observation is valid: *X is G_δ-dense in $\widetilde{X}$ if and only if X meets each nonvoid closed G_δ subset of $\widetilde{X}$.*

5.1. PROPOSITION. *If X is G_δ-dense in $\widetilde{X}$ and Y is G_δ-dense in $\widetilde{Y}$, then $X\times Y$ is G_δ-dense in $\widetilde{X}\times\widetilde{Y}$.*

5.2. THEOREM. *If $X\times Y$ is C^*-embedded in $vX\times vY$, then $X\times Y$ is C-embedded in $vX\times vY$.*

Proof. According to 1.18 of [3] it suffices to show that each nonvoid closed G_δ subset Z of $vX\times vY$ which misses $X\times Y$ satisfies a certain condition. Since X and Y are G_δ-dense in vX and vY respectively, there are by 5.1 no such sets Z.

5.3. THEOREM. *Let* card Y *be nonmeasurable and suppose that either*

(a) *Y is compact;*

or

(b) *the projection from $X\times Y$ to X is closed;*

or

(c) *$X\times Y$ is C^*-embedded in $X\times\beta Y$.*

Then $v(X\times Y)=vX\times vY$.

Proof. It suffices to deduce the desired conclusion from hypothesis (c), the implication (a)$\Rightarrow$(b) being well-known and the implication (b)$\Rightarrow$(c) being given by 3.1. According to 5.2 it is enough to show that each bounded continuous real-valued function on $X\times Y$ extends continuously to $vX\times vY$. By (c) we can extend to $X\times\beta Y$, and 2.8 takes us from there to the space $vX\times\beta Y$, which contains $vX\times vY$.

References

[1] W. W. Comfort and Kenneth A. Ross, *Pseudocompactness and uniform continuity in topological groups*, Pacific J. Math., 16 (1966), pp. 483-496.

[2] Zdeněk Frolík, *The topological product of two pseudocompact spaces*, Czech. Math. J. 10 (1960), pp. 339-349.

[3] Leonard Gillman and Meyer Jerison, *Rings of continuous functions*, Princeton, N. J., 1960.

[4] Irving Glicksberg, *Stone-Čech compactifications of products*, Trans. Amer. Math. Soc. 90 (1959), pp. 369-382.

[5] Edwin Hewitt, *Rings of real-valued continuous functions I*, Trans. Amer. Math. Soc. 64 (1948), pp. 45-99.

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On defining well-orderings

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0. Introduction. Let $L_{\alpha\omega}$ be the extension of the finitary first-order predicate logic obtained by allowing conjunctions and disjunctions of μ-sequences of formulas ($\mu<\alpha$). The purpose of this note is to show that *even allowing arbitrary many non-logical constants the notion of a well-ordered relation cannot be expressed by a set of sentences of the infinitary first-order language $L_{\alpha\omega}$* (i.e. that for all α, $\mathbf{W}\notin\mathbf{PC}_\Delta(L_{\alpha\omega})$ where $\mathbf{W}$ is the class of all non-empty well-orderings) (0).

The method used can be summarized as follows: First we determine an upper bound for the Hanf-number of $L_{\alpha\omega}$ (1). Then we show that if for some α, $\mathbf{W}\in\mathbf{PC}_\Delta(L_{\alpha\omega})$, then there would exist cardinals $\varkappa$, λ, and a sentence Φ of $L_{\varkappa\omega}$ such that (i) Φ does not have arbitrarily large models, and (ii) Φ has a model of cardinality λ and λ is equal to the upper bound previously obtained for the Hanf-number of $L_{\varkappa\omega}$.

I. The language $L_{\alpha\omega}$. It is convenient for our purposes to define the language $L_{\alpha\omega}$ in a slightly different (but clearly equivalent) way to that suggested in the introduction.

DEFINITION 1.1 (2).

(.1) T is a *pseudo-α-tree* if and only if T is a set of finite sequences of ordinals smaller than α such that (i) $0\in T$, (ii) if $\langle\mu_0,\ldots,\mu_{n-2},\mu_{n-1}\rangle\in T$, then $\langle\mu_0,\ldots,\mu_{n-2}\rangle\in T$ and for all $\delta<\mu_{n-1}$, $\langle\mu_0,\ldots,\mu_{n-2},\delta\rangle\in T$, and (iii) if $s\in T$, then $|\{\mu\colon s^\frown\langle\mu\rangle\in T\}|<\alpha$.

