that

(7) $$f(x) = f\big(x(S', a)\big)$$

for every $x \in A$. Indeed, the function f' satisfying $f = f'(p_{S'}|A)$, whose existence follows from (7), is sequentially continuous and hence continuous, the space $p_{S'}(A) \subset \underset{s\in S'}{P} X_s$ satisfying the first axiom of countability.

First, we shall show that (7) holds for $x \in Q$. The set $D = \{s \in S\colon p_s(x) \neq p_s(a)\}$ is finite in this case, and the set $D \cap S'$ is contained in S_k for some integer k. From (3) it follows, $\underset{s\in S_k}{P} X_s$ satisfying the first axiom of countability, that there exists a sequence $x_1, x_2, \ldots$ of points of Q_{k+1} such that $p_{S_k}(x) \in \lim p_{S_k}(x_i)$.

By (6), for the points

$$x_i' = x_i\big(S\setminus(D\setminus S'), x\big)$$

we have $f(x_i') = f(x_i)$. Since $x \in \lim x_i'$ and $x(S', a) \in \lim x_i$, the validity of (7) follows from the sequential continuity of f.

We shall now show that (7) holds for every $x \in A$. Since $A \subset \Sigma(a)$, there exists a sequence $x_1, x_2, \ldots$ of points of Q convergent to x. We have $f(x_i) = f\big(x_i(S', a)\big)$ and $x(S', a) \in \lim x_i(S', a)$ and the validity of (7) follows from the sequential continuity of f.

We now formulate the lemma of [15], which we shall use in the proof of Theorem 4. This lemma is an improvement of the well-known theorem of S. Ulam [21] on real-valued σ-additive measures. Let us remember that the function Φ from the family $\mathbf{S}$ of sets to a Hausdorff space Y is called *sequentially continuous* if $\lim S_n = S$ in $\mathbf{S}$ implies $\lim f(S_n) = f(S)$ in Y (³).

LEMMA 3. *If Φ is a sequentially continuous function from the family $\mathbf{S}$ of all subsets of a set S whose power does not exceed the first inaccessible aleph* (⁴) *to a Hausdorff space Y, in which every point is a sequentially G_δ-set transforming every finite subset of S to a fixed point $y_0 \in Y$, then $\Phi(S) = y_0$.*

THEOREM 4. *If A is a subset of the Cartesian product $\underset{s \in S}{P} X_s$ of T_1-spaces with countable bases, where $\overline{\overline{S}}$ does not exceed the first inaccessible aleph, invariant under composition, and $f\colon A \to Y$, is a sequentially continuous function from A to a Hausdorff space Y in which every point is a sequentially G_δ-set, then there exists a factorization $f = f'(p_{S'}|A)$, where $S' \subset S$ is countable and f' is a continuous function from $p_{S'}(A)$ to Y.*

Moreover, if $\Sigma(a) \subset A$ for some $a \in A$, then it suffices to suppose that $x(T, a) \in A$ for $x \in A$ and $T \subset S$, X_s's are T_1, separable, and satisfy the first axiom of countability, and every finite product $X_{s_1} \times X_{s_2} \times \ldots \times X_{s_k}$, where $s_1, s_2, \ldots, s_k \in S$, has the Lindelöf property.

Proof. Let a be an arbitrary point of A. The set

$$A_0 = \{x(Z, a)\colon x \in A \text{ and } \overline{\overline{Z}} \leqslant \aleph_0\} \subset A \cap \Sigma(a)$$

is invariant under composition, and by Theorem 3 there exist a countable $S' \subset S$ and a continuous $f_0\colon p_{S'}(A_0) \to Y$ such that $f|A_0 = f_0(p_{S'}|A_0)$. To show that S' satisfies the conclusion of the theorem it suffices to prove that (7) holds for every $x \in A$. Let Φ_x be a function from the family of all subsets of $S \setminus S'$ to Y defined by

$$\Phi_x(T) = f(x(S' \cup T, a)) \quad \text{for} \quad T \subset S \setminus S'.$$

If T is finite, then $x(S' \cup T, a) \in A_0$ and $f(x(S' \cup T, a)) = f(x(S', a))$, i.e. $\Phi_x(T) = f(x(S', a)) = y(x)$. Since, as can easily be verified, Φ_x is sequentially continuous, we obtain from Lemma 3

$$f(x) = f(x(S, a)) = \Phi_x(S \setminus S') = y(x) = f(x(S', a)).$$

The second part of theorem can be deduced analogously from the second part of Theorem 3.

(³) $S = \lim S_n$ means that S_n converges to S in the sense of the theory of sets, i.e. that $S = \bigcup_{n=1}^{\infty} \bigcap_{m=0}^{\infty} S_{n+m} = \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty} S_{n+m}$.

(⁴) *An aleph* $\aleph_\lambda$ is said to be *inaccessible* provided that $\lambda > 0$ is a limit ordinal and that $\sum_{s \in S} \mathfrak{m}_s < \aleph_\lambda$ whenever $\overline{\overline{S}} < \aleph_\lambda$ and $\mathfrak{m}_s < \aleph_\lambda$ for $s \in S$.

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