On the conditions of monotonicity of functions *

by

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G. P. Tolstov [2] and Z. Zahorski [3] have obtained the following theorems on the monotonicity of functions:

TOLSTOV'S THEOREM. *Let f be a function which is approximately continuous in the interval (a, b). Suppose that its approximate derivative exists, except possibly at a countable set of points, and takes non-negative values a.e. in (a, b).*

Then the function f is continuous and non-decreasing in (a, b).

ZAHORSKI'S THEOREM. *Let f be a function satisfying in (a, b) the conditions*: (a) *in every interval $\langle p, q\rangle \subset (a, b)$ the function f takes all values from the interval $\langle f(p), f(q)\rangle$*, (b) *the derivative of the function f exists except at a countable set of points*, (c) *$f'(x) \geqslant 0$ a.e. in (a, b).*

Then the function f is continuous and non-decreasing in (a, b).

The theorems just quoted give some sufficient conditions for a function to be monotone. The sets of functions satisfying the conditions given by the authors of these theorems are not exclusive but neither includes the other. Z. Zahorski has proved that both sets are parts of the family Z consisting of all functions f satisfying in an interval (a, b) the following conditions:

1. *f is a function of the first class of Baire,*

2. *in every interval $\langle p, q\rangle \subset (a, b)$ the function f takes all values from the interval $\langle f(p), f(q)\rangle$,*

3. *the approximative derivative of the function f exists except at a countable set in (a, b) and*

4. *$f'_{\mathrm{ap}}(x) \geqslant 0$ a.e. in (a, b).*

Z. Zahorski has noticed that there exist functions satisfying three of the conditions 1-4 chosen arbitrarily, which, however, are not monotone. Simultaneously he raises the following question:

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THE PROBLEM OF ZAHORSKI. *Is every function of the family* Z *continuous and non-decreasing?*

In the present paper we shall formulate some theorems giving certain sufficient conditions of the monotonicity of functions. One of them (theorem 1) is a positive answer to the question of Z. Zahorski stated above [1].

In paper [3] mentioned above, Z. Zahorski has described a class $\mathfrak{J}$ of functions satisfying conditions 1 and 2. Of the properties of the functions of the class $\mathfrak{J}$ the following two will be employed in the sequel. (a) A function f belongs to the class $\mathfrak{J}$ if and only if each of the sets of the form $\{x\colon f(x) > M\}$ and $\{x\colon f(x) < m\}$ is an F_σ set and consists only of bilateral condensation points. (b) If $f \in \mathfrak{J}$ and g is a continuous function, then the sum of these functions belongs to $\mathfrak{J}$.

I. LEMMA 1. *Let a function, measurable in* (a, b), f *possess at each point of a residual set* $H \subset (a, b)$ *an approximate derivative. Further, let* M *and* N *be real numbers such that* $M > N$.

Then at most one of the sets $A = \{x\colon f'_{\text{ap}}(x) \geqslant M\}$, $B = \{x\colon f'_{\text{ap}}(x) \leqslant N\}$ *can be dense in* (a, b).

Proof. Suppose, on the contrary, that $\bar{A} = \bar{B} = \langle a, b\rangle$ and let $H \supset (a, b) - \bigcup F_n$, where $\{F_n\}$ is a sequence of closed nowhere dense sets. Moreover, let M' and N' be two numbers fixed throughout the argument and such that $N < N' < M' < M$.

Now let $x_1 \in A - F_1$. There exist a number $t_1 > 0$ such that $a < x_1 - 4t_1 < x_1 + 4t_1 < b$ and sets $A_1 \subset (x_1 - 4t_1, x_1)$ and $B_1 \subset (x_1, x_1 + 4t_1)$ such that

(1) $$|A_1| > 3t_1, \quad |B_1| > 3t_1,$$

(2) $$\langle x_1 - 4t_1, x_1 + 4t_1\rangle \subset (a, b) - F_1$$

and

(3) $$\frac{f(x) - f(x_1)}{x - x_1} > M' \quad \text{for} \quad x \in A_1 \cup B_1.$$

Now let $x_2 \in B \cap (x_1 - t_1, x_1 + t_1) - F_2$. There exist a number $t_2 > 0$ and sets $A_2 \subset (x_2 - 4t_2, x_2)$ and $B_2 \subset (x_2, x_2 + 4t_2)$ such that

(4) $$|A_2| > 3t_2, \quad |B_2| > 3t_2,$$

(5) $$\langle x_2 - 4t_2, x_2 + 4t_2\rangle \subset (x_1 - t_1, x_1 + t_1) - F_2$$

and

(6) $$\frac{f(x) - f(x_2)}{x - x_2} < N' \quad \text{for} \quad x \in A_2 \cup B_2.$$

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[1] Independently of my results the solution of Zahorski's problem has been obtained by A. Brückner. His result has not yet been published.

