Autohomeomorphisms on E^n

by

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1. Introduction. Let H denote the family of all autohomeomorphisms on E^n (homeomorphisms on euclidean n-dimensional space onto itself). If $f, g \in H$ and if d is the usual metric in E^n, let

$$\varrho(f, g) = \sup d\big(f(x), g(x)\big) \quad \text{for all } x \in E^n .$$

For any countable dense sets A and B in E^n, let

$$H(A, B) = \{h \in H\colon\ h(A) = B\} .$$

It is well known that $H(A, B)$ is non-empty (for example, [2], p. 44). Section 2, below, is devoted to the proof that there exist members of $H(A, B)$ arbitrarily close (in the sense of ϱ) to the identity. Various applications of this result are given in Section 3. Since ϱ is not a metric, we assume there that H is given the compact-open topology. It then follows that $H(A, B)$ is dense in H and this fact is a key to other applications.

2. Main Theorem. The proof of Theorem 1 depends upon two lemmas. The argument is related to that in Hurewicz and Wallman but involves Cauchy sequences.

THEOREM 1. *If A and B are countable dense sets in E^n, then for any $\varepsilon > 0$ there is an $h \in H$ such that $h(A) = B$ and $\varrho(h, \text{identity}) \leqslant \varepsilon$.*

LEMMA 1. *Let A and B be countable dense sets in E^1 and let $\varepsilon > 0$ be given. Then A and B may be ordered into sequences $A = \{\alpha_i\}$ and $B = \{\beta_i\}$ such that for each i, j*

(1) $$\varepsilon/2 < \beta_i - \alpha_i < \varepsilon ,$$

(2) $$\alpha_i < \alpha_j \quad \textit{if and only if} \quad \beta_i < \beta_j ,$$

(3) $$\alpha_i < \alpha_j \quad \textit{implies} \quad \beta_j - \alpha_j < \beta_i - \alpha_i .$$

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Proof. The proof is by induction and we may assume that A and B are ordered into sequences $\{a_i\}$ and $\{b_i\}$ arbitrarily except that

$$\varepsilon/2 < b_1 - a_1 < \varepsilon \quad \text{and} \quad b_2 < b_1 .$$

Define $\alpha_1 = a_1$, $\beta_1 = b_1$, $\delta_1 = \beta_1 - \alpha_1$. Corresponding to $\beta = b_2$, let α denote any a_i $(i > 1)$ such that

$$\delta_1 < \beta - \alpha < \varepsilon .$$

Define $\alpha_2 = \alpha$, $\beta_2 = b_2 = \beta$, $\delta_2 = \beta_2 - \alpha_2$, and note that $\alpha_2 < \alpha_1$ and $\varepsilon/2 < \delta_1 < \delta_2 < \varepsilon$.

Now assume that $\alpha_1, \ldots, \alpha_{2m}$ and $\beta_1, \ldots, \beta_{2m}$ have been selected, so that conditions (1), (2), (3) hold. Let α denote the first a_i not previously selected. There are three cases to consider.

(i) If $\alpha < \alpha_i$ for $i = 1, \ldots, 2m$, let j be the subscript such that $\alpha_j - \alpha$ is the minimum, and let β denote any b_i not previously selected such that

$$\delta_j < \beta - \alpha < \varepsilon \quad \text{and} \quad \beta < \beta_j .$$

Define $\alpha_{2m+1} = \alpha$, $\beta_{2m+1} = \beta$, $\delta_{2m+1} = \beta_{2m+1} - \alpha_{2m+1}$, and note that for $i = 1, \ldots, 2m$

$$\beta_{2m+1} < \beta_i \quad \text{and} \quad \varepsilon/2 < \delta_i < \delta_{2m+1} < \varepsilon .$$

(ii) If $\alpha_i < \alpha$ for $i = 1, \ldots, 2m$, let k be the subscript such that $\alpha - \alpha_k$ is the minimum, and let β denote any b_i not previously selected such that

$$\varepsilon/2 < \beta - \alpha < \delta_k \quad \text{and} \quad \beta_k < \beta .$$

Define $\alpha_{2m+1} = \alpha$, $\beta_{2m+1} = \beta$, $\delta_{2m+1} = \beta_{2m+1} - \alpha_{2m+1}$, and note that for $i = 1, \ldots, 2m$

$$\beta_i < \beta_{2m+1} \quad \text{and} \quad \varepsilon/2 < \delta_{2m+1} < \delta_i < \varepsilon .$$

