Characterization of the sets of angular and global convergence, and of the sets of angular and global limits, of functions in a half-plane *

by

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In the Euclidean plane provided with a Cartesian coordinate system, let $\mathcal{H}$ denote the upper half-plane and call the horizontal x-axis R. By an *angle* at a point $x \in R$ we mean a set of the form

$$\Delta(x, a, \beta) = \{x + re^{i\theta}: 0 < r < +\infty, \ a < \theta < \beta\}$$
$$(0 < a < \beta < \pi, \ i = \sqrt{-1}).$$

We shall be concerned with functions f that are defined and single-valued in $\mathcal{H}$ and assume finite real values. We call a point $x_0 \in R$ a *point of global convergence* of f provided that there exists a finite real number y_0 for which

$$\lim_{\substack{z \to x_0 \\ z \in \mathcal{H}}} f(z) = y_0 ;$$

y_0 is then termed the *global limit* of f at x_0. A point $x \in R$ is called a *point of angular convergence* of f provided that there exists a finite real number y for which

$$\lim_{\substack{z \to x \\ z \in \Delta}} f(z) = y \quad \text{for every angle } \Delta \text{ at } x ;$$

y is then termed the *angular limit* of f at x. The set A_0 of all points of global convergence of f will be called the *set of global convergence* of f, and the set A of all points of angular convergence of f will be referred to as the *set of angular convergence* of f. Then clearly $A_0 \subset A$ ("$\subset$" stands for set inclusion, not necessarily proper). If, for every $x \in A$, $\varphi(x)$ denotes the angular limit of f at x, then φ is a single-valued real-valued function

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defined on A, which we call the *boundary function* of f. We designate the set $\varphi(A)$ as the *set of angular limits* of f and the set $\varphi(A_0)$ as the *set of global limits* of f.

In this setting, our theorems afford a complete characterization, for a function continuous in $\mathcal{H}$, of the sets of angular and global convergence, of the limit function, and of the sets of angular and global limits.

THEOREM 1. *Let f be an arbitrary real-valued function with domain $\mathcal{H}$. Let A_0 be the set of global convergence, let A be the set of angular convergence, and let φ be the boundary function of f. Then A_0 is a G_δ, A is an $F_{\sigma\delta}$, φ is of Baire class one on A and continuous (relative to A) on A_0, and $A-A_0$ is a set of first category.*

Proof. Throughout the proof, k, m and n denote natural numbers. Set

$$S_n = \{x+iy:\ -\infty < x < +\infty,\ 0 < y < 1/n\}.$$

That A_0 is a G_δ is pointed out by Hausdorff ([2], p. 275). We now prove that A is an $F_{\sigma\delta}$. Set

$$\Delta(x, n) = \Delta(x, 1/n, \pi - 1/n).$$

For each (k, m, n) the set $F_{k,m,n}$ of points $x \in R$ such that

$$|f(z')-f(z'')| < 1/k \quad \text{if} \quad z', z'' \in \Delta(x, n) \cap S_m$$

is closed. Clearly

$$A = \bigcap_{k,n} \bigcup_m F_{k,m,n},$$

and so A is an $F_{\sigma\delta}$.

To prove that φ is of the first class on A, it is sufficient to prove that for each real number y each of the sets $\{x \in A:\ \varphi(x) > y\}$ and $\{x \in A:\ \varphi(x) < y\}$ is an F_σ relative to A ([2], p. 248). Let y be a real number, and set $\Delta_x = \Delta(x, \pi/4, 3\pi/4)$. For each (k, n) the set

$$F_{k,n} = \left\{x:\ f(\Delta_x \cap S_n) \subset \left\{y':\ y' \geqslant y + \frac{1}{k}\right\}\right\}$$

is closed. Thus since

$$\Big(\bigcup_{k,n} F_{k,n}\Big) \cap A = \{x \in A:\ \varphi(x) > y\},$$

the set $\{x \in A: \varphi(x) > y\}$ is an F_σ relative to A. Similarly, $\{x \in A: \varphi(x) < y\}$ is an F_σ relative to A.