Repeating the above argument we obtain sequences of numbers $\{x_n\}$ and $\{t_n\}$, and sequences of sets $\{A_n\}$ and $\{B_n\}$ such that for every natural n we have

(7) $$A_n \subset (x_n - 4t_n, x_n), \quad B_n \subset (x_n, x_n + 4t_n),$$

(8) $$|A_n| > 3t_n, \quad |B_n| > 3t_n,$$

(9) $$\langle x_{n+1} - 4t_{n+1}, x_{n-1} + 4t_{n+1}\rangle \subset (x_n - t_n, x_n + t_n) - F_{n+1},$$

(10) $$\frac{f(x) - f(x_{2n-1})}{x - x_{2n-1}} > M' \quad \text{for} \quad x \in A_{2n-1} \cup B_{2n-1}$$

and

(11) $$\frac{f(x) - f(x_{2n})}{x - x_{2n}} < N' \quad \text{for} \quad x \in A_{2n} \cup B_{2n}.$$

Let x_0 be the common point (in view of (9) the only one) of all the intervals $\langle x_n - t_n, x_n + t_n\rangle$ and let E be an arbitrary set with a density point x_0 and let the limit

(12) $$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{when} \quad x \in E$$

exist. Such a set exists because by (9) $x_0 \in H$. In view of (7), (8), and (9) there exist two sequences $\{i_n\}$ and $\{j_n\}$ of positive integers such that the set E has common points with each of the sets

(13) $$\begin{aligned} A'_n &= A_{2i_n} \cap \langle x_{2i_n} - 4t_{2i_n}, x_{2i_n} - t_{2i_n}\rangle, \\ B'_n &= B_{2i_n} \cap \langle x_{2i_n} + t_{2i_n}, x_{2i_n} + 4t_{2i_n}\rangle, \\ A''_n &= A_{2j_n-1} \cap \langle x_{2j_n-1} - 4t_{2j_n-1}, x_{2j_n-1} - t_{2j_n-1}\rangle, \\ B''_n &= B_{2j_n-1} \cap \langle x_{2j_n-1} + t_{2j_n-1}, x_{2j_n-1} + 4t_{2j_n-1}\rangle. \end{aligned}$$

Now let $x_{1,n} \in E \cap A'_n$, $x_{2,n} \in E \cap B'_n$, $x_{3,n} \in E \cap A''_n$, and $x_{4,n} \in E \cap B''_n$. By (7), (10), and (11) we have for $n = 1, 2, \ldots$

(14) $$\frac{f(x_{2,n}) - f(x_{1,n})}{x_{2,n} - x_{1,n}} < N'$$

and

(15) $$\frac{f(x_{4,n}) - f(x_{3,n})}{x_{4,n} - x_{3,n}} > M',$$

which contradicts (12), since $\lim\limits_{n\to\infty} x_{i,n} = x_0$ and $x_{i,n} \in E$ for $i = 1, 2, 3, 4$. The contradiction obtained proves lemma 1.

LEMMA 2. *Let the function* $f \in \mathfrak{J}$ *possess an approximate derivative in the interval* (a, b) *except at a denumerable set of points. Moreover, let* $f'_{\text{ap}}(x) \geqslant M$, *where* $M > 0$, *also except at a denumerable set of points.*

Then the function f *is non-decreasing in* (a, b).

Proof. Let $\{p_n\}$ be a sequence of points of the interval (a, b) beyond which there exists an $f'_{ap}(x)$ and $f'_{ap}(x) \geqslant M$. Suppose that the lemma is not true and let $a_1 = a < x_1' < x_1'' < b = b_1$ and $f(x_1'') < f(x_1')$. In view of $f \in \mathfrak{J}$ we may assume that $p_1 \notin \langle x_1', x_1'' \rangle$. Since $f \in \mathfrak{J}$ the set $(x_1', x_1'') \cap \{x\colon f(x) > f(x_1'')\}$ is non-denumerable. Therefore there exists a point $x_1''' \in (x_1', x_1'')$ such that $f(x_1''') > f(x_1'')$ and there exists an $f'_{ap}(x_1''') \geqslant M$. Let d_1 be a positive number such that

$$|(x_1''', x_1''' + d_1) \cap \{x\colon f(x) \geqslant f(x_1''')\}| \geqslant \tfrac{7}{8} d_1,$$
$$x_1''' + d_1 < x_1'', \qquad f(x_1''' + d_1) \geqslant f(x_1''').$$

