sional space over the ring of integers). The quotient space $C^t/M(R)$ is isomorphic to the cartesian direct product of the abelian groups $G_i/G_{i+1}\cdot(G_i\cap N)$ for $i=1,\ldots,c$. The finiteness of the space $C^t/M(R)$ is equivalent to the rank of $M(R)$ equal t ($s=t$ and $a_{ii}\neq 0$ for $i=1,\ldots,t$).

The effectiveness of the construction of the matrix $M(R)$ (cf. 3.2 and 4.1) gives an algorithm to decide whether or not the rank of $M(R)$ is equal to t. This gives an algorithm to decide for any presentation for which the set X of generators and the set R of relations is finite, whether or not the group so presented is finite, if it is ensured that the nilpotency of the group is equal to or less than a given number c.

4.5. Final remarks. This final section is a continuation of the remarks contained in section 4.1.

In sections 4.2-4.4 the following two algorithms were described.

I. The algorithm for deciding the inclusion problem and the word problem relative to the class of all finite presentations of nilpotent groups of a given nil (described in 4.2-4.3).

II. The algorithm for deciding the finiteness problem relative to the same class as in I (described in 4.4).

The base, for both I, and II, was the algorithm of constructing a normal base for a subgroup of a nilpotent free group of a given nil. The algorithm was described in section 3.2, where the subgroup theorem was proved by giving the explicit method of the construction. A possibility of programming this algorithm for a computer was discussed in section 4.1.

The author believes that algorithms I and II can also be programmed for a computer.

The author does not know how deep is the interest of topology and other branches, in the practical possibility of deciding the word problem and the finiteness problem in such a narrow class of presentations as that for which the algorithms I and II are applicable. But he is glad that he has been able to construct algorithms, practical as he hopes, for a larger class of groups than the class of Abelian groups.

References

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On topologies for F^i

by

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The terminology and propositions referred to by number are those of [1].

Let X be a space with geometry G of length $m-1\geqslant 0$. The purpose of this paper is to investigate possible topologies on F^i, the set of i-flats of G. Two possible topologies of F^i are defined as follows:

I. Let $\{f^\nu\}_{\nu\in N}$ be a net of i-flats. Define $\overline{\lim} f^\nu=\{x|\ x$ is a limit point for some net $\{x_\nu\}_{\nu\in N},\ x_\nu\in f^\nu\}$ and $\underline{\lim} f^\nu=\{x|$ there is a net $\{x_\nu\}_{\nu\in N},\ x_\nu\in f^\nu$, with $x_\nu\to x\}$. We say that $f^\nu\to f$ if f is an i-flat and $\overline{\lim} f=\underline{\lim} f=f$.

II. Define $L_i(X)\subset X^{i+1}$ by $L_i(X)=\{(x_0,\ldots,x_i)\in X^{i+1}|\ \{x_0,\ldots,x_i\}$ is linearly independent in $X\}$. If $z=(x_0,\ldots,x_i)\in L_i(X)$, let z^* denote $\{x_0,\ldots,x_i\}$. For $w,z\in L_i(X)$, define $w\sim z$ if $f_i(w^*)=f_i(z^*)$. $\sim$ is an equivalence relation. Let $Y_i=L_i(X)/\sim$ with the quotient topology. There is a natural map $p\colon Y_i\to F^i$ defined by $p(y)=f(y^*)$. p is obviously 1-1 and onto, hence topologize F^i so as to make p a homeomorphism.

II is clearly equivalent to

II′. Let $\{f^\nu\}_{\nu\in N}$ be a net of i-flats. Then $f^\nu\to f$ iff there is a basis $\{x_0^\nu,\ldots,x_i^\nu\}$ for each f^ν such that $(x_0^\nu,\ldots,x_i^\nu)\to(x_0,\ldots,x_i)=x$ in $L_i(X)$ and x^* is a basis for f.

That topology I is not necessarily the same as topology II is shown by the following example:

EXAMPLE 1. Let $X=\{(x,y)\in R^2|\ x^2+y^2<1\}\cup\{(x,y)|\ 1\leqslant x\leqslant 2,\ y=0\}$ and let X have geometry G_X induce from R^2 (with the usual Euclidean geometry). Consider the sequence of 1-flats $\{f^n\}_{n\in I}$ where $f^n=\left\{(x,y)|\ y=\frac{1}{n}x\right\}\cap X$. Then in topology I, this sequence fails to converge, but in topology II, $f^n\to f=\{(x,y)|\ y=0\}\cap X$.