It is clear that φ is continuous on A_0, and the fact that $A-A_0$ is of first category follows from a theorem of Collingwood ([1], p. 1241, Theorem 4). Thus the proof of Theorem 1 is complete.

THEOREM 2. *Let A_0 be a G_δ in R, let A be an $F_{\sigma\delta}$ containing A_0, and let φ be a real-valued function of Baire class one on A that is continuous (relative to A) on A_0. Then there exists a function f, real-valued and continuous in $\mathcal{H}$ such that*

(1) *A is the set of angular convergence and φ is the boundary function of f, and*

(2) *at each $x \in A_0$, f has the global limit $\varphi(x)$.*

Moreover, if, in addition, $A-A_0$ is a set of first category, then there exists a function f, real-valued and continuous in $\mathcal{H}$, such that (1) *and*

(3) *A_0 is the set of global convergence of f.*

Proof. We assume the hypotheses of the theorem except that we at present do not assume that $A-A_0$ is a set of first category.

We first prove the following

LEMMA. *Let U' be an open set, and let H' be an F_σ such that*

$$A_0 \subset U' \subset H' \quad \textit{and} \quad A \subset H'.$$

Let ψ be a function of the first class on A that is continuous (relative to A) on A_0, and let ε be a positive number. Then there exist an open set U containing A_0, an F_σ set H containing $U \cup A$, and a function Ψ defined on H such that

$$U \subset U', \quad H \subset H',$$

Ψ is of the first class on H and is continuous on U, the range of Ψ on $H-U$ is an isolated set, and

$$|\Psi(x) - \psi(x)| < \varepsilon \quad \textit{if} \quad x \in A.$$

Proof of the lemma. For each $x \in A_0$, let I_x be an open interval with midpoint x such that the diameter of $\psi(I_x \cap A)$ is less than $\varepsilon/4$. Set

$$U = \Big(\bigcup_{x \in A_0} I_x\Big) \cap U',$$

$$\Psi^*(x) = \sup\{\psi(x'):\ I_{x'} \ni x\} \quad (x \in U),$$

$$\Psi_*(x) = \inf\{\psi(x'):\ I_{x'} \ni x\} \quad (x \in U).$$

Then Ψ^* is lower semi-continuous and Ψ_* is upper semi-continuous (on U). Let Ψ be a continuous function on U such that

$$\Psi_*(x) \leqslant \Psi(x) \leqslant \Psi^*(x) \quad (x \in U)$$

([2], p. 248). By a simple calculation we find that

$$|\Psi(x) - \psi(x)| < \varepsilon \quad \text{if} \quad x \in U \cap A.$$

Let $\mathfrak{J}$ be the family of open intervals with rational endpoints and length less than $\varepsilon/4$. Since ψ is of the first class on A, each of the sets

$$\psi^{-1}(I) = \{x \in A\colon \psi(x) \in I\} \qquad (I \in \mathfrak{J})$$

is an F_σ relative to A ([2], p. 248). For each $I \in \mathfrak{J}$, let H_I be an (absolute) F_σ such that

$$H_I \cap A = \psi^{-1}(I).$$

Set

$$H^* = \Big(\bigcup_{I \in \mathfrak{J}} H_I\Big) \cap H'.$$

Then H^* is an F_σ. Let a_I denote the midpoint of the interval I, and set

$$\psi^*(x) = \sup\{a_I\colon H_I \ni x,\ I \in \mathfrak{J}\} \qquad (x \in H^*),$$
$$\psi_*(x) = \inf\{a_I\colon H_I \ni x,\ I \in \mathfrak{J}\} \qquad (x \in H^*).$$

Then in the notation of [2], p. 235, ψ^* is of class $(F_\delta, *)$, and ψ_* is of class $(*, G_\delta)$. Thus ([2], pp. 242, 243) there exists a function $\hat{\psi}$ of the first class on H^* such that

$$\psi_*(x) \leqslant \hat{\psi}(x) \leqslant \psi^*(x) \qquad (x \in H^*).$$

By a simple calculation,

$$|\hat{\psi}(x) - \psi(x)| \leqslant 3\varepsilon/4 \quad \text{if} \quad x \in A.$$

