On the axiom of determinateness (II)

by

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This paper is a continuation of part I [9]. It contains an alternative form of the axiom of determinateness (A) of H. Steinhaus and this author [12], which has the same main consequences as (A). But a theorem proved here (Theorem 5) shows that the consistency of this new form is a conjecture which is at present much better founded than that of (A). In sections 1 and 2 the consequences of the new form are derived, in section 3, several theorems on the determinateness of some positional games are proved and the final section 4 contains miscellaneous remarks and problems.

The axiom of determinateness recalls an old saying on two kinds of truth quoted by Niels Bohr [1]: "To the one kind belong statements so simple and clear that the opposite assertions obviously could not be defended. The other kind, the so-called »deep truths«, are statements in which the opposite also contains deep truth".

0. Notation. We adopt the notation introduced in part I [9], § 2 with the following additions.

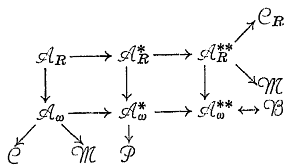
R denotes the set of real numbers.

$\mathcal{C}_R$ denotes the following weak form of the axiom of choice: *For every family* $\boldsymbol{F}$ *of disjoint non-empty sets, with* $|\boldsymbol{F}| \leqslant 2^{\aleph_0}$ *and* $|\bigcup_{X \in \boldsymbol{F}} X| \leqslant 2^{\aleph_0}$, *there exists a selector*. Of course $\mathcal{C}_R$ implies the still more special $\mathcal{C}$ of [9] (by a misprint the word *disjoint* is missing there).

Concerning the definition of the games $G^*_X(P)$ and $G^{**}_X(P)$ given in [9], let us add for clarity that such a game does not change whether we assume that the players when making a choice know the sequence of previous choices (¹) or that they know only their concatenation. In fact the existence of a winning strategy in one of these senses implies the existence of such a strategy for the same player in the other sense. This proposition is easy and a case of it (for $G^*_{\{0,1\}}(P)$) was proved in [2], Lemma 4.2 and another case (for $G^{**}_X(P)$) was proved by S. Świerczkowski (about 1956, unpublished).

(¹) As it is always supposed in the theory of games with perfect information (see [8]).

1. Introduction. The following diagram of implications holds.



It shows that the proposition $\mathcal{A}_R^*$ also implies $\mathcal{C}$, $\mathcal{M}$, $\mathcal{B}$ and $\mathcal{P}$, which we consider as the main consequences of $\mathcal{A}_\omega$ ((A) is equivalent to $\mathcal{A}_\omega$). In view of Theorem 5 below the consistency of $\mathcal{A}_R^*$ (in a set theory without the axiom of choice) is a better founded conjecture than that of $\mathcal{A}_\omega$. But notice that neither $\mathcal{A}_R$ nor some stronger proposition considered in [9], § 7, Remark 1, have been still disproved. Perhaps the consistency of $\mathcal{A}_R^*$ or at least $\mathcal{A}_R^{**}$ could be proved by the method of Cohen.

All the arrows of the above diagram, except $\mathcal{A}_R \to \mathcal{A}_R^* \to \mathcal{A}_R^{**} \to \mathcal{C}_R$ & $\mathcal{M}$, were proved in [9]. The proofs of $\mathcal{A}_R \to \mathcal{A}_R^*$ and $\mathcal{A}_R^* \to \mathcal{A}_R^{**}$ are quite similar to those of $\mathcal{A}_\omega \to \mathcal{A}_\omega^*$ and $\mathcal{A}_\omega^* \to \mathcal{A}_\omega^{**}$ of [9] and thus they are left to the reader. $\mathcal{A}_R^{**} \to \mathcal{C}_R$ & $\mathcal{M}$ will be proved in the next section (Theorems 1 and 2). Our proof of $\mathcal{A}_R^{**} \to \mathcal{M}$ is analogous to the proof of $\mathcal{A}_\omega^* \to \mathcal{B}$ (see [9], [15]) and is simpler than that of $\mathcal{A}_\omega \to \mathcal{M}$ (see [13]).

The main results of this paper were announced in [10].