Example 2 shows that the topology defined by II is not always T_2 even when X is.

EXAMPLE 2. Let $X = R^2$ with geometry defined as follows: Define $q\colon R^2 \to R^2$ by

$$q\big((x, y)\big) = (x, y), \quad (x, y) \in R^2 - \{(0,0), (1,0), (0,1), (1,1)\},$$
$$q\big((0,0)\big) = (0,1),$$
$$q\big((0,1)\big) = (0,0),$$
$$q\big((1,0)\big) = (1,1),$$
$$q\big((1,1)\big) = (1,0).$$

If G represents the usual Euclidean geometry on R^2, let X have geometry $q(G)$. Then the sequence of 1-flats $\{f^n\}_{n \in I}$ where $f^n = \{(x, y) |\ y = 1/n\}$ converges to both the images under q of the lines whose equations are $y = 0$ and $y = 1$. If $f^n \to f$ in topology I, then $f^n \to f$ in topology II, hence if topology II is T_2, topology I is also.

THEOREM 1. *If $F^0 = \big\{\{x\} |\ x \in X\big\}$, the following statements are equivalent:*

(a) *X is T_2 and F^i with topology* II *is T_2, $0 \leqslant i \leqslant m$.*

(b) *If $x_j^\nu \to x_j$, $j = 0, \dots, i$, and $\{x_0, \dots, x_i\}$ is linearly independent, then there is some $\nu_0 \in N$ such that $\nu > \nu_0$ implies $\{x_0^\nu, \dots, x_i^\nu\}$ is linearly independent, $0 \leqslant i \leqslant m$.*

(c) *If $\{x_0, \dots, x_i\}$ is linearly independent, there is an open neighborhood of each x_j, U_j, $j = 0, \dots, i$ such that for each set $S = \{y_0, \dots, y_i\}$, $y_j \in U_j$, S is linearly independent, $0 \leqslant i \leqslant m$.*

Proof. (b) and (c) are clearly equivalent. Before completing the proof, we prove the following

LEMMA 1. *If $\dim f(\{y_0, \dots, y_{k+1}\}) = k$, then $S = \{y_0, \dots, y_{k+1}\}$ contains at least two bases for $f_k(S)$.*

Proof. If $k = 0$, then $\{y_0\}$ and $\{y_1\}$ are both bases for $f_0(S)$. Suppose that lemma 1 has been proved for $k-1 \geqslant 0$. $S = \{y_0, \dots, y_{k+1}\}$ contains at least one basis for $f_k(S)$, i.e. a maximal linearly independent subset; hence we may suppose that $\{y_0, \dots, y_k\}$ in such a basis. If $\dim g(S - \{y_0\}) = k$, then $g(S - \{y_0\}) = f_k(S)$ and $S - \{y_0\}$ is another basis for $f_k(S)$. If $\dim g(S - \{y_0\}) = k-1$, then by the induction assumption, $S - \{y_0\}$ contains a basis B for $g(S - \{y_0\})$ which includes y_{k+1}, whence $B \cup \{y_0\}$ is a basis for $f_k(S)$ distinct from $\{y_0, \dots, y_k\}$.

Proof of theorem 1 completed. (a) implies (b). If (b) does not hold, we can find, for each element ν of a directed set N and some integer k, a set $S = \{x_0, \dots, x_k\}$ such that $x_j^\nu \to x_j$, $j = 0, \dots, k$, with $S = \{x_0, \dots, x_k\}$ linearly independent, but $\dim f(\{x_0^\nu, \dots, x_k^\nu\}) = k-1$. By lemma 1, $\{x_0^\nu, \dots, x_k^\nu\}$ contains two distinct bases for $f(\{x_0^\nu, \dots, x_k^\nu\})$ for

each $\nu \in N$; hence we can find a subnet of $f(\{x_0^\nu, \dots, x_k^\nu\})$, say $\{f^{\nu_\mu}\}_{\mu \in M}$, such that $S_{\nu_\mu} - \{x_p^{\nu_\mu}\}$ and $S_{\nu_\mu} - \{x_q^{\nu_\mu}\}$ are bases for each f^{ν_μ} for fixed p and q, $p \neq q$. Then $f^{\nu_\mu} \to f_{k-1}(S - \{x_p\})$ and $f^{\nu_\mu} \to f_{k-1}(S - \{x_q\})$ and hence F^{k-1} could not be T_2.