Put

(16) $$a_2' = \sup\{x\colon x \geqslant x_1''' + d_1;\ x_1''' + d_1 \leqslant t \leqslant x \Rightarrow f(t) \geqslant f(x_1''')\}.$$

Now let x_1^{IV} be a number different from all the numbers p_n such that $f(x_1^{IV}) < f(a_2')$ and $a_2' < x_1^{IV} < \min(x_1'', a_2' + d_1/8)$. Thus there exists a number $d_1' > 0$ such that $x_1^{IV} - d_1' > a_2'$ and $f(x_1^{IV} - d_1') \leqslant f(x_1^{IV})$ and the measure of the set $\{x\colon f(x) \geqslant f(x_1^{IV})\} \cap (x_1^{IV} - d_1', x_1^{IV})$ is not less than $3d_1'/4$. Now put

(17) $$b_2 = \inf\{x\colon x \leqslant x_1^{IV} - d_1';\ x \leqslant t \leqslant x_1^{IV} - d_1' \Rightarrow f(t) \leqslant f(x_1^{IV})\}$$
$$a_2 = \max(a_2', b_2 - d_1'/4).$$

Moreover, putting

(18) $$A_1 = \{x\colon f(x) \geqslant f(x_1''')\} \cap (a_2 - d_1, a_2),$$
$$B_1 = \{x\colon f(x) \leqslant f(x_1^{IV})\} \cap (b_2, b_2 + d_1'),$$

we have

(19) $$f(x') > f(x'') \quad \text{for} \quad x' \in A_1,\ x'' \in B_1$$

and

(20) $$|A_1| \geqslant \tfrac{3}{4} d_1, \qquad |B_1| \geqslant \tfrac{3}{4} d_1'.$$

Now let x_2' and x_2'' satisfy the inequality $a_2 < x_2' < x_2'' < b_2$ and let also $p_2 \in (x_2', x_2'')$ and

(21) $$f(x_2') > f(x_2'').$$

The points x_2' and x_2'' with the properties given above exist for $f \in \mathfrak{J}$ and by (17) $\{x\colon f(x) > f(b_2)\} \cap (a_2, b_2) \neq 0$.

Proceeding analogously we find numbers a_3, b_3, d_2, and d_2' and sets $A_2 \subset (a_3 - d_2, a_3)$ and $B_2 \subset (b_3, b_3 + d_2')$ such that

(22) $$b_3 - a_3 \leqslant \min(d_2/4, d_2'/4),$$

(23) $$x_2' < a_3 - d_2 < a_3 < b_3 < b_3 + d_2' < x_2'',$$

(24) $$|A_2| \geqslant \tfrac{3}{4} d_2, \qquad |B_3| \geqslant \tfrac{3}{4} d_2',$$

and

(25) $$f(x') < f(x'') \quad \text{for} \quad x' \in A_2 \text{ and } x'' \in B_2.$$

It is seen from the above considerations that one can define recurrently number sequences $\{a_n\}$, $\{b_n\}$, $\{d_n\}$, and $\{d_n'\}$ and set sequences $\{A_n\}$ and $\{B_n\}$ such that for all natural n we have

(26) $$a < a_n < a_{n+1} - d_n < a_{n+1} < b_{n+1} < b_{n+1} + d_n' < b_n < b,$$

(27) $$b_{n+1} - a_{n+1} \leqslant \min(d_n/4, d_n'/4),$$

(28) $$p_n \notin \langle a_{n+1}, b_{n+1} \rangle,$$

(29) $$A_n \subset (a_{n+1} - d_n, a_{n+1}), \qquad B_n \subset (b_{n+1}, b_{n+1} + d_n'),$$

(30) $$|A_n| \geqslant \tfrac{3}{4} d_n, \qquad |B_n| \geqslant \tfrac{3}{4} d_n'$$

and

(31) $$f(x'') < f(x') \quad \text{for} \quad x' \in A_n,\ x'' \in B_n.$$

Let x_0 be the—in view of (26) and (27)—unique point common to all the intervals $\langle a_n, b_n \rangle$. In view of (28) there exists a derivative $f'_{ap}(x_0) \geqslant M$. Consequently there exists a set E whose density at x_0 equals 1 and such that

(32) $$\frac{f(x) - f(x_0)}{x - x_0} \geqslant \frac{M}{2} \quad \text{for} \quad x \in E \text{ and } x \neq x_0.$$

By (26), (27), (29), and (30) there exists a sequence of positive integers $\{n_k\}$ such that

(33) $$E \cap A_{n_k} \neq 0 \quad \text{and} \quad E \cap B_{n_k} \neq 0 \quad \text{for} \quad k = 1, 2, \ldots$$

Now if $x' \in E \cap A_{n_k}$, $x'' \in E \cap B_{n_k}$, then by (26), (29), and (31) $x' < x''$ and $f(x'') < f(x')$, which is impossible in view of (32). Thus lemma 2 has been proved.