Let $\tilde{\psi}$ be a function of the first class on H^* that has an isolated range and satisfies

$$|\tilde{\psi}(x) - \hat{\psi}(x)| < \varepsilon/4 \qquad (x \in H^*)$$

([2], p. 247). Then

$$|\tilde{\psi}(x) - \psi(x)| < \varepsilon \quad \text{if} \quad x \in A.$$

Let $H = H^* \cup U$, and extend the definition of Ψ to all of H by

$$\Psi(x) = \tilde{\psi}(x) \quad \text{if} \quad x \in H - U.$$

Since a function is of the first class if and only if the preimage of each open set is an F_σ, it follows easily that Ψ is of the first class on H. Since Ψ clearly has the desired properties, the proof of the lemma is complete.

Let $\{U_n^*\}$ be a sequence of open sets, and let $\{H_n^*\}$ be a sequence of sets F_σ such that $U_n^* \subset H_n^*$,

$$A_0 = \bigcap_{n=1}^{\infty} U_n^* \quad \text{and} \quad A = \bigcap_{n=1}^{\infty} H_n^*.$$

We now define sequences $\{U_n\}$, $\{H_n\}$ and $\{\varphi_n\}$ inductively. Applying the lemma, we let U_1 be an open set, let H_1 be an F_σ, and let φ_1 be a function defined on H_1 such that

$$A_0 \subset U_1 \subset H_1, \quad A \subset H_1,$$
$$U_1 \subset U_1^*, \quad H_1 \subset H_1^*,$$

φ_1 is of the first class on H_1 and continuous on U_1, the range of φ_1 on $H_1 - U_1$ is isolated, and

$$|\varphi_1(x) - \varphi(x)| < \frac{1}{4^3} \quad \text{if} \quad x \in A.$$

Suppose now that U_j, H_j and φ_j are defined for $j = 1, \ldots, n-1$ $(n > 1)$ so that $(j = 1, \ldots, n-1)$

(4) U_j is an open set, H_j is an F_σ,

(5) $A_0 \subset U_j \subset H_j, \quad A \subset H_j,$

(6) $U_j \subset U_j^*, \quad H_j \subset H_j^*,$

(7) φ_j is a function of the first class on H_j that is continuous on U_j,

(8) the range of φ_j on $H_j - U_j$ is isolated,

(9) $$\Big|\varphi(x) - \sum_{k=1}^{j} \varphi_k(x)\Big| < \frac{1}{4^{j+2}} \quad \text{if} \quad x \in A,$$

and

(10) for $j > 1$ and $x \in H_j$, $|\varphi_j(x)| < \dfrac{1}{4^j}$.

Then $\varphi - \sum_{j=1}^{n-1} \varphi_j$ is of the first class on A and continuous (relative to A) on A_0. Thus, from the lemma, there exists an open set U'_n containing A_0, an F_σ set H'_n containing $U'_n \cup A$, and a function φ_n defined on H'_n, such that

$$U'_n \subset U_n^*, \quad H'_n \subset H_n^*,$$

φ_n is of the first class on H'_n and continuous on U'_n, the range of φ_n on $H'_n - U'_n$ is isolated, and

(11) $$\Big|\varphi_n(x) - \Big(\varphi(x) - \sum_{j=1}^{n-1} \varphi_j(x)\Big)\Big| < \frac{1}{4^{n+2}} \quad \text{if} \quad x \in A.$$