2. Consequences of $\mathcal{A}_R^{**}$. The axiom of choice will not be used in this section.

THEOREM 1. $\mathcal{A}_R^{**} \to \mathcal{C}_R$.

Proof. With no loss of generality we can suppose that the family $\boldsymbol{F}$ is of the form $\{X_t\colon t \in R\}$ and that $0 \notin \bigcup_{X \in \boldsymbol{F}} X \subseteq R$. We consider the game $G_R^{**}(P)$, where $(t_0, t_1, \ldots) \in P$ if and only if $t_i \notin X_{t_0}$ for every $i > 0$. Clearly player I has no winning strategy in that game, and thus by $\mathcal{A}_R^{**}$ player II has a winning strategy, say σ. Let $(t, t_1, t_2, \ldots)$ be the unique play in which player I made the first choice (t) (a one-term sequence) and all his later choices were (0), while II played according to σ. Hence $(t, t_1, t_2, \ldots) \notin P$ and then $t_i \in X_t$ for some $i > 0$. Such t_i with i minimal when t runs over R constitute a selector for $\boldsymbol{F}$. Q.E.D.

Remark 1. Proposition (7.3) of [9] can be completed as follows. *Each of the six propositions $\mathcal{A}_X$, $\mathcal{A}_X^*$ and $\mathcal{A}_X^{**}$, where $|X| = \aleph_1$ or $|X| = \mathfrak{k}$* (²), *is inconsistent.* This can be proved by an easy modification of the proof of the first part of (7.3), similar to the above proof of Theorem 1.

(²) Concerning $\mathfrak{k}$, see [9] § 3 and Remark on p. 222.

THEOREM 2. $\mathcal{A}_R^{**} \to \mathcal{M}$.

Proof. For any set $X \subseteq J$ (³) we denote by $\boldsymbol{P}(X)$ the set of all perfect subsets of X of positive measure and diameter $\leqslant \frac{1}{2}$(diameter of X). For any set $S \subseteq J$ we define a game $\Gamma(S)$: Player I chooses any $F_0 \in \boldsymbol{P}(J)$, then player II chooses any $F_1 \in \boldsymbol{P}(F_0)$, and so on, I making the even choices and II the odd choices and the nth choice $F_n \in \boldsymbol{P}(F_{n-1})$. Of course there is a unique point $p \in \bigcap_{n<\omega} F_n$ and player I wins if $p \in S$ and II wins if $p \notin S$.

First we show that

(i) *$\mathcal{A}_R^{**}$ implies that the game $\Gamma(S)$ is determined.*

This is an exercise in coding the game $\Gamma(S)$ by means of a game $G_R^{**}(P)$, which is perhaps obvious, but for the convenience of the reader we perform it in detail.

For every perfect set $F \subseteq J$ of positive measure, let f_F be a function which maps in a one-to-one way $\boldsymbol{P}(F)$ onto R (such a function can be effectively constructed). To every sequence $F_0, F_1, \ldots$ such that $F_n \in \boldsymbol{P}(F_{n-1})$ for $n < \omega$ (⁴) we define a sequence $\varphi(F_0, F_1, \ldots) = (r_0, r_1, \ldots) \in R^\omega$ by putting

$$r_n = f_{F_{n-1}}(F_n) \quad \text{for} \quad n < \omega .$$

Thus φ is one-to-one. We set

$$P_S = \varphi\{(F_0, F_1, \ldots)\colon F_n \in \boldsymbol{P}(F_{n-1}) \text{ for } n < \omega \text{ and } \bigcap_{n<\omega} F_n \subseteq S\} .$$

Now it is visible that the games $G_R(P_S)$ and $\Gamma(S)$ are equivalent. To show that $\mathcal{A}_R^{**}$ implies their determinateness it is enough to prove that the determinateness of $G_R^{**}(P_S)$ implies that of $G_R(P_S)$.

Suppose thus that σ is a winning strategy for player II in the game $G_R^{**}(P_S)$ (the case where $G_R^{**}(P_S)$ is a win for I can be treated similarly). We transform σ into a winning strategy $\overline{\sigma}$ for player II in the game $G_R(P_S)$. Thus $\overline{\sigma}$ has to be a real-valued function defined on all finite sequences of real numbers such that for every $r = (r_0, r_1, \ldots) \in R^\omega$ we have

$$(1) \qquad \big(r_0, \overline{\sigma}(r_0), \ldots, r_n, \overline{\sigma}(r_0, \ldots, r_n), \ldots\big) \notin P_S .$$

$\overline{\sigma}$ is defined as follows.