LEMMA 3. *Let the function $f \in \mathfrak{J}$ possess an approximate derivative in the interval (a, b), except perhaps at a denumerable set of points. Let also $f'_{ap}(x) \geqslant M > 0$ almost everywhere in (a, b). Moreover, let $\beta \in (a, b)$ be such a number that f is not monotonic in any interval of the form $(\beta, \beta + h)$.*

Then there exists for arbitrary $\varepsilon > 0$ and $\delta > 0$ a point $x_0 \in (\beta, \beta + \delta)$ such that (i) *$f(x_0) > f(\beta) - \varepsilon$,* (ii) *$f$ is not monotonic in any of the intervals $(x_0 - h, x_0)$, and* (iii) *at least one of the conditions* (a) *f is monotonic in the interval $(x_0, x_0 + h)$, for certain $h > 0$ or* (b) *there exists an $f'_{ap}(x_0)$, holds.*

Proof. Suppose that there exist a number $\varepsilon > 0$ and an interval $(\beta, \beta + \delta)$ which does not include the point x_0 satisfying conditions (i)-(iii). Since $f \in \mathfrak{J}$, there exists in the interval $(\beta, \beta + \delta)$ a non-denumerable set of points satisfying the inequality $f(x) > f(\beta) - \varepsilon/2$. Thus there exist among them points at which f'_{ap} exists. Let x_1 be one of them. Since it satisfies conditions (i) and (iii), it cannot fulfill condition (ii). Thus

x_1 lies in an interval of monotonicity of the function f, not being the left-end point of this interval. Denote by (a_1, b_1) the largest of such intervals. Since evidently $\beta < a_1 < \beta+\delta$, we have $f(a_1) \leqslant f(\beta)-\varepsilon$.

In a similar way we may find in the interval $(\beta, \beta+\delta)$ an infinite set of intervals $\{(a_n, b_n)\}$ such that (A) each of the intervals $\langle a_n, b_n\rangle$ is the maximal interval of monotonicity of the function f (is not included in another interval of monotonicity of function f), (B) $f(a_n) \leqslant f(\beta)-\varepsilon$, $f(b_n) \geqslant f(\beta)-\varepsilon/2$, (C) the set $E = \langle\beta, \beta+\delta\rangle - \bigcup (a_n, b_n)$ possesses a continuum of components, (D) the sequence (a_n, b_n) includes any interval included in $(\beta, \beta+\delta)$ and satisfying conditions (A) and (B); (E) for $n = 2, 3, \ldots$ point b_n is an accumulation point of the sets $\{a_k\}$ and $\{b_k\}$.

Let A be the boundary of the set E. Each point of the set A different from all the numbers a_n is an accumulation point of the set $\{b_n\}$. Thus removing from the set A its isolated points (if such exist) we obtain a perfect set B. Consequently the reduced function f/B should have in B a non-denumerable amount of continuity points. Thus there exists a point $x_0 \in B-\{a_n\}$ at which f/B is continuous and $f'_{\rm ap}$ exists. Since, according to what has been said above, x_0 is an accumulation point of the set $\{b_n\}$, we have $f(x_0) \geqslant f(\beta)-\varepsilon/2$. Thus point x_0 satisfies conditions (i)-(iii). The contradiction obtained proves lemma 3.

THEOREM 1. *Let a function $f \in \mathfrak{J}$ possess an approximate derivative in (a, b), except perhaps on a denumerable set of points, and let $f'_{\rm ap}(x) \geqslant 0$ almost everywhere in (a, b).*

Under these assumptions f is non-decreasing and continuous in (a, b).

Proof. Since $f \in \mathfrak{J}$, it suffices to prove the continuity of the function f.

Assume first that $f'_{\rm ap}(x) \geqslant M > 0$ almost everywhere in (a, b) and let $\{p_n\}$ be a sequence of points beyond which $f'_{\rm ap}$ exists. Suppose that f is not non-decreasing in (a, b). Thus by lemmas 1 and 2 the maximal interval of monotonicity of the function f including its end-points in (a, b) exists. Denote it by $\langle a_1, b_1\rangle$. Let δ_1 be a positive number such that (i) $4\delta_1 < b_1-a_1$, (ii) (a) the approximate derivative of function f at the point $b_1+\delta_1$ exists, or (b) f is monotonic in one interval whose left end-point is $b_1+\delta_1$, (iii) $f(b_1+\delta_1) > f(b_1)-(b_1-a_1)M/8$, (iv) the function f is not monotonic in any interval whose right end-point is $b_1+\delta_1$. The existence of such a number δ_1 immediately follows from lemma 3.