It follows from (9) and (11) that

$$|\varphi_n(x)| < \frac{1}{4^n} \quad \text{if} \quad x \in A.$$

Set

$$U_n = \left\{x \in U_n' \colon |\varphi_n(x)| < \frac{1}{4^n}\right\}$$

and

$$H_n'' = \left\{x \in H_n' \colon |\varphi_n(x)| < \frac{1}{4^n}\right\}.$$

Then U_n is open, and since H_n'' is an F_σ relative to the F_σ set H_n', it is an (absolute) F_σ. Also, $A_0 \subset U_n' \cap A \subset U_n$, and $A \subset H_n''$. Set

$$H_n = U_n \cup H_n''.$$

Since $U_n' \cap H_n'' \subset U_n$, it follows that

$$H_n - U_n \subset H_n' - U_n';$$

and we see that the range of φ_n on $H_n - U_n$ is isolated. Clearly,

$$|\varphi_n(x)| < \frac{1}{4^n} \quad \text{if} \quad x \in H_n.$$

Thus U_j, H_j and φ_j $(j = 1, \ldots, n)$ satisfy conditions (4) through (10) for $j = 1, \ldots, n$.

We note that the above description works equally well for the case $n = 2$, although statement (10) is vacuous for this case. Thus we may suppose that we have U_j, H_j and φ_j $(j \geqslant 1)$ defined so that for each j, statements (4) through (10) hold.

Observe that from (5) and (6) we have

(12) $$A_0 = \bigcap_{n=1}^{\infty} U_n, \qquad A = \bigcap_{n=1}^{\infty} H_n.$$

For each open interval $I = (x_1, x_2)$ and each real number t satisfying $0 < t \leqslant 1$, define the cross cut $C(I, t)$ of $\mathcal{H}$ as follows. Set

$$a = \frac{x_2 + x_1}{2} + it\frac{x_2 - x_1}{2},$$

and let $C(I, t) = C_1 \cup C_2$, where C_j is the shorter of the two arcs (including a and excluding x_j) with endpoints a and x_j of the circle through a that is tangent to R at x_j. The definition of the modified cross cut $C_n(I, t)$ agrees with that of $C(I, t)$ except that whenever $x_j \notin H_n$, we replace C_j by the rectilinear segment (including a and excluding x_j) joining a and x_j. Let $\Delta(I, t)$ and $\Delta_n(I, t)$ be the interiors of the Jordan curves $C(I, t) \cup \bar{I}$ and $C_n(I, t) \cup \bar{I}$, respectively (the bar denotes closure).

We now think of n as fixed, and suppose for the sake of our notation that U_n has infinitely many components; modifications for the case

in which U_n has only finitely many components will be obvious. Let $\{U_{n,j}\}_{j=1}^{\infty}$ be an enumeration of the components of U_n, and let $\{F_{n,j}\}_{j=1}^{\infty}$ be a sequence of closed sets such that $F_{n,j} \subset F_{n,j+1}$ $(j \geqslant 1)$ and

$$H_n - U_n = \bigcup_{j=1}^{\infty} F_{n,j}.$$

Set $F_{n,1}' = F_{n,1}$,

$$F_{n,j}' = F_{n,j} \cup [\bigcup_{k=1}^{j-1} \bar{U}_{n,k}] \qquad (j > 1),$$

and let $\{V_{n,j,k}\}_k$ be an enumeration of the (possibly only finitely many) components of $R - F_{n,j}'$. With each $V_{n,j,k}$ we associate a number $t_{n,j,k}$ such that

$$0 < t_{n,j,k} \leqslant \tfrac{1}{4},$$

(13) the diameter of the circles in the definition of $C(V_{n,j,k}, 4t_{n,j,k})$ is greater than 1,

and

(14) if $V_{n,j,k_1} \subset V_{n,j-1,k_2}$ $(j > 1)$, then $4t_{n,j,k_1} < t_{n,j-1,k_2}$.

Note that $U_{n,j} \cap F_{n,j}' = \emptyset$, and let $k_{n,j}$ be the natural number such that, with the notation

$$V(n, j) = V_{n,j,k_{n,j}},$$

it is the case that $U_{n,j} \subset V(n, j)$. If $U_{n,j}$ and $V(n, j)$ have an endpoint x_0 in common, then $x_0 \in F_{n,j}$, and thus $x_0 \in H_n$. It follows that there exists a positive number $t_{n,j}$ such that $(t_{n,j} \leqslant 1)$