(0) This answers a problem raised by Professor Mostowski, namely, whether there existed an α such that $\mathbf{W}\in\mathbf{PC}_\Delta(L_{\alpha\omega})$.

(1) See Definition 4.1.

(2) Standard set-theoretical terminology will be used. In particular an *ordinal* is the set of smaller ordinals (small greek letters: μ, δ, ϱ, ξ, shall denote ordinals; ω is the smallest infinite ordinal). $\mu+\delta$ is the ordinal *sum* of μ and δ. A *cardinal* is an initial ordinal and if X is a set then $|X|$ is the cardinal of X. A function whose domain is an ordinal will also be called a *sequence*. If f and g are sequences, then $f^\frown g$ is the concatenation of f and g (i.e. the sequence h such if ξ, ζ are the domains of f and g respectively then $h=\{(\mu,f(\mu))\colon\mu\in\xi\}\cup\{(\xi+\mu,g(\mu))\colon\mu\in\zeta\}$.

(.2) If T is a pseudo-α-tree, $s \in T$, and there does not exist an ordinal μ such that $s^\frown\langle\mu\rangle \in T$, then s is an *uppermost element* of T.

(.3) If T is a pseudo-α-tree, $s \in T$ and $s' = s^\frown\langle\mu\rangle \in T$, then s is an *immediate predecessor* of s' (in T).

(.4) If T is a pseudo-α-tree, then B is a *branch of* T just in case that B is a maximal subset of T such that no two (different) elements in B have the same immediate predecessor in T.

(.5) T is an *α-tree* just in case that T is a pseudo-α-tree in which every branch is finite.

The following lemma is immediate (because an α-tree is well-founded).

Lemma 1.2. *If T is an α-tree, then to each element $s \in T$ can be associated an ordinal $d_T(s)$, called the depth of s in T, such that*

(i) *if s is an uppermost element of T, then $d_T(s) = 0$,*

(ii) *if s is not an uppermost element of T, and $s^\frown\langle\mu\rangle \in T$ if and only if $\mu < \delta$, then $d_T(s) = \bigcup\{d_T(s^\frown\langle\mu\rangle)+1\colon\ \mu < \delta\}$.*

Definition 1.3. Φ is a formula of $L_{\alpha\omega}$ ($\Phi \in L_{\alpha\omega}$) just in case that there exists an α-tree T and a function f whose domain is T and such that

(i) $f(0) = \Phi$,

(ii) if s is an uppermost element of T, then $f(s)$ is an atomic formula (of the usual finitary first-order predicate logic),

(iii) if s is not an uppermost element of T, $s^\frown\langle\mu\rangle \in T$ if and only if $\mu < \delta$ ($\delta > 0$), and for each $\mu < \delta$, $f(s^\frown\langle\mu\rangle) = \Psi_\mu$ then either (a) $f(s) = \bigvee\langle\Psi_0, \ldots, \Psi_\mu, \ldots\rangle_{\mu<\delta}$ or (b) $f(s) = \bigwedge\langle\Psi_0, \ldots, \Psi_\mu, \ldots\rangle_{\mu<\delta}$ or (c) $\delta = 1$ and $f(s) = \neg\Psi_0$, or (d) $\delta = 1$ and for some individual variable x either $f(s) = (\mathrm{E}x)\Psi_0$ or $f(s) = (\forall x)\Psi_0$ [3].

It is straightforward to verify that to each formula $\Phi \in L_{\alpha\omega}$ there corresponds a unique α-tree T and a function f satisfying the conditions stated in Definition 1.3, and hence $\langle T, f\rangle$ will be called the *formation of* Φ. Furthermore in order to make the formulas easier to read we shall let

$$\textstyle\bigvee_{\mu<\delta}\Psi_\mu = \bigvee\langle\Psi_\mu\rangle_{\mu<\delta} \quad \text{and} \quad \bigwedge_{\mu<\delta}\Psi_\mu = \bigwedge\langle\Psi_\mu\rangle_{\mu<\delta}.$$

The *set of subformulas* of a formula $\Phi \in L_{\alpha\omega}$, $\mathrm{SF}(\Phi)$ can then simply be defined as the range of f where $\langle T, f\rangle$ is the formation of Φ. The set of *free variables of a formula* Φ, $\mathrm{FV}(\Phi)$, is defined in the usual way, and a *sentence* is a formula without free variables. An important, yet trivial, result is that *if Φ is a sentence of $L_{\alpha\omega}$, then for every subformula Ψ of Φ,*

(3) $\bigvee\langle\Psi_\mu\rangle_{\mu<\delta}$ is the *disjunction* of the sequence $\langle\Psi_\mu\rangle_{\mu<\delta}$; $\bigwedge\langle\Psi_\mu\rangle_{\mu<\delta}$ is the *conjunction* of the sequence $\langle\Psi_\mu\rangle_{\mu<\delta}$.