Now if $f'_{\rm ap}(b_1+\delta_1) \geqslant 0$ or (ii) (b) holds then there exist a number $h_1 > 0$ and a set $A_1 \subset (b_1+\delta_1, b_1+\delta_1+h_1) \subset (a, b)$ such that (a) $2h_1 < \delta_1$, (b) $2|A_1| > h_1$, (c) for $x \in A_1$ we have $f(x) > f(b_1)-(b_1-a_1)M/8$. If, on the other hand, $f'_{\rm ap}(b_1+\delta_1) < 0$, then there exist a number $h_1 > 0$ and a set $A_1 \subset (b_1+\delta_1-h_1, b_1+\delta_1)$ such that conditions (a), (b), and (c) are satisfied and the function f is not monotonic in any interval whose right end-point is $b_1+\delta_1-h_1$.

Let $\langle a_2, b_2\rangle \subset (d_1-h_1, d_1)$ (where $d_1 = b_1+\delta_1$ or $d_1 = b_1+\delta_1-h_1$ according to whether $f'_{\rm ap}(b_1+\delta_1) \geqslant 0$ or $f'_{\rm ap}(b_1+\delta_1) < 0$) is an interval in which f is not monotonic and which does not include the point p_1.

Introducing the notation $B_1 = \langle a_1, (a_1+b_1)/2\rangle$, we have

$$(34)\qquad |B_1| > \frac{x-a_1}{4}, \quad |A_1| > \frac{d_1-x}{4} \quad \text{for} \quad a_2 \leqslant x \leqslant b_2,$$

and

$$(35)\qquad \frac{f(x'')-f(x')}{x''-x'} \geqslant \frac{M}{2} \quad \text{for} \quad x' \in B_1,\ x'' \in A_1.$$

Since $\langle a_2, b_2\rangle$ is not a interval of monotonicity of function f, there exists, according to lemma 2, a point $x_2 \in (a_2, b_2)$ such that there exists an $f'_{\rm ap}(x_2) < M/10$.

Thus there exist numbers c_2, d_2, a_3, b_3 and sets A_2 and B_2 satisfying the conditions

$$(36)\qquad a_2 \leqslant c_2 < a_3 < b_3 < d_2 \leqslant b_2,$$

$$(37)\qquad B_2 \subset (c_2, a_3), \quad A_2 \subset (b_3, d_2),$$

$$(38)\qquad |B_2| > \frac{x-c_2}{2}, \quad |A_2| > \frac{d_2-x}{2} \quad \text{for} \quad a_3 \leqslant x \leqslant b_3$$

and

$$(39)\qquad \frac{f(x'')-f(x')}{x''-x'} < \frac{M}{8} \quad \text{for} \quad x' \in B_2,\ x'' \in A_2.$$

The interval $\langle a_3, b_3\rangle$ can be chosen in such a way that the function f is not monotonic in it and that $p_2 \notin \langle a_3, b_3\rangle$.

In fact, there exists a positive number δ_2 such that we have $p_2 \notin \langle x_2-\delta_2, x_2+\delta_2\rangle \dot{\subset} (a_2, b_2)$,

$$\left|\left\{x:\ \frac{f(x)-f(x_2)}{x-x_2} < \frac{M}{8}\right\} \cap (x_2-\delta_2, x_2)\right| > \frac{3}{4}\delta_2,$$

and

$$\left|\left\{x:\ \frac{f(x)-f(x_2)}{x-x_2} < \frac{M}{8}\right\} \cap (x_2, x_2+\delta_2)\right| > \frac{3}{4}\delta_2.$$

Thus is suffices to put

$$(40)\qquad a_3 = x_2-\tfrac{1}{8}\delta_2, \quad b_3 = x_2+\tfrac{1}{8}\delta_2, \quad c_2 = x_2-\delta_2, \quad d_2 = x_2+\delta_2$$

and

$$(41)\qquad \begin{aligned} A_2 &= \left\{x:\ \frac{f(x)-f(x_2)}{x-x_2} < \frac{M}{8}\right\} \cap (b_3, d_2), \\ B_2 &= \left\{x:\ \frac{f(x)-f(x_2)}{x-x_2} < \frac{M}{8}\right\} \cap (c_2, a_3). \end{aligned}$$