(15) $$C_n(U_{n,j}, t_{n,j}) \subset \Delta\big(V(n, j), t_{n,j,k_{n,j}}\big).$$

For each $V_{n,j,k}$, set

(16) $$f_n(z) = \begin{cases} 0 & \text{if} \quad z \in C(V_{n,j,k}, 3t_{n,j,k}), \\ \dfrac{1}{4^n} & \text{if} \quad z \in C(V_{n,j,k}, 2t_{n,j,k}), \end{cases}$$

and for each $U_{n,j}$, set $(z = x + iy)$

(17) $$f_n(z) = \begin{cases} 0 & \text{if} \quad z \in C_n(U_{n,j}, \tfrac{3}{4}t_{n,j}), \\ 1/4^n & \text{if} \quad z \in C_n(U_{n,j}, \tfrac{1}{2}t_{n,j}), \\ \varphi_n(x) & \text{if} \quad z \in \bar{\Delta}(U_{n,j}, \tfrac{1}{4}t_{n,j}) \cap \mathcal{H}. \end{cases}$$

Since $F_{n,j-1}' \subset F_{n,j}'$ $(j > 1)$, it follows from (14) that definition (16) is possible; and from (15) we see that (17) is compatible with (16). At this point we have defined $f_n(z)$ on a set which is closed relative to $\mathcal{H}$ and is contained in the open set

$$\Delta_n = [\bigcup_{j,k} \{\Delta(V_{n,j,k}, 4t_{n,j,k}) - \bar{\Delta}(V_{n,j,k}, t_{n,j,k})\}] \cup [\bigcup_j \Delta_n(U_{n,j}, t_{n,j})].$$

We now suppose that the range of φ_n on H_n-U_n is an infinite set; modifications for the case in which it is finite will be obvious. Let $\{a_{n,j}\}_{j=1}^{\infty}$ be an enumeration of the range of φ_n on H_n-U_n. Since the range of φ_n on H_n-U_n is isolated, for each (n, j) the set

$$H_{n,j}=\{x \in H_n-U_n\colon\ \varphi_n(x)=a_{n,j}\}$$

is an F_σ relative to the F_σ set H_n-U_n, and is therefore an (absolute) F_σ. Let $\{F_{n,j,k}\}_{k=1}^{\infty}$ be a sequence of closed sets such that

(18) $$F_{n,j,k}\subset F_{n,j,k+1} \qquad (k\geqslant 1)$$

and

$$H_{n,j}=\bigcup_{k=1}^{\infty}F_{n,j,k}.$$

Then for (n, k) fixed, $\{F_{n,j,k}\}_{j=1}^{k}$ is a pairwise disjoint family of closed sets, and we can find a positive number $r_{n,k}$ such that $r_{n,k}<1/k$ and, with the notation

$$S_{n,j,k}=\bigcup_{x\in F_{n,j,k}}\{\zeta\colon\ |\zeta-(x+ir_{n,k})|\leqslant r_{n,k}\} \qquad (j=1, \ldots, k),$$

it is the case that $\{S_{n,j,k}\}_{j=1}^{k}$ is a pairwise disjoint family of closed sets. For each (n, j, k) $(j\leqslant k)$ set

(19) $$f_n(z)=a_{n,j} \quad \text{if} \quad z\in S_{n,j,k}\cap(\mathcal{H}-\Delta_n).$$

By (16), (17), and (19), f_n is defined on a set S_n that is closed relative to $\mathcal{H}$, and f_n is continuous on S_n. From (10), (16), (17), and (19),

$$|f_n(z)|\leqslant\frac{1}{4^n} \qquad (z\in S_n,\ n>1).$$

Thus, by Tietze's theorem, f_n has a continuous extension f_n to all of $\mathcal{H}$; and in the case $n>1$, we can require that

(20) $$|f_n(z)|\leqslant\frac{1}{4^n} \qquad (z\in\mathcal{H},\ n>1).$$

Clearly,

(21) if $x\in U_n$, then f_n has the global limit $\varphi_n(x)$ at x.