$\mathrm{FV}(\Psi)$ is finite. By a formula in *negation normal form*, n.n.f., we understand a formula Φ of $L_{\alpha\omega}$ such that Ψ is an atomic formula whenever $\neg\Psi$ is a subformula of Φ.

For the remaining part of this paper we shall make the

Assumption I. *α is an infinite regular cardinal* [4].

A simple consequence of the assumption is that for every formula Φ of $L_{\alpha\omega}$, $|\mathrm{SF}(\Phi)| < \alpha$.

We assume that if is known under what conditions a relational system $\mathfrak{A} = \langle A, R_i\rangle_{i\in I}$ and a sequence s of elements from A satisfy a formula Φ of $L_{\alpha\omega}$; we shall express this condition by $(\mathfrak{A}, s) \models \Phi$ [5]. If for every sequence s, $(\mathfrak{A}, s) \models \Phi$, then $\mathfrak{A}$ is a *model* of Φ. If $\mathfrak{A} = \langle A, R_0, R_i\rangle_{i\in I}$ and $\mathfrak{B} = \langle A, R_0\rangle$, then $\mathfrak{B}$ *is the reduct of* $\mathfrak{A}$. If $\mathbf{K}$ is a class of (similar) relational systems then (i) $\mathbf{K} \in \mathbf{EC}(L_{\alpha\omega})$ ($\mathbf{K} \in \mathbf{EC}_\Delta(L_{\alpha\omega})$) just in case that $\mathbf{K}$ is the class of all models of a sentence of $L_{\alpha\omega}$ (a set of sentences of $L_{\alpha\omega}$), and (ii) $\mathbf{K} \in \mathbf{PC}(L_{\alpha\omega})$ ($\mathbf{K} \in \mathbf{PC}_\Delta(L_{\alpha\omega})$) if and only if there exists a class $\mathbf{M}$ of relational systems such that $\mathbf{M} \in \mathbf{EC}(L_{\alpha\omega})$ ($\mathbf{M} \in \mathbf{EC}_\Delta(L_{\alpha\omega})$) and $\mathbf{K}$ is the class of reducts of $\mathbf{M}$.

Finally it can easily be checked that (as a consequence of De Morgan's law) every formula of $L_{\alpha\omega}$ is semantically equivalent to a formula of $L_{\alpha\omega}$ in n.n.f.

II. α-relational systems.

Definition 2.1. A relational system $\mathfrak{A} = \langle A, R_0, R_1, a_\xi, R_2, \ldots\rangle_{\xi<\alpha}$ is an *α-relational system* just in case that

(i) for all $\xi \leqslant \alpha$, $\xi \in A$,

(ii) $R_0 = \{\xi\colon\ \xi \leqslant \alpha\} \subseteq A$,

(iii) $R_1 = \{\langle\mu, \varrho\rangle\colon\ \mu \leqslant \alpha,\ \varrho \leqslant \alpha \text{ and } \mu < \varrho\}$,

(iv) for each $\xi \leqslant \alpha$, $a_\xi = \xi$ (considered as a distinguished element of $\mathfrak{A}$, not as a unary relation).

Definition 2.2. By an *α-language* we understand a finitary first-order language such that its basic symbols include

(a) a unary relation symbol $\mathcal{A}$,

(b) a binary relation symbol $\precsim$,

(c) for each $\xi \leqslant \alpha$ an individual constant symbol k_ξ.

We shall need to make use of the following result.

(4) A cardinal α is regular just in case there does not exist a function f whose domain is an ordinal $\xi \in \alpha$, whose range is included in α and such that $\alpha = \bigcup_{\mu\in\xi} f(\mu)$ (cf. Bachmann [1]).

(5) See, for example, Karp [4].

THEOREM 2.3 (Helling/Morley). *If a set T of α-sentences, $|T| \leqslant \alpha$, has an α-model (i.e. a model which is isomorphic to an α-relational system) of cardinality $\beth_{(2^\alpha)^+}$ then T has α-models of arbitrary large cardinalities* (6).