(iii) Suppose that for some j, k, $\alpha_k < \alpha < \alpha_j$, where α_k and α_j are the closest elements to α. By the induction hypothesis, $\beta_k < \beta_j$ (no other β_i lies between them) and $\delta_j < \delta_k$. Let β denote any b_i not previously selected such that

$$\delta_j < \beta - \alpha < \delta_k \quad \text{and} \quad \beta_k < \beta < \beta_j .$$

Define $\alpha_{2m+1} = \alpha$, $\beta_{2m+1} = \beta$, $\delta_{2m+1} = \beta_{2m+1} - \alpha_{2m+1}$, and note that if $\alpha_i < \alpha_{2m+1}$ then

$$\beta_i < \beta_{2m+1} \quad \text{and} \quad \varepsilon/2 < \delta_{2m+1} < \delta_i < \varepsilon ,$$

and if $\alpha_{2m+1} < \alpha_i$ then

$$\beta_{2m+1} < \beta_i \quad \text{and} \quad \varepsilon/2 < \delta_i < \delta_{2m+1} < \varepsilon .$$

A similar argument suffices for the construction of α_{2m+2} and β_{2m+2}. This completes the induction and the proof.

It is important to observe that, at each step in the construction above, an interval is established within which the appropriate α or β may be selected.

LEMMA 2. *Let A and B be countable dense sets in E^n and let $\varepsilon > 0$ be given. Then A and B may be ordered into sequences $A = \{\alpha_i\}$ and $B = \{\beta_i\}$ such that for each i, j*

(1) $$\varepsilon/2 < d(\alpha_i, \beta_i) < \varepsilon ,$$

(2) $$d(\beta_j, \beta_i) < d(\alpha_j, \alpha_i) .$$

Proof. Let the coordinate representation of a point p in E^n be $(p^1, p^2, \ldots, p^n)$. Again the proof is by induction and we may assume that A and B are ordered into sequences $\{a_i\}$ and $\{b_i\}$ arbitrarily except that for each coordinate ν

$$\varepsilon/2\sqrt{n} < b_1^\nu - a_1^\nu < \varepsilon/\sqrt{n} \quad \text{and} \quad b_2^\nu < b_1^\nu .$$

Define $\alpha_1 = a_1$, $\beta_1 = b_1$. Corresponding to $\beta = b_2$, use Lemma 1 coordinate by coordinate to determine an n-dimensional interval within which α $(\neq \alpha_1)$ can be selected so that each coordinate pair (α^ν, β^ν) satisfies Lemma 1 with ε replaced by $\varepsilon/\sqrt{n}$. (We assume here that the coordinate axes are in general position with respect to A and B; [2], p. 45.) The inductive step follows also by use of Lemma 1 coordinate by coordinate. For each i, j, and $\nu = 1, 2, \ldots, n$,

$$\varepsilon/2\sqrt{n} < \beta_i^\nu - \alpha_i^\nu < \varepsilon/\sqrt{n} \quad \text{implies} \quad \varepsilon/2 < d(\alpha_i, \beta_i) < \varepsilon ,$$
$$\beta_j^\nu - \alpha_j^\nu < \beta_i^\nu - \alpha_i^\nu \quad \text{implies} \quad d(\beta_j, \beta_i) < d(\alpha_j, \alpha_i) ,$$

and the lemma is proved.

Proof of Theorem 1. We may assume that $A = \{\alpha_i\}$ and $B = \{\beta_i\}$ are ordered, so that conditions (1) and (2) of Lemma 2 hold. For each i, define $h(\alpha_i) = \beta_i$. For each $x \in E^n - A$, let $\{x_j\}$ be any subsequence of A converging to x. By (2), $\{h(x_j)\}$ converges to a point $h(x)$ independently of the subsequence chosen, and by (1), $\varepsilon/2 \leqslant d\big(x, h(x)\big) \leqslant \varepsilon$ for all $x \in E^n$. By (2) again, h is non-expansive (i.e. for all $u, v \in E^n$ $d\big(h(u), h(v)\big) \leqslant d(u, v)$) hence h is continuous. To show that h is one-to-one, suppose that $d(u, v) > 0$. Then for some coordinate ν, $u^\nu \neq v^\nu$. By (2) of Lemma 1, $\big(h(u)\big)^\nu \neq \big(h(v)\big)^\nu$; hence $h(u) \neq h(v)$. By Brouwer's theorem on invariance of domain, h is a homeomorphism on E^n into E^n, and $h(E^n)$ is open in E^n. Since the image of each unbounded sequence in E^n is unbounded, $h(E^n)$ is also closed in E^n, whence $h \in H$.