Repeating the above considerations we find number sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ and set sequences $\{A_n\}$ and $\{B_n\}$ such that $c_1 = a_1$ and for positive integer n we have the relations:

$$a_n \leqslant c_n < a_{n+1} < b_{n+1} < d_n \leqslant b_n\,, \tag{42}$$

$$A_n \subset (b_{n+1}, d_n)\,, \quad B_n \subset (c_n, a_{n+1})\,, \tag{43}$$

$$4|A_n| > d_n - x\,, \quad 4|B_n| > x - c_n \quad \text{for} \quad a_{n+1} \leqslant x \leqslant b_{n+1}\,, \tag{44}$$

$$\frac{f(x'')-f(x')}{x''-x'} \geqslant \frac{M}{2} \quad \text{for} \quad x' \in B_{2n-1},\ x'' \in A_{2n-1}\,, \tag{45}$$

$$\frac{f(x'')-f(x')}{x''-x'} < \frac{M}{8} \quad \text{for} \quad x' \in B_{2n},\ x'' \in A_{2n}\,, \tag{46}$$

$$\lim_{n\to\infty} (b_n - a_n) = 0 \tag{47}$$

and

$$p_n \notin \langle a_{n+1}, b_{n+1}\rangle\,. \tag{48}$$

Now let x_0 be a common point of the intervals $\langle a_n, b_n\rangle$. By (48) the function f possesses at the point x_0 an approximate derivative. In view of (44), (45), and (46), however, (as in the proof of lemma 1) one can prove that such a derivative does not exist. The contradiction obtained proves the validity of the theorem under the additional assumption that $f'_{\mathrm{ap}}(x) \geqslant M > 0$ almost everywhere in (a, b).

Now suppose that there exists a function f satisfying the assumption of theorem 1 which, however, is not non-decreasing. Thus there exist two points x' and x'' and a positive number M such that $x' < x''$ and $f(x') - f(x'') = 2M(x''-x')$. So the function $g(x) = f(x) + Mx$ satisfies the assumption of theorem 1 together with the additional condition under which the assertion of theorem 1 has already been proved. However, it is not non-decreasing for $g(x') > g(x'')$. Thus theorem 1 has been proved.

THEOREM 2. *Let the function f possess a derivative in the interval (a, b), except perhaps on a denumerable set. Moreover, let f take in every interval $\langle p, q\rangle \subset (a, b)$ all the values of the interval with the end-points $f(p)$ and $f(q)$. Further, let such a number $M > 0$ exist that the set given by the inequality $f'(x) \geqslant M$ is dense in (a, b). Finally let the equation $f'(x) = 0$ have in (a, b) at most a denumerable number of solutions.*

Under these assumptions f is a function non-decreasing and continuous in (a, b).

Proof. First let us observe that f is a function of the first class of Baire. This can be seen from the following argument. The discontinuity points of the function f may be only those points at which the derivative does not exist and those at which the derivative is infinite. The latter, however, would then be the asymmetry points of the function f. Hence the function has at most a denumerable set of discontinuity points and thus is a function of the first class of Baire.

By lemmas 1 and 2 there exists an open set $G \subset (a, b)$ dense in (a, b) and such that in each of its components f is non-decreasing.

Put

$$\varphi(x) = \int_a^x \operatorname{dist}(x, F)\,dx \quad \text{for} \quad a \leqslant x \leqslant b\,, \tag{49}$$

where F denotes the complement of G to (a, b). Further, let ψ be the converse function of φ. We shall show that the function

$$g(t) = f\big(\psi(t)\big) \tag{50}$$

satisfies in the interval $\big(\varphi(a), \varphi(b)\big)$ the assumptions of Zahorski's theorem (and consequently also those of theorem 1).

Indeed, the fact that the function f takes the intermediate values follows from the corresponding properties of the function f and from the continuity of the function ψ. Further, we have $g'(t) = f'\big(\psi(t)\big)\psi'(t)$ whenever the two equalities $f'\big(\psi(t)\big) = 0$ and $\psi'(t) = \infty$ do not hold simultaneously. The latter equality can be fulfilled, in view of our assumptions, only in an at most denumerable set. Since $\psi'(t)$ exists everywhere in the interval $\big(\varphi(a), \varphi(b)\big)$ and $f'(x)$ in (a, b) except on a denumerable set of points, $g'(t)$ exists in $\big(\varphi(a), \varphi(b)\big)$ except on a denumerable set of points. It is clear that $g'(t) \geqslant 0$ at all these points of the set $\varphi(G)$ at which it is defined. Since, in turn, the measure of the set $\varphi(G)$ equals $\varphi(b) - \varphi(a)$, $g'(t) \geqslant 0$ almost everywhere in $\big(\varphi(a), \varphi(b)\big)$.