M. Helling has shown that the above theorem, for the case $\alpha = \omega$, is an immediate consequence of a result of Morley [5] (in the case $\alpha = \omega$, $\beth_{(2^\alpha)^+}$ is replaced by $\beth_{\omega_1}$). Since Helling's result has not been published, we give, with Helling's permission, the proof of Theorem 2.3 which is obtained by (essentially) replacing ω by α in his proof.

Proof of Theorem 2.3. Let T' be the set of α-sentences obtained by adding to T all the sentences of the form (7):

(i) $(\forall xy)(x \precsim y \to Ax \wedge Ay)$,

(ii) $k_\mu \precsim k_\delta$ for all $\mu < \delta \leqslant \alpha$,

(iii) $(\forall xy)[x \precsim y \to \neg(x = y) \wedge \neg(y \precsim x)]$,

(iv) $(\forall xy)[Ax \wedge Ay \to x \precsim y \vee x = y \vee y \precsim x]$,

(v) $(\forall xyz)(x \precsim y \wedge y \precsim z \to x \precsim z)$,

(vi) $\neg(\mathrm{E}x)(k_\mu \precsim x \wedge x \precsim k_{\mu+1}) \qquad \mu < \alpha$,

(vii) $(\forall x)(Ax \to k_0 = x \vee k_0 \precsim x)$,

(viii) $(\forall x)(Ax \to x \precsim k_\alpha \vee x = k_\alpha)$.

Then let Σ be the following set of α-formulas:

$$\Sigma = \{Ax \wedge \neg(x = k_\xi) \colon \xi \leqslant \alpha\}.$$

Assume then that T has an α-model of cardinality at least $\beth_{(2^\alpha)^+}$ and let $\mathfrak{A}$ be such an α-model. Hence $\mathfrak{A}$ is a model of T' in which no element satisfies all the formulas in Σ. Hence by Morley's result [5], T' has arbitrary large models $\mathfrak{B}$ in which no element of $\mathfrak{B}$ satisfies all the formulas in Σ. But any model of T' in which no element satisfies all the formulas in Σ must be an α-model of T.

III. Reduction of $L_{\alpha\omega}$ to an α-language. We shall show in this section that each sentence Φ of $L_{\alpha\omega}$ can be replaced by a set $\mathrm{TR}(\Phi)$ of (finitary) α-sentences such that Φ has a model of cardinality $\varkappa$ ($\varkappa > \alpha$) if and only if $\mathrm{TR}(\Phi)$ has an α-model of cardinality $\varkappa$. The gist of the method is that a formula $\Phi = \bigvee_{\mu<\delta} \Psi_\mu$, such that $\mathrm{FV}(\Phi) \subseteq \{x_0, \ldots, x_n\}$

(6) $\beth_\mu$ is the μ-th Beth number and is defined by $\beth_\mu = \omega \cup \bigcup_{\xi<\mu} |2^{\beth_\xi}|$. Also if π is a cardinal then π^+ is the least cardinal strictly greater than π.

(7) $\wedge$, $\vee$, $\to$ and $\leftrightarrow$ are defined as usual in terms of $\neg$, $\bigwedge$ and $\bigvee$.

is in the sense mentioned above, equivalent to the following set of formulas

(i) $(\forall x_0, \ldots, x_n)(Pk_\mu x_0, \ldots, x_n \leftrightarrow \Psi_\mu), \qquad \mu < \delta$,

(ii) $(\mathrm{E}y)(y \precsim k_\delta \wedge Pyx_0, \ldots, x_n)$ (8).

LEMMA 3.1. *If Φ is a sentence of $L_{\alpha\omega}$ in* n.n.f. *and $\langle T, f\rangle$ is the formation of Φ, then there exists a function Tr_Φ whose domain is T, whose range consists of sequences (of length $<\alpha$) of formulas of an α-language and such that:*

(.1) *if $f(s)$ is either an atomic or negation of an atomic formula, then* $\mathrm{Tr}_\Phi(s) = \langle f(s)\rangle$,