First, of course, σ is a map of finite sequences of finite sequences of real numbers into finite sequences of real numbers. For every $r \in R^\omega$ we put

$$s(r) = (r_0)^\frown \sigma\big((r_0)\big)^\frown \ldots {}^\frown (r_n)^\frown \sigma\big((r_0), \ldots, (r_n)\big)^\frown \ldots \text{ (⁵)} .$$

(³) J denotes the closed unit interval.

(⁴) We always assume that $F_{-1} = J$.

(⁵) $^\frown$ denotes concatenation of sequences; (r_i) denotes the sequence having only one term r_i.

Since σ is a winning strategy for II, we have

(2) $$s(r) \notin P_S \quad \text{for every } r \in R^\omega .$$

Let $(F_0, F_1, \ldots) = \varphi^{-1}(s(r))$ and let $k(r_0, \ldots, r_n)$ be defined as the number of terms of the finite sequence

$$(r_0)^\frown\sigma((r_0))^\frown \ldots {}^\frown(r_n)^\frown\sigma((r_0), \ldots, (r_n)),$$

and $k(\emptyset) = 0$ (6). Finally we put for any $n < \omega$

$$\bar{\sigma}(r_0, \ldots, r_n) = f_{F_{k(r_0,\ldots,r_{n-1})}}(F_{k(r_0,\ldots,r_n)-1}).$$

Now we have to prove (1). Let $(R_0, R_1, \ldots) = \varphi^{-1}(r_0, \bar{\sigma}(r_0), \ldots$ $\ldots, r_n, \bar{\sigma}(r_0, \ldots, r_n), \ldots)$. It is clear that $R_0, R_1, \ldots$ is a subsequence of $F_0, F_1, \ldots$ Then by (2) we have $\bigcap_{n<\omega} R_n = \bigcap_{n<\omega} F_n$ non $\subseteq S$ and (1) follows. This concludes the proof of (i).

To show Theorem 2 it is enough to prove on account of $\mathcal{A}_R^{**}$ that every set $S \subseteq J$ either is of measure 0 ($|S| = 0$) or has a positive interior measure ($|S|_i > 0$). Indeed, if there were any non-measurable sets $X \subseteq J$, it would be easy to construct effectively by means of X a set $S \subseteq J$ with $|S|_i = 0$ and exterior measure $|S|_e = 1$. Therefore by (i) Theorem 2 will be proved if we show the following propositions:

(ii) *$\Gamma(S)$ is a win for player* I *if and only if $|S|_i > 0$.*

(iii) *$\Gamma(S)$ is a win for player* II *if and only if $|S| = 0$.*

It is obvious that $\Gamma(S)$ is a win for player I (II) if $|S|_i > 0$ ($|S| = 0$). But it is the converse implications that are essential for us. Their proof requires the axiom of choice $\mathcal{C}_R$, but on account of Theorem 1 we can use $\mathcal{C}_R$. We are going to show (iii) (the proof of (ii) is analogous).

Let σ be a winnig strategy for player II in the game $\Gamma(S)$. For every finite sequence $F_0, \ldots, F_{2n-1}$ $(n < \omega)$ (4), with $F_{2i} \in \boldsymbol{P}(F_{2i-1})$ and $F_{2i+1} = \sigma(F_0, F_2, \ldots, F_{2i})$ for $i < n$, and every perfect or empty set $P \subseteq F_{2n-1}$, with $|F_{2n-1} - P| > 0$ we put

$$\varkappa(P, F_0, \ldots, F_{2n-1}) = \sup\{|\sigma(F_0, F_2, \ldots, F_{2n})|: F_{2n} \in \boldsymbol{P}(F_{2n-1}) \text{ and } F_{2n} \cap P = 0\}$$

and

$$\boldsymbol{K}(P, F_0, \ldots, F_{2n-1}) = \{F_{2n}: F_{2n} \in \boldsymbol{P}(F_{2n-1}), F_{2n} \cap P = 0 \text{ and } |\sigma(F_0, F_2, \ldots, F_{2n})| \geqslant \tfrac{1}{2}\varkappa(P, F_0, \ldots, F_{2n-1})\}.$$