3. Applications. In this section we assume that H is assigned the compact-open topology: a sub-basis for this topology is the family of all sets

$$W(K, U) = \{h \in H:\ h(K) \subset U\}$$

where K is compact and U is open in E^n.

THEOREM 2. *For any countable dense sets A and B in E^n, $H(A, B)$ is dense in H.*

Proof. Any member M of a basis in the space H can be represented as a *finite* intersection

$$M = \bigcap W(K_i, U_i)\,.$$

If $f \in M$ then, for each i, $f(K_i)$ is compact and $\operatorname{dist}\big(f(K_i), E^n - U_i\big) > \varepsilon_i > 0$. Let ε be the least of the positive numbers ε_i. Now $f(A) = A'$ is countable and dense in E^n; hence by Theorem 1 there is a $g \in H$ such that $g(A') = B$ and $\varrho(g, \mathit{identity}) \leqslant \varepsilon$. Then $gf \in H(A, B) \cap M$.

Suppose now that $n \geqslant 1$ and that $\dim X < n$ where $X \subset E^n$. In [1], S. W. Hahn showed that for some $h \in H$, $h(X)$ misses the set of rational points in E^n. The corollaries below extend this result.

COROLLARY 1. *If B is any countable dense set in E^n, then the set $H_B = \{h \in H:\ h(X) \subset E^n - B\}$ is dense in H.*

Proof. Since $\dim X < n$, $E^n - X$ contains a countable dense set A. But $H(A, B) \subset H_B$, so H_B is dense in H.

COROLLARY 2. *Under the additional hypothesis that X is an F_σ-subset of E^n, H_B is a dense G_δ-subset of H.*

Proof. The argument is similar to one given in [3]. Let $X = \bigcup X_i$, $i = 1, 2, \ldots$, where, for each i, X_i is compact and $\dim X_i < n$, and let $B = \{b_1, b_2, \ldots\}$. For each i, j, let

$$H_{ij} = \{h \in H:\ b_j \notin h(X_i)\}\,.$$

Note that (1) H_{ij} is open, since it is a sub-basic set in the c-o topology, and (2) H_{ij} is dense, since by Theorem 2 it contains a dense set. The space H is metrizable as a complete space, and since $H_B = \bigcap H_{ij}$ (all i and j), Baire's Theorem applies.

We consider finally an application to mappings (continuous functions). Let F denote the family of all mappings on E^n into itself under the compact-open topology. Let L denote the subfamily of maps which lower dimension in the sense

$$L = \{f \in F:\ \dim f(E^n) < n\}\,.$$

THEOREM 3. *If B is a countable dense set in E^n and if $F_B = \{f \in F:\ f(E^n) \subset E^n - B\}$, then $F_B \subset L \subset \overline{F}_B$ (closure of F_B).*

Proof. If $f \in F_B$ then $f(E^n)$ contains no set open in E^n, hence $\dim f(E^n) < n$. If $f \in L$ then $\dim f(E^n) < n$ and by Corollary 1, in any neighborhood of the identity there is an $h \in H$ such that $h\big(f(E^n)\big) \subset E^n - B$. Hence in any neighborhood of f there is a mapping $hf \in F_B$.

Theorem 3 leaves unsettled an interesting conjecture: that $L = \overline{F}_B$. The conjecture is true if and only if $F - L$ is open, which means (roughly): if $f(E^n)$ contains a spherical neighborhood in E^n, then so also does the image of any map sufficiently close to f. We prove this only for the case $n = 1$.

COROLLARY 3. *In the notation above, if $n = 1$ then $L = \overline{F}_B$.*

Proof. In this case, the set L consists of all constant maps. If $f \in F - L$ then there exist disjoint open intervals U_1 and U_2 in $f(E^1)$. Let K_i be any compact subset of $f^{-1}(U_i)$. Then $f \in W(K_1, U_1) \cap W(K_2, U_2) \subset F - L$. Thus L is closed under the compact-open topology.

References

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