(.2) *if $f(s) = C_{\mu<\delta} f(s^\frown\langle\mu\rangle)$ (where C is either $\bigvee$ or $\bigwedge$), $\mathrm{Tr}_\Phi(s^\frown\langle\mu\rangle) = \langle\Psi^\mu_\xi\rangle_{\xi<\delta_\mu}$, $\varrho = \bigcup_{\mu<\delta}\delta_\mu$, $\mathrm{FV}(f(s)) = \{x_0, \ldots, x_{n-1}\}$ $(n < \omega)$ then $\mathrm{Tr}_\Phi(s)$ is a well-ordering of the set consisting of the following formulas:*

(i) $(\forall x_0, \ldots, x_{n-1})(P_s k_\mu k_\xi x_0, \ldots, x_{n-1} \leftrightarrow \Psi^\mu_\xi)$ *where* $\mu < \delta$, $\xi < \delta_\mu$,

(ii) $(\forall x_0, \ldots, x_{n-1})(P_s k_\mu k_\xi x_0, \ldots, x_{n-1} \leftrightarrow \Psi^\mu_0)$ *and* $\mu < \delta$, $\delta_\mu \leqslant \xi < \varrho$,

(iii.a) $(\forall yz)(y \precsim k_\delta \wedge z \precsim k_\varrho \to P_s yzx_0, \ldots, x_{n-1})$ *in the case when* $C = \bigwedge$,

(iii.b) $(\mathrm{E}y)\big(y \precsim k_\delta \wedge (\forall z)(z \precsim k_\varrho \to P_s yzx_0, \ldots, x_{n-1})\big)$ *in the case when* $C = \bigvee$.

(.3) *if $f(s) = (\mathrm{Q}x_0) f(s^\frown\langle 0\rangle)$ (where Q is either $\forall$ of E), $\mathrm{Tr}_\Phi(s^\frown\langle 0\rangle) = \langle\Psi_\xi\rangle_{\xi<\delta}$, $\mathrm{FV}\big(f(s)^\frown\langle 0\rangle\big) = \{x_1, \ldots, x_{n-1}\}$, then $\mathrm{Tr}_\Phi(s)$ is a well-ordering of the set consisting of the following formulas:*

(i) $(\forall x_0, \ldots, x_{n-1})(P_s k_\xi x_0, \ldots, x_{n-1} \leftrightarrow \Psi_\xi)$ *where* $\xi < \delta$.

(ii.a) $(\forall x_0 y)(y \precsim k_\delta \to P_s yx_0, \ldots, x_{n-1})$ *in the case* $\mathrm{Q} = \forall$.

(ii.b) $(\mathrm{E}x_0)(\forall y)(y \precsim k_\delta \to P_s yx_0, \ldots, x_{n-1})$ *in the case* $\mathrm{Q} = \mathrm{E}$.

Proof. Usual definition by induction on a well-founded relation.

From Lemma 3.1 we then obtain (using a proof by induction on the depth of s in T).

LEMMA 3.2. *If Φ is a sentence of $L_{\alpha\omega}$ in* n.n.f. *$\langle T, f\rangle$ is the formation of Φ, $s \in T$, $f(s) = \Psi$, Γ is the range of $\mathrm{Tr}_\Phi(s)$, where Tr_Φ is an in Lemma* 3.1, *then*

(.1) *given any relation system $\mathfrak{A} = \langle A, R_i\rangle_{i\in I}$ and a sequence h of elements of A such that $(\mathfrak{A}, h) \models \Psi$ then there exists as α-relational system $\mathfrak{B}$ of the form $\mathfrak{B} = \langle A \cup (\alpha+1), (\alpha+1), \epsilon_{\alpha+1}, \xi, N_s, R_i\rangle_{\xi\leqslant\alpha, s\in T, i\in I}$ such that for all $\Delta \in \Gamma$, $(\mathfrak{B}, h) \models \Delta$,*

(8) And correspondingly for $\bigwedge$. This method of replacing infinite formulas by finite formulas has been used by E. Engeler in [2].

(.2) *if* h *is a sequence of elements of the non-empty set* A, $(A \cap (\alpha+1) = 0)$ $\mathfrak{B}$ *is an* α-*relational system of the form* $\mathfrak{B} = \langle A \cup (\alpha+1), (\alpha+1), \epsilon_{\alpha+1}, \xi, N_s, R_i\rangle_{\xi<\alpha, s\in T, i\in I}$ *and for all* $\Delta \in \Gamma$, $(\mathfrak{B}, h) \models \Delta$, *then* $(\mathfrak{A}, h) \models \Psi$ *where* $\mathfrak{A} = \langle A, R_i\rangle_{i\in I}$.