By $\mathcal{C}_R$ there exists a third function K which is a selector for $\boldsymbol{K}$, i.e.

$$K(P, F_0, \ldots, F_{2n-1}) \in \boldsymbol{K}(P, F_0, \ldots, F_{2n-1}).$$

(6) $\emptyset$ denotes the empty sequence.

Now K permits us to argue similarly to Oxtoby [15] (this idea goes back to Świerczkowski and probably even to Banach (see [14])). For every $n < \omega$ we define a denumerable family $\boldsymbol{A}_n$ of mutually disjoint choices of player II by means of σ immediately following some choices of player I which are either his first choices (if $n = 0$) or his choices immediately following any choice of II belonging to $\boldsymbol{A}_{n-1}$ (if $n > 0$). Moreover, this will be done in such a way that putting

$$A_n = \bigcup_{C \in \boldsymbol{A}_n} C$$

we have

(3) $$|A_n| = 1 \quad \text{for every } n < \omega .$$

Clearly this will already prove (iii) since, $\boldsymbol{A}_n$ being a family of disjoint sets and σ being a winning strategy, it follows that $\bigcap_{n<\omega} A_n \cap S = 0$, which implies $|S| = 0$.

We start our inductive definition of $\boldsymbol{A}_n$ putting $\boldsymbol{A}_{-1} = \{J\}$. Suppose that $\boldsymbol{A}_0, \ldots, \boldsymbol{A}_{n-1}$ are already defined, satisfy the conditions above and, moreover, we have a sequence $\boldsymbol{B}_0, \ldots, \boldsymbol{B}_{n-1}$ such that

(4) for every $C \in \boldsymbol{A}_{n-1}$ there exists a unique sequence $B_i \in \boldsymbol{B}_i$ $(i < n)$ such that $C = \sigma(B_0, \ldots, B_{n-1})$ and $\sigma(B_0, \ldots, B_i) \in \boldsymbol{A}_i$ for all $i < n$.

Let $\boldsymbol{A}_{n-1} = \{C_0, C_1, \ldots\}$. For every C_i we take the corresponding sequence $B_0^{(i)}, \ldots, B_{n-1}^{(i)}$ and we put for all $j < \omega$

$$B^{(i,j)} = K\Big(\bigcup_{k<j} \sigma(B_0^{(i)}, \ldots, B_{n-1}^{(i)}, B^{(i,k)}), B_0^{(i)}, \sigma(B_0^{(i)}), \ldots$$
$$\ldots, B_{n-2}^{(i)}, \sigma(B_0^{(i)}, \ldots, B_{n-2}^{(i)}), B_{n-1}^{(i)}, C_i\Big).$$

Of course $B^{(i,j)} \subseteq C_i$ and by the definition of K (look at $\boldsymbol{K}$ and $\varkappa$) it is easy to verify that

(5) $$\Big|C_i - \bigcup_{j<\omega} \sigma(B_0^{(i)}, \ldots, B_{n-1}^{(i)}, B^{(i,j)})\Big| = 0$$

and

$$\sigma(B_0^{(i)}, \ldots, B_{n-1}^{(i)}, B^{(i,j)}) \cap \sigma(B_0^{(i)}, \ldots, B_{n-1}^{(i)}, B^{(i,j')}) = 0 \quad \text{for} \quad j \neq j'.$$

We put $\boldsymbol{B}_n = \{B^{(i,j)}: i, j < \omega\}$ and $\boldsymbol{A}_n = \{\sigma(B_0^{(i)}, \ldots, B_{n-1}^{(i)}, B^{(i,j)}): i, j < \omega\}$. Then by the inductive assumption we find that $\boldsymbol{A}_n$ is disjoint, that $\boldsymbol{B}_n$ satisfies (4) with n replaced by $n+1$, and that (5) implies (3). This concludes our inductive definition of $\boldsymbol{A}_n$ satisfying the required properties and (iii) is proved.