Suppose now that $x\in H_n-U_n$. From (13), if $x\notin V_{n,j,k}$, then

$$\Delta(V_{n,j,k}, 4t_{n,j,k})\cap\{\zeta\colon\ |\zeta-(x+i)|<1\}=\emptyset.$$

Keeping n fixed, we note that x is in only finitely many of the sets $V_{n,j,k}$. Thus, in particular, we have from (15) that for all sufficiently large j,

$$\Delta_n(U_{n,j}, t_{n,j})\cap\{\zeta\colon\ |\zeta-(x+i)|<1\}=\emptyset.$$

Hence, for some sufficiently small positive number h,

$$\Delta_n\cap\{\zeta\colon\ |\zeta-(x+ih)|<h\}=\emptyset.$$

We see from (18) that for some sufficiently large k,

$$x\in\bigcup_{j=1}^{k}F_{n,j,k}.$$

Thus it follows from (19) that

(22) if $x\in H_n$, then f_n has the angular limit $\varphi_n(x)$ at x.

Suppose now that $x\notin H_n$. If $x\in\bigcup_{j=1}^{\infty}\overline{U}_{n,j}$, then it follows from the definition of $C_n(I, t)$ and (17) that there exists an angle at x in which the oscillation of f_n at x is at least $1/4^n$. If $x\notin\bigcup_{j=1}^{\infty}\overline{U}_{n,j}$, then for each j, $x\notin F'_{n,j}$, and there exists k_j such that $x\in V_{n,j,k_j}$. Thus from (14) and (16), f_n has an oscillation at x of at least $1/4^n$ in each angle at x. Hence,

(23) if $x\notin H_n$, then there exists an angle at x in which the oscillation of f_n at x is at least $1/4^n$.

Set

$$f(z)=\sum_{n=1}^{\infty}f_n(z) \qquad (z\in\mathcal{H}).$$

From (20) we obtain $(m\geqslant 1,\ z\in\mathcal{H})$

(24) $$\sum_{n=m+1}^{\infty}|f_n(z)|\leqslant\frac{1}{3}\cdot\frac{1}{4^m};$$

in particular, f is continuous in $\mathcal{H}$.

It is clear from (9), (12), (21), and (24) that (2) holds. Similarly it is clear from (9), (12), (22), and (24) that if $x\in A$, then f has the angular limit $\varphi(x)$ at x. Suppose that $x\notin A$. We wish to prove that f does not have an angular limit at x. Applying (12), let m be the least natural number n such that $x\notin H_n$. If $m>1$, we have from (22) that $\sum_{n=1}^{m-1}f_n$ has an angular limit at x. Thus from (23) and (24), there exists an angle at x in which the oscillation of f at x is positive, and we have established (1).

We now assume that $A-A_0$ is a set of first category. Then $A-A_0$ is contained in an F_σ set H of first category. By considering the set

$$U_n\cup[H\cap(H_n-U_n)],$$

we see that we may suppose H_n to have been chosen so that $H_n - U_n$ is a set of first category. Let $x \in A - A_0$, and let m be the least natural number n such that $x \notin U_n$. Then $x \in H_m - U_m$. Let H_m° denote the interior of H_m. Since

$$H_m^\circ - \overline{U}_m \subset H_m - U_m ,$$

the open set $H_m^\circ - \overline{U}_m$ is empty; in particular, $x \notin H_m^\circ - \overline{U}_m$. Thus, either x is an accumulation point of $R - H_m$, or x is an accumulation point of U_m. In the first case, it follows from (23) that the global oscillation of f_m at x is at least $1/4^m$. But from the definition of m and (21), for $m > 1$, $\sum_{n=1}^{m-1} f_n$ has a global limit at x. Thus from (24), f does not have a global limit at x. In the second case, it follows from (17) and a similar argument that f does not have a global limit at x, and we have established (3). This completes the proof of the theorem.

By taking $A_0 = A$ or $A_0 = \emptyset$ in Theorem 2, we obtain the following corollaries.