DEFINITION 3.3. If Φ is a sentence of $L_{\alpha\omega}$ in n.n.f., $\langle T, f\rangle$ is the formation of Φ, Tr_Φ is as in Lemma 3.1, then $\mathrm{TR}(\Phi)$ is the range of Tr_Φ (0).

Using Lemma 3.2 we then obtain:

THEOREM 3.4. *If* Φ *is a sentence of* $L_{\alpha\omega}$ *in* n.n.f., $\varkappa$ *is a cardinal greater than or equal to* α, *then* Φ *has a model of cardinality* $\varkappa$ *if and only if* $\mathrm{TR}(\Phi)$ *has an* α-*model of cardinality* $\varkappa$.

IV. Hanf-numbers of $L_{\alpha\omega}$.

DEFINITION 4.1. The Hanf-number of $L_{\alpha\omega}$, $H(L_{\alpha\omega})$, is the least cardinal $\varkappa$ such that for every sentence Φ of $L_{\alpha\omega}$, if Φ has a model of cardinality at least $\varkappa$, then Φ has arbitrary large models.

THEOREM 4.2.

$$H(L_{\alpha\omega}) \leqslant \beth_{(2^\alpha)^+}.$$

Proof. Suppose that Φ is a sentence of $L_{\alpha\omega}$ which has a model of cardinality $\varkappa \geqslant \beth_{(2^\alpha)^+}$. Hence if Φ^* is a sentence of $L_{\alpha\omega}$ in n.n.f. which is semantically equivalent to Φ, then Φ^* has a model of cardinality $\varkappa$. Thus by Theorem 3.4 $\mathrm{TR}(\Phi^*)$ has an α-model of cardinality $\varkappa$. Thus by Theorem 2.3 $\mathrm{TR}(\Phi^*)$ has arbitrary large α-models. Hence by Theorem 3.4 Φ^* (and thus also Φ) has arbitrary large models.

V. Non-definability of well-orderings.

DEFINITION 5.1. Or_μ is the formula defined by the recursion

$$\mathrm{Or}_\mu = \big[\textstyle\bigwedge_{\xi<\mu}(\mathrm{E}y)\big(y < x \wedge (\mathrm{E}x)(x = y \wedge \mathrm{Or}_\xi)\big)\big]$$
$$\big[(\forall y)\big(y < x \rightarrow \textstyle\bigvee_{\xi<\mu}(\mathrm{E}x)(x = y \wedge \mathrm{Or}_\xi)\big)\big].$$

The following lemma can easily be proved by induction on μ (cf. [6]).

LEMMA 5.2.

(.1) *If* $\mu < \alpha$, *then* Or_μ *is a formula of* $L_{\alpha\omega}$.

(.2) *If* $\mathfrak{A} = \langle A, R\rangle$, *is a linearly ordered system and* $a \in A$, *then* $(\mathfrak{A}, \langle a\rangle) \models \mathrm{Or}_\mu$ *if and only if the set of* R-*predecessors of* a *has order type* μ.

DEFINITION 5.3.

$$\mathbf{W} = \{\langle A, R\rangle\colon A \neq 0 \text{ and } R \text{ well-orders } A\}.$$

Finally we come to the main theorem.

THEOREM 5.4. *For every* α, $\mathbf{W} \notin \mathbf{PC}_\Delta(L_{\alpha\omega})$.

Proof. Suppose that on the contrary that for some α:

$$(*) \qquad \mathbf{W} \in \mathbf{PC}_\Delta(L_{\alpha\omega}).$$

Let then Γ be the set of sentences of $L_{\alpha\omega}$ which relatively characterize the notion of a well-ordering (i.e. such that $\mathbf{W}$ is the class of reducts of models of Γ). Suppose that in addition to the binary relation symbol $<$, the non-logical constants occurring in Γ are P_i, $i \in I$. It is clear that we may assume that $I \cap \omega = 0$. Next let $\pi = |\Gamma| \cup \alpha$ and $\beta = \pi^+$. Furthermore let Φ be the conjunction of the sentences in Γ. Then Φ is a sentence of $L_{\beta\omega}$.