This concludes the proof of Theorem 2.

3. Determinateness of some games. The axiom of choice is assumed throughout this section. X will denote a discrete space and X^ω will have the usual product topology. The results proved here refine proposition (3.4) of [9] and the parts of Theorem 4 of [9] which concern $\mathcal{A}_X^*$ and $\mathcal{A}_X^{**}$ (7).

THEOREM 3. (a) *$G_X^{**}(P)$ is a win for* I *if and only if $V-P$ is of the first category for a non-empty open set $V \subseteq X^\omega$;*

(b) *$G_X^{**}(P)$ is a win for* II *if and only if P is of the first category in X^ω.*

The proof is an easy modification of the idea of Świerczkowski and Oxtoby (and probably Banach) already used in the proof of (iii) above (by a construction of analogous families $\boldsymbol{A}_n$, in this case not necessarily denumerable).

Theorem 3 clearly implies the following corollary.

COROLLARY 4. *$\mathcal{A}_X^{**}(P)$ holds true for every set $P \subseteq X^\omega$ having the property of Baire.*

Remark 2. Following the way indicated above and in Theorem 7 below *one can prove without using the axiom of choice that $\mathcal{A}_R^{**}$ is equivalent to the conjunction of $\mathcal{C}_R$ and the statement that every set $P \subseteq X^\omega$, where $|X| \leqslant 2^{\aleph_0}$, has the property of Baire.*

We do not know any similar equivalence involving measurability in place of the property of Baire; however, the well-known analogy of these two properties can be further magnified as follows. Consider a topology on R in which a set $A \subseteq R$ is open if and only if every $a \in A$ is a metric density point of A, i.e. $\lim_{h\to 0+} |A \cap \langle a-h, a+h\rangle|/2h = 1$ (8). It is easy to see that every set $B \subseteq R$ is Lebesgue-measurable if and only if it has the property of Baire with respect to this topology.

Remark 3. A proof that the sets of the first category are denumerably additive, which already implies that sets having the property of Baire form a denumerably additive Boolean algebra which thus includes all Borel sets, requires the axiom of choice. But for a separable space this can be established on account of $\mathcal{C}$ only (9).

(7) It was erroneously stated in [9] that this theorem does not require the axiom of choice; in fact even for X finite or denumerable the proof of $\mathcal{A}_X(P)$ for all $P \in F_{\sigma\delta} \cup G_{\delta\sigma}$ is based on $\mathcal{C}$. For a similar result see Theorem 8 below. By a mistake it was not mentioned in [9], § 3 that $\mathcal{C}$ was used in the proofs of the propositions (o), ..., (iii) (in fact the proofs were based on the 01-laws of the theories of measure and category, which require $\mathcal{C}$).

(8) This topology was studied in [5]. It is regular but not locally compact and I do not know if it is completely regular? It has the Souslin property (see [7], § 1) and is connected but is not separable and has totally disconnected open sets. In this space first category, nowhere density and measure 0 coincide.

A set P in a topological space S is called *analytic* if it is the result of applying the operation $(\mathcal{A})$ to a system of closed sets, i.e. $P = \bigcup_i \bigcap_{n<\omega} F_{i_0,\ldots,i_n}$, where $i = (i_0, i_1, \ldots)$ runs over all sequences of natural numbers and all $F_{i_0,\ldots,i_n}$ are closed subsets of S. Recall that a Borel set in a complete separable metric space is analytic (10).

THEOREM 5. *$\mathcal{A}_X^*(P)$ holds true for every analytic set $P \subseteq X^\omega$.*

Proof. We will first prove two auxiliary statements.

(i) *If each of the games $G_X^*(P_n)$ $(n < \omega)$ is a win for player* II *then $G_X^*(\bigcup_{n<\omega} P_n)$ is also a win for player* II.

In fact, let σ_n be a winning strategy for II in the game $G_X^*(P_n)$. Let σ be a strategy for II such that in each play each of the strategies σ_n is applied infinitely many times, the sequences between the consecutive choices by means of σ_n being treated as if they were made by player I. Clearly σ is a winning strategy in $G_X^*(\bigcup_{n<\omega} P_n)$.