COROLLARY 1. *Let A_0 be a G_δ in R, and let φ be a continuous function on A_0. Then there exists a function f, continuous in $\mathcal{H}$, such that A_0 is the set of angular convergence as well as the set of global convergence of f, and φ is the boundary function of f.*

COROLLARY 2. *Let A be an $F_{\sigma\delta}$ of first category in R, and let φ be a function of Baire class one on A. Then there exists a function f, continuous in $\mathcal{H}$, such that A is the set of angular convergence and φ is the boundary function of f, and the set of global convergence of f is empty.*

THEOREM 3. *Let S and T be sets of real numbers with $S \subset T$. Then S and T are analytic sets if and only if there exists a function f, real-valued and continuous in $\mathcal{H}$, such that S is the set of global limits and T is the set of angular limits of f.*

Proof. Suppose first that f is an arbitrary real-valued function with domain $\mathcal{H}$, and denote the sets of global and angular convergence of f by A_0 and A, respectively. Then according to Theorem 1, A_0 is a G_δ, A is an $F_{\sigma\delta}$, and the boundary function φ of f is of Baire class one on A. Since the Baire image of a Borel set is an analytic set ([2], p. 266), the sets $\varphi(A_0)$ of global limits of f and $\varphi(A)$ of angular limits of f are analytic.

Suppose now that S and T are analytic sets with $S \subset T$. We wish to show that there exists a function f, continuous in $\mathcal{H}$, having S as its set of global limits and T as its set of angular limits. This is obvious if $T = \emptyset$; so assume that $T \neq \emptyset$. Let C be a perfect nowhere dense set of positive real numbers. Then there exists ([3], p. 388) a real-valued function $\tau(x)$ of Baire class one on C such that

(25) $$\tau(C) = T .$$

Let R_- denote the set of non-positive real numbers. If $S \neq \emptyset$, let ([4], p. 82) $\sigma(x)$ be a real-valued function R_- such that

(26) σ is continuous on the left at every point $x \in R_-$,

(27) $$\sigma(R_-) = S ,$$

and

(28) for every $y \in S$ there exists a point x of continuity of σ such that $\sigma(x) = y$.

Let D be the set of points of discontinuity (including 0) of σ. Because of (26), D is at most countable, and σ is of Baire class one on R_-.

Now let

$$A_0 = R_- - D, \quad A = C \cup R_- \quad \text{if} \quad S \neq \emptyset ;$$
$$A_0 = \emptyset, \quad A = C \quad \text{if} \quad S = \emptyset ;$$

and define

$$\varphi(x) = \begin{cases} \sigma(x) & \text{for } x \in R_- \\ \tau(x) & \text{for } x \in C \end{cases} \quad \text{if} \quad S \neq \emptyset ;$$

$$\varphi(x) = \tau(x) \quad \text{for } x \in C \quad \text{if} \quad S = \emptyset .$$

It is easily verified that all the hypotheses of Theorem 2 are satisfied, and so there exists a function f, real-valued and continuous in $\mathcal{H}$, such that A is the set of angular convergence, A_0 is the set of global convergence, and φ is the boundary function of f. In view of (27) and (28), S is the set of global limits of f, and (25) and (27) imply that T is the set of angular limits of f. This completes the proof of the theorem.

COROLLARY 3. *A necessary and sufficient condition that a set T of real numbers be an analytic set is that there exist a function f, real-valued and continuous in $\mathcal{H}$, having T as its set of angular limits.*

Poprougénko has shown ([4], p. 82) that *a necessary and sufficient condition that a non-empty set T of real numbers be an analytic set is that there exist a harmonic function f in $\mathcal{H}$ having T as its set of global limits.* It would be interesting to know whether this result remains valid with "global" replaced by "angular".

References

[1] E. F. Collingwood, *Cluster sets of arbitrary functions*, Proc. Nat. Acad. Sci. 46 (1960), pp. 1236-1242.

[2] F. Hausdorff, *Mengenlehre*, 2d ed., Berlin-Leipzig 1927.

[3] C. Kuratowski, *Topologie I*, 2d ed., Mon. Matemat. 20, Warszawa-Wrocław 1948.

[4] G. Poprougénko, *Sur l'analyticité des ensembles (A)*, Fund. Math. 18 (1932), pp. 77-84.

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