Next let P_1, P_2, P_3, P_4, P_5 be binary relation symbols, P_6 a ternary relation symbol, P_7 a unary relation symbol and $k_0, k_1, \ldots, k_6$ be individual constants (none occurring in Γ). Then we let Ψ be the conjunction of the following sentences of $L_{\beta\omega}$ (⁹):

(1) $(\forall xy)\big(P_1xy \leftrightarrow x < y \wedge \neg(\mathrm{E}z)(x < z \wedge z < y)\big)$,

(2) $(\forall x)\Big(P_7x \leftrightarrow k_0 < x \wedge (\forall y)\big(y < x \rightarrow (\mathrm{E}z)(y < z \wedge z < x)\big)\Big)$,

(3) $(\mathrm{E}x)(x = k_0 \wedge \mathrm{Or}_0)$,

(4) $(\mathrm{E}x)(x = k_1 \wedge \mathrm{Or}_\omega)$,

(5) $(\mathrm{E}x)(x = k_2 \wedge \mathrm{Or}_\pi)$,

(6) $(\forall xy)\big(x < k_3 \wedge y < k_3 \rightarrow [(\forall z)(P_2xz \leftrightarrow P_2yz) \rightarrow x = y]\big)$,

(7) $(\forall xy)(P_2xy \rightarrow y < k_2)$,

(8) $(\forall xy)\big(x < k_4 \wedge y < k_4 \rightarrow [(\forall z)(P_3xz \leftrightarrow P_3yz) \rightarrow x = y]\big)$,

(9) $(\forall xy)(P_3xy \rightarrow y < k_3)$,

(10) $(\forall xy)\big(x < k_5 \wedge y < k_5 \rightarrow [(\forall z)(P_4xz \leftrightarrow P_4yz) \rightarrow x = y]\big)$,

(11) $(\forall xy)(P_4xy \rightarrow y < k_4)$,

(12) $(\forall wxy)\big(x < w \wedge y < w \rightarrow [(\forall z)(P_6wxz \leftrightarrow P_6wyz) \rightarrow x = y]\big)$,

(13) $(\forall xyz)(P_5xy \wedge P_5xz \rightarrow y = z)$,

(14) $(\forall x)\big(x < k_5 \vee x = k_5 \rightarrow (\mathrm{E}y)(P_5xy)\big)$,

(⁹) The author is indebted to Hanf [3] for the use of these sentences. In [3] Hanf deals with another type of infinite languages, namely certain infinitary languages in which the notion of a well-ordering can be expressed by a sentence of the language. For those languages Hanf shows that the Hanf-number must be exceedingly large. Reading [3], the author realized that (because the Hanf-numbers for $L_{\alpha\omega}$ are relatively small) the arguments used by Hanf could be used (in the reverse direction) to show that well-orderings could not be characterized (nor relatively characterized) in the languages $L_{\alpha\omega}$.

(15) $(\forall xy)(P_5xy \to x < k_5 \vee x = k_5)$,

(16) $P_5k_0k_1 \wedge P_5k_5k_6$,

(17) $(\forall uvwx)\big(P_1uv \wedge P_5uw \wedge P_5vx \to (\forall yz)(P_6xyz \to z < w)\big)$,

(18) $(\forall vw)\big(P_7v \wedge P_5vw \to (\forall x)[x < w \to (\mathrm{E}yz)(y < v \wedge P_5yz \wedge x < z)]\big)$,

(19) $(\forall x)(x < k_6 \vee x = k_6)$.

Next let $\varkappa'' = |2^{\pi}|$, $\varkappa' = |2^{\varkappa''}|$ and $\varkappa = |2^{\varkappa'}|$. Note that from Theorem 4.2 we obtain that:

(+) $$H(L_{\beta\omega}) \leqslant \beth_{\varkappa}.$$

Hence to show that the assumption (*) leads to a contradiction it suffices to show (because $\Phi \wedge \Psi$ is a sentence of $L_{\beta\omega}$) that:

(I) *The cardinality of the models of $\Phi \wedge \Psi$ are strictly smaller than $\beth_{\varkappa^+}$.*

(II) *There exists a model of $\Phi \wedge \Psi$ of cardinality $\beth_{\varkappa}$.*

Concerning (I). Suppose that $\mathfrak{A}$ is a model of $\Phi \wedge \Psi$. Then because $\mathfrak{A}$ is then a model of Φ we may assume that for some ordinal ϱ, $\mathfrak{A}$ is of the form:

$$\mathfrak{A} = \langle \varrho, \epsilon_{\varrho}, R_1, \ldots, R_7, \xi_0, \ldots, \xi_6, R_i\rangle_{i\in I}.$$

Because of the sentences (3)-(5) we immediately obtain:

$$\xi_0 = 0, \quad \xi_1 = \omega, \quad \xi_2 = \pi.$$

From (19) we deduce that $\xi_6+1 = \varrho$.