Now let $P = \bigcup_i \bigcap_{n<\omega} F_{i_0,\ldots,i_n}$ where all $F_{i_0,\ldots,i_n}$ are closed subsets of X^ω and let us suppose, which does not diminish the generality of the considerations, that $F_{i_0,\ldots,i_n} \subseteq F_{i_0,\ldots,i_{n-1}}$ for every i and n. We put $P_\emptyset = P$ (6) and

$$P_{i_0,\ldots,i_n} = \bigcup_j \bigcap_{m<\omega} F_{i_0,\ldots,i_n,j_0,\ldots,j_m}.$$

(ii) *If $p \in X^n$ is not a lost position for player* I *in the game $G_X^*(P_{i_0,\ldots,i_{k-1}})$, then there exists a number $i_k < \omega$ and a sequence $q \in X^m$ with $m < \omega$ such that for every $x \in X$ the sequence $p^\frown q^\frown(x)$ (5) is not a lost position for* I *in the game $G_X^*(P_{i_0,\ldots,i_k})$.*

Since $P_{i_0,\ldots,i_{k-1}} = \bigcup_{h<\omega} P_{i_0,\ldots,i_{k-1},h}$, this proposition follows of course from (i).

Suppose that $G_X^*(P)$ is not a loss for I. The theorem will be proved if we show that $G_X^*(P)$ is a win for I. Let $i\big(p, (i_0, \ldots, i_{k-1})\big)$ denote the first i_k which satisfies the conclusion of (ii) and $\vartheta\big(p, (i_0, \ldots, i_{k-1})\big)$ be any sequence q corresponding to this i_k as in (ii) (if p were a lost position for I the definition of $i(\ )$ and $\vartheta(\ )$ would not matter). Now we define inductively two other functions s and σ.

$$s(\emptyset) = \big(i(\emptyset, \emptyset)\big); \text{ (6)}$$
$$\sigma(\emptyset) = \vartheta\big(\emptyset, s(\emptyset)\big);$$
$$\cdots\cdots\cdots\cdots$$

(9) See [9], Appendix 1. Separable = having a denumerable basis of open sets.

(10) See [6], § 33, I. This fact requires only $\mathcal{C}$.

$$s(x_0, \ldots, x_n) = s(x_0, \ldots, x_{n-1})^\frown\Big(i\big(\sigma(\emptyset)^\frown(x_0)^\frown\ldots^\frown\sigma(x_0, \ldots, x_{n-1})^\frown(x_n),\; s(x_0, \ldots, x_{n-1})\big)\Big);$$

$$\sigma(x_0, \ldots, x_n) = \vartheta\big(\sigma(\emptyset)^\frown(x_0)^\frown\ldots^\frown\sigma(x_0, \ldots, x_{n-1})^\frown(x_n),\; s(x_0, \ldots, x_n)\big);$$

. . . .

I claim that σ is a winning strategy for player I. Indeed for every $\boldsymbol{x} = (x_0, x_1, \ldots) \in X^\omega$ let

$$\tau(\boldsymbol{x}) = \sigma(\emptyset)^\frown(x_0)^\frown\ldots^\frown\sigma(x_0, \ldots, x_{n-1})^\frown(x_n)^\frown\ldots$$

It is enough to show that $\tau(\boldsymbol{x}) \in P$ for every $\boldsymbol{x}$ and in fact $\tau(\boldsymbol{x}) \in \bigcap_{n<\omega} F_{s(x_0,\ldots,x_n)}$. If this were not the case there would exist a neighbourhood of $\tau(\boldsymbol{x})$ disjoint with some set $F_{s(x_0,\ldots,x_n)}$, which is contrary to the fact that by the definition of σ there are arbitrarily long initial segments of $\tau(\boldsymbol{x})$ which are non-lost positions for player I in the game $G_X^*(P_{s(x_0,\ldots,x_n)})$. Q.E.D.