By Zahorski's theorem, g is a non-decreasing function in the interval $\big(\varphi(a), \varphi(b)\big)$, and hence follows the monotonicity of the function f in the interval (a, b), which completes the proof.

Remark. In theorem 2 the assumption of denumerability of the set $\{x\colon f'(x) = 0\}$ can be replaced by the assumption of denumerability of the set $\{x\colon f'(x) = N\}$ for some $N \in (0, M)$. To prove this it suffices to consider the function $f(x) - Nx$.

THEOREM 3. *Let f possess a derivative in (a, b) except perhaps on a denumerable set and let the set $\{x\colon f'(x) \geqslant 0\}$ be dense in (a, b). Moreover, suppose that there exists a sequence of positive numbers $\{M_n\}$ which is convergent to zero and such that each of the sets $\{x\colon f'(x) = M_n\}$ is denumerable. Finally let f take in any interval $\langle p, q\rangle \subset (a, b)$ all the values from the interval with the end-points $f(p)$ and $f(q)$.*

Under these assumptions the function f is continuous and non-decreasing in the interval (a, b).

Proof. Suppose that there exist two points x' and x'' such that $x' < x''$ and $f(x') > f(x'')$. There exists such a number M_n (among the numbers of the sequence mentioned in the assumptions of the theorem) that $f(x')-f(x'') > (x''-x')M_n$. Thus the function $f(x)+M_n x$ satisfies the assumptions of theorem 2 without being a non-decreasing function. The contradiction obtained proves the validity of theorem 3.

II. Let T be a topology of the set of real numbers stronger than the natural (i.e. a family of sets closed with respect to the summation of an arbitrary and to the multiplication of a finite number of sets). Considering in the sequel a symbol or a term related with topology T we shall subscribe to it the letter "T". For instance, $\bar{A}_T$ denotes the closure of the set A with respect to the topology T. The symbol $f'_T(x)$ denotes that the T-derivative of the function f at the point x is understood as the T-limit of the appropriate difference quotient, i.e. a limit found in terms of neighbourhoods of the point x in the sense of the topology T.

A topology T stronger than the natural topology of the set of real numbers is said to satisfy condition (W) if for any point x and its T-neighbourhood U (i.e. a set from the family T including the point x) there exists a number $\delta > 0$ such that the set $(x-\delta, x+\delta)-U$ has no T-accumulation point. The condition (W) is equivalent to the condition (W'): If $x_n \in (E_n)'_T$ for $n = 1, 2, \ldots$ and $x_n - x \to 0$ as $n \to \infty$, then $x \in (\bigcup E_n)'_T$. (The equivalence of the conditions (W) and (W') has been proved in paper [1]. The definitions of T-limiting value and of the T-asymmetry point of a function, as well as the theorem on the denumerability of the set of T-asymmetry points of an arbitrary function with a topology T satisfying condition (W), which will be applied in the proof of lemma 6, can be found in the same paper.)

As will be shown later, in Zahorski's theorem the derivative of the function f may be replaced by its T-derivative with an arbitrary topology T satisfying condition (W). But first we shall prove a number of lemmas explaining the "action" of condition (W).

LEMMA 4. *Let the topology T satisfy condition* (W) *and let the value of the function f at each point of the interval (a, b) be one of its T-limit values at this point. If the T-limit g of the function f at some point $x_0 \in (a, b)$ exists, then the usual limit of the function f at the point x_0 also exists and its value equals g.*

Proof. Suppose that there exists a sequence $\{x_n\}$ such that $x_n \neq x_0$, $x_n - x_0$ converges to zero, and the sequence $\{f(x_n)\}$ converges to the limit $g' \neq g$. Assume e.g. $g' < g$ and let $0 < \varepsilon < g - g'$. Omitting, if necessary, a finite number of elements of the sequence $\{x_n\}$, we may assume that $f(x_n) < g - \varepsilon$. By assumption there exist sets E_n such that for $n = 1, 2, \ldots$

$$x_n \in (E_n)'_T \tag{51}$$

holds and

$$f(x) < g - \varepsilon \quad \text{for} \quad x \in E_n. \tag{52}$$

Since T satisfies condition (W), putting $E = \bigcup E_n$ we have

$$x_0 \in E'_T. \tag{53}$$

It follows from (52) and (53) that the number g cannot be the T-limit of the function f at point x_0. Thus the existence of a sequence $\{x_n\}$ possessing the above mentioned properties is impossible, which proves the validity of the lemma.