If we let

$$f = \big\{\langle\mu, X\rangle\colon \mu < \xi_3 \text{ and } X = \{\zeta\colon \langle\mu, \zeta\rangle \in R_2\}\big\},$$

then from (6) and (7) we conclude that f is a (1-1) function whose domain is ζ_3 and whose range is included in the power set of π $(= \xi_2)$. Thus $|\xi_3| \leqslant |2^{\pi}| = \varkappa''$. Similarly using (8)-(11) we obtain that $|\xi_4| \leqslant \varkappa'$ and that $|\xi_5| \leqslant \varkappa$. That is we have shown that:

(i) $$\xi_5 < \varkappa^+.$$

From sentences (1) and (2) we obtain that $\langle\mu, \delta\rangle \in R_1$ if and only if $\mu+1 = \delta$ and that $\mu \in R_7$ if and only if μ is a non-zero limit ordinal.

Next let for each μ $(\mu \in \varrho)$:

$$g_{\mu} = \big\{\langle\delta, X\rangle\colon \delta < \mu \text{ and } X = \{\zeta\colon \langle\mu, \delta, \zeta\rangle \in R_6\}\big\}.$$

Then from sentence (12) we obtain that for each μ, g_{μ} is a (1-1) function whose domain is μ.

Sentences (13)-(16) inform us that R_5 is a function whose domain is ξ_5+1 and such that:

(ii) $$R_5(0) = \omega \quad \text{and} \quad R_5(\xi_5) = \xi_6.$$

Sentences (17) and (18) are then use to prove by induction on μ $(\mu \leqslant \xi_5)$ that:

(iii) $$|R_5(\mu)| \leqslant \beth_{\mu}.$$

Combining (i), (ii) and (iii) we then obtain that

$$|\varrho| = |R_5(\xi_5)| \leqslant \beth_{\xi_5} < \beth_{\varkappa^+}.$$

Thus (I) is shown.

Concerning (II). Let $\delta = \beth_{\varkappa}+1$. Since Φ relatively characterizes the well-orderings, there must exist relations S_i, $i \in I$ such that $\langle\delta, \epsilon_{\delta}, S_i\rangle_{i\in I}$ is a model of Φ. Let then:

$$\eta_0 = 0,\ \eta_1 = \omega,\ \eta_2 = \pi,\ \eta_3 = |2^{\pi}|,\ \eta_4 = |2^{\eta_3}|,\ \eta_5 = |2^{\eta_4}|,\ \eta_6+1 = \delta.$$

$$S_1 = \{\langle\mu, \mu+1\rangle\colon \mu \in \delta\},$$

$$S_5 = \{\langle\mu, \epsilon_{\mu}\rangle\colon \mu \leqslant \eta_5\},$$

$$S_7 = \{\mu\colon 0 < \mu < \delta \text{ and } \mu \text{ is a limit ordinal}\}.$$

Since for each $\mu \leqslant \eta_5$ there exists a (1-1) function f_{μ} mapping the power set of $\beth_{\mu}$ onto $\beth_{\mu+1}$ relations S_2, S_3, S_4 and S_6 can then be found so that

$$\langle\delta, \epsilon_{\delta}, S_1, \ldots, S_7, \eta_0, \ldots, \eta_6, S_i\rangle_{i\in I}$$

is a model of $\Phi \wedge \Psi$ of cardinality $\beth_{\varkappa}$.

References

[1] H. Bachmann, *Transfinite Zahlen*, Berlin 1955.

[2] E. Engeler, *Unendliche Formeln in der Modelltheorie*, Zeits. für math. Logik und Grund. der Math. 7 (1961), pp. 154-160.

[3] W. P. Hanf, *Some fundamental problems concerning languages with infinitely long expressions*, Doctoral Dissertation, University of California, Berkeley, California.

[4] C. Karp, *Languages with expressions of infinite length*, Studies in Logic, Amsterdam 1964.

[5] M. Morley, *Omitting classes of elements*, article in *The Theory of Models*, Studies in Logic, Amsterdam 1966.

[6] D. Scott, *Logic with denumerably long formulas and finite trings of quantifiers*, artcile in *The Theory of Models*, Studies in Logic, Amsterdam 1966.

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