Remark 4. In a T_1 topological space (i.e. singletons are closed) every set which is analytic or analytic complement has the property of Baire (see [6], § 11, VII). By [9], Theorem 3, it is consistent with the usual axioms of set theory that there are sets $P \in \boldsymbol{CA}$ for which $\mathscr{A}^*_{\{0,1\}}(P)$ fails and sets $P \in \boldsymbol{PCA} \cap \boldsymbol{CPCA}$ for which $\mathscr{A}^{**}_{\{0,1\}}(P)$ fails. Hence Theorem 5 and Corollary 4 are sharp.

Remark 5. If N is a discrete denumerable space and X_c is the set X with the smallest T_1 topology, i.e. only finite sets and X are closed, and $f\colon N^\omega \to X_c^\omega$ is a continuous mapping, then $f(N^\omega)$ is analytic in X^ω.

THEOREM 6. *In the case where there exists a well ordering of X Theorem 3 and Corollary 4 can be proved without using the axiom of choice and in the case $|X| \leqslant 2^{\aleph_0}$ they require only $\mathcal{C}_R$. In the case where X is at most denumerable Theorem 5 requires only $\mathcal{C}$.*

The proof follows by a simple analysis of the proofs of the corresponding results.

4. Miscellanea. 1.(a) $\mathcal{C}_R$ implies the following selfrefinement, which is a weak form of the principle of dependent choices.

$$\bigwedge_{r\in R}\bigvee_{s\in R}(r, s) \in A \rightarrow \bigvee_{r\in R^\omega}\bigwedge_{n<\omega}(r_n, r_{n+1}) \in A\,.$$

(b) $\mathcal{C}$ or $\mathcal{C}_R$ does not seem to imply that a denumerable union of denumerable sets is denumerable or even that there exists a choice set for a denumerable set of disjoint pairs.

2.(a) There are games which I have not been able to represent in the form $G_X^*(P)$; however, they fullfil statements analogous to Theorem 5. Such is for instance, as I have realized recently, a game given as example 3 in [8]. A slight simplification of that game for which the same result holds is the following. A set $P \subseteq R$ is given. Player I chooses any $x_0 \in R$ and then II chooses any $y_0 < x_0$ and then I any x_1 with $y_0 < x_1 < x_0$ and again II any y_1 with $y_0 < y_1 < x_1$, etc., always between the last two choices. I wins if $\lim x_n \in P$ and II wins in the other case. It is easy to see that I has a winning strategy if P has a perfect subset and that II has a winning strategy if P is at most denumerable [11]. This implies that this game is determined if P is analytic.

(b) Other facts on positional games and sets having the property L of Lusin (see [6], § 36, VII) are given in [18].

(c) Positional games with recursive strategies were considered by M. O. Rabin [16].

(d) If we assume the axiom of choice, there are of course non-determined games of the form $G^*_{\{0,1\}}(P)$ or $G^{**}_{\{0,1\}}(P)$ (see e.g. [9] propositions (3.3) and (3.4)). But even if we assume this axiom the problem of the existence of non-determined games of the form described above in 2.(a) or of several related types (see [8], § 2.3 and [3] and [4] where other references are given) is open.

3. Finally I want to mention another set of problems on positional games; however, it is quite remote from the main subject of this paper.

(a) An infinite set X is given. I cuts X into two parts and II chooses one of them, then I cuts the chosen part into two parts and II chooses one of them, etc. After ω steps I pays to II the cardinality of the intersection of the chosen parts. It is clear that I has a strategy for paying not more than 1 if and only if $|X| \leqslant 2^{\aleph_0}$. Of course II has a strategy to get at least 1, but is it possible, for X sufficiently large, that he has a strategy to get at least 2? (if there exists a denumerably additive 01-valued measure m on all subsets of X, with $m(X) = 1$, and vanishing on singletons then the answer is positive; but the existence of such X and m does not follow from the usual axioms of set theory).

(b) A related problem [11] probably involves similar difficulties.