COROLLARY. *If a real function f of a real variable is T-continuous at any point of some interval and the topology T satisfies condition* (W), *then f is a continuous function, in the usual sense, at all points of this interval.*

LEMMA 5. *If T satisfies condition* (W), *then the interval is a T-connected set.*

Proof. If the interval could be presented as the union of two non-empty T-open sets, then the characteristic function of any of them would be T-continuous in this interval without being continuous in it.

LEMMA 6. *Let the topology T satisfy condition* (W) *and let $f'_T(x)$ exist at all points of the interval (a, b) except at a denumerable set of points. Then the function f is continuous at all points of the interval (a, b) except at a denumerable set of points.*

Proof. If $f'_T(x_0)$ is a finite number, then, as can easily be seen, f is T-continuous at x_0.

Now let $f'_T(x_0) = +\infty$ and suppose that f is not T-continuous at the point x_0. Thus at this point there exist T-limit values different from $f(x_0)$. However, since in this case the T-limit values on the right cannot be smaller than $f(x_0)$ and the T-limit values on the left cannot be greater than $f(x_0)$, x_0 is a T-asymmetry point of the function f. We should obtain a similar result in the case where $f'_T(x_0) = -\infty$.

Since the set of T-asymmetry points of the function f is an at most denumerable set, the lemma has been proved.

LEMMA 7. *Under the assumptions of lemma 6 let the function f take in every interval $\langle p, q\rangle \subset (a, b)$ all the intermediate values between $f(p)$ and $f(q)$.*

Then the function f possesses the usual derivative everywhere in (a, b) except in a denumerable set. More precisely, $f'(x)$ exists at those points $x \in (a, b)$ at which $f'_T(x)$ exists.

Proof. Suppose that at some point $x_0 \in (a, b)$ $f'_T(x_0)$ exists while $f'(x_0)$ does not exist. E.g., let the lower left derivative of the function f

at this point be smaller than $f'_T(x_0)$. Then there exist two numbers M and N and two increasing sequences $\{x'_n\}$ and $\{x''_n\}$ which converge to x_0 such that $MN > 0$, $M < N < f'_T(x_0)$, and for $n = 1, 2, \ldots$ we have

$$\frac{f(x'_n)-f(x_0)}{x'_n-x_0} < N, \tag{54}$$

$$\frac{f(x''_n)-f(x_0)}{x''_n-x_0} > M. \tag{55}$$

Choosing, if necessary, appropriate subsequences we may assume that for $n = 1, 2, \ldots$

$$x'_n < x''_n < x'_{n+1} < x''_{n+1} \tag{56}$$

holds.

Now if $N > 0$, then in view of (54), (55), and (56) we have

$$f(x'_n) < f(x_0)+N(x'_n-x_0) < f(x_0)+M(x''_n-x_0) < f(x''_n). \tag{57}$$

Thus the function f takes in the interval (x'_n, x''_n) all the values included between $f(x_0)+M(x''_n-x_n)$ and $f(x''_n)$. So in view of lemma 6 there exists in the interval (x'_n, x''_n) such a point x_n that the function f is T-continuous in it and the inequality

$$f(x_0)+M(x''_n-x_0) < f(x_n) < f(x_0) \tag{58}$$

holds.

Hence we obtain

$$\frac{f(x_n)-f(x_0)}{x_n-x_0} < M. \tag{59}$$

Similarly, if $N < 0$, there exists in the interval (x''_n, x'_{n+1}) a point x_n satisfying inequality (59) and such that f is T-continuously at it.

Thus there exists a sequence $\{U_n\}$ of T-neighbourhoods of the points x_n such that for $n = 1, 2, \ldots$ we have

$$\frac{f(x)-f(x_0)}{x-x_0} < M \quad \text{for} \quad x \in U_n. \tag{60}$$

Since, evidently, $x_n - x_0$ converges to zero, by condition (W) x_0 is a point of T-accumulation of the set defined by inequality (60). This, however, is impossible for $M < f'_T(x_0)$. Lemma 7 has thus been proved.

THEOREM 4. *Let f be a function taking all the intermediate value in an interval included in (a, b). Moreover, suppose that in some topology T, satisfying condition* (W), *$f'_T(x)$ exist everywhere in (a, b) except in a denumerable set of points, Finaly, let be $f'_T(x) \geqslant 0$ almost everywhere in (a, b).*

Then the function f is continuous and non-decreasing in (a, b).

Proof. The validity of theorem 4 follows from the fact that, in view of lemma 7, the function f satisfies the assumptions of Zahorski's theorem.

Remark. By lemma 7 in theorems 2 and 3, $f'(x)$ may be replaced by $f'_T(x)$ in an arbitrary topology satisfying condition (W). We omit the corresponding results and their immediate proofs.

References

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