(c) Similar problems were stated by Banach in the Scottish Book (Problems 67, 1) and 2) (1935)). They were solved by J. Schreier [17]. In [17] the game theoretical form of the result is not stated and it has never appeared in printed form. Let us formulate it here. An infinite set X of power $\mathfrak{m}$ being given, we define two games Γ_1 and Γ_2. In Γ_1 player I chooses any $X_0 \subseteq X$ with $|X_0| = \mathfrak{m}$, then player II chooses any $X_1 \subseteq X_0$ with $|X_1| = \mathfrak{m}$, etc.; they construct a sequence $X_0 \supseteq X_1 \supseteq \ldots \ldots \supseteq X_i \supseteq \ldots$ $(i < \omega)$ of sets of power $\mathfrak{m}$. In Γ_2 they construct in the same

[11] The same statements are true for the game $G^*_{\{0,1\}}(P)$ (see [2], Theorems 4.1 and 4.2, repeated as proposition (3.3) in [9]).

order a sequence of disjoint sets $X_i \subseteq X$ ($i < \omega$) such that $|X - \bigcup_{i<j} X_i| = \mathfrak{m}$ for every $j < \omega$. In Γ_1 player II wins if $\bigcap_{i<\omega} X_i = 0$ and in Γ_2 if $\bigcup_{i<\omega} X_i = X$. Schreiers' argument proves that in both games, Γ_1 and Γ_2, player II has a winning strategy.

(d) A similar problem stated by S. Ulam [19] is still open.

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A duality property of nerves

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1. Our main aim in this paper is to prove the following

1.1. THEOREM. *Let Y be a normal space, and $\mathfrak{U} = \{U_\alpha \mid \alpha \in \mathfrak{A}\}$ a nbd-finite* (1) *covering of Y by open F_σ-sets. Assume that*

(a) *The order* (2) *of $\mathfrak{U}$ is $\leqslant n$*

and

(b) *For each $k \geqslant 1$, the intersection of every k sets of $\mathfrak{U}$ is $(n-k)$-connected* (3).

Then each canonical map $\varkappa$ of Y into the nerve (4) *$N(\mathfrak{U})$ of $\mathfrak{U}$ has a right homotopy inverse* (5) *$g\colon N(\mathfrak{U}) \to Y$. Moreover, $\varkappa$ and g can be chosen so that $g \circ \varkappa$ is $\mathfrak{U}$-close* (6) *to the identity map of Y.*

In [8], pp. 142-145, Weil derived the above conclusion from the two assumptions: (a') No restriction on the order of $\mathfrak{U}$, and (b') Every finite intersection of sets of $\mathfrak{U}$ is ∞-connected; thus, in 1.1 we strengthen one of his hypotheses and weaken the other. Our proof of 1.1 will be a modification of his; note that the above version does not require the

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(1) $\mathfrak{U}$ is nbd-finite if each point has a neighborhood meeting at most finitely many members of $\mathfrak{U}$.

(2) The largest integer n such that there are $(n+1)$ members of with non-empty intersection, i.e., the dimension of the nerve of $\mathfrak{U}$.

(3) X is k-connected if $\pi_i(X) = 0$ for $0 \leqslant i \leqslant k$; it is ∞-connected if $\pi_i(X) = 0$ for all $i \geqslant 0$; $\pi_0(X) = 0$ denotes that X is path-connected.

(4) We realize the nerve of a covering $\mathfrak{U}$ as a rectilinear polytope in a real vector space spanned by linearly independent vectors in a fixed one-to-one correspondence with the non-empty $U_\alpha \in \mathfrak{U}$. The vertex corresponding to U_α is the unit point on the corresponding vector, and is denoted by p_α. The topology of $N(\mathfrak{U})$ is the CW-topology ([9], p. 223). A continuous $\varkappa\colon Y \to N(\mathfrak{U})$ is called *cannonical* if $\varkappa^{-1}(\mathrm{St}\, p_\alpha) \subset U_\alpha$ for each $\alpha \in \mathfrak{A}$.

(5) That is, $\varkappa \circ g \simeq 1$; equivalently, Y dominates $N(\mathfrak{U})$.

(6) Two maps $f, g\colon X \to Y$ are $\mathfrak{U}$-close if for each $x \in X$ there is a $U_\alpha \in \mathfrak{U}$ containing both $f(x)$ and $g(x)$. Under certain conditions (for example, if each finite intersection of the closures of the U_α is an AR (normal) ([8], p. 142) or if Y is an ANR and the U_α are "sufficiently small" ([5], p. 243)) $\mathfrak{U}$-closed maps are homotopic.