(b) implies (a). Suppose that F^i is not T_2. Then we can find a net of i-flats $\{f^\nu\}_{\nu \in N}$ such that $f^\nu \to f$ and $f^\nu \to f'$, $f \neq f'$. Then for each $\nu \in N$ we have bases $\{x_0^\nu, \dots, x_i^\nu\}$ and $\{y_0^\nu, \dots, y_i^\nu\}$ of f^ν such that $x_j^\nu \to x_j$, $y_j^\nu \to y_j$, $j = 0, \dots, i$, and $\{x_0, \dots, x_i\}$ and $\{y_0, \dots, y_i\}$ are bases of f and f', respectively. We may suppose $y_0 \notin f$. Then $\{x_0, \dots, x_i, y_0\}$ is linearly independent, but (b) is not satisfied.

If $F^0 = \{\{x\} |\ x \in X\}$, F^0 with topology II is clearly homeomorphic to X.

LEMMA 2. *Suppose that X and G form an m-arrangement, $S = \{x_0, \dots, x_m\}$ is a linearly independent subset of X and $y \in \operatorname{Int} C(S)$.* (a) *Then for any face of $C(S)$, $F^i C(S)$, there is at least one $m-1$-flat f which contains y and which does not intersect $F^i C(S)$.* (b) *Moreover, $f \cap C(S) = C(T)$, where $T = f \cap (\bigcup_{i=1}^{m} \overline{x_0 x_i})$ is a linearly independent set of $m-1$ points.*

Proof. Lemma 2 is trivially verified for $m = 1$. Assume that it has been proved for $m-1 \geqslant 0$. Let $S = \{x_0, \dots, x_m\}$ be a linearly independent subset of X, and $y \in \operatorname{Int} C(S)$. We may suppose that $i = 0$ and $y \in \overline{x_1 z}$

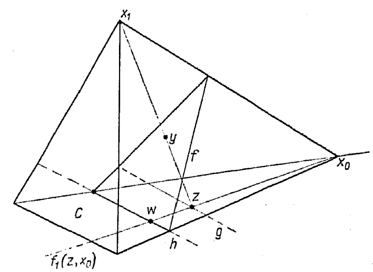

Fig. 1

for some $z \in F^1 C(S)$ (3.6). By the induction assumption, we have some $m-2$-flat $g \subset f_{m-1}\big(F^1 C(S)\big)$ which contains z but does not intersect $F^0\big(F^1 C(S)\big)$. g disconnects $F^1 C(S)$ into two components, one containing x_0 and the other containing $F^0\big(F^1 C(S)\big)$ (4.12, 3.25, and 3.26); label this latter component C.

A simple argument shows that $f_1(x_0, z) \cap C \cap \operatorname{Int} F^1 C(S) \neq 0$; hence choose w in this intersection. Again using the induction assumption, there is an $m-2$-flat h which contains w and does not intersect $F^0\big(F^1 C(S)\big)$. Let $f = f_{m-1}(h \cup \{y\})$. f disconnects X into two convex components A and B. We may suppose $x_0 \in A$. For $i = 2, \dots, m$, $f \cap \operatorname{Int} \overline{x_0 x_i} \neq \emptyset$; hence

$x_i \in B$, $i = 2, \ldots, m$. Since $f \cap \operatorname{Int}\overline{zx_1} \neq \emptyset$, it follows by (3.23) that either $f \cap \overline{x_0 z} \neq \emptyset$, or $f \cap \overline{x_0 x_1} \neq \emptyset$. If $\overline{x_0 z} \cap f \neq \emptyset$, then $f_1(x_0, w) \subseteq h$, which implies $h \cap F^0(F^1 C(S)) \neq \emptyset$, a contradiction; hence it must be that $f \cap \overline{x_0 x_1} \neq \emptyset$. If $x_1 \in f$, then $\overline{x_1 z} \subseteq f$ and hence again $f_1(x_0, w) \subseteq h$, whence we have that f intersects $\overline{x_0 x_1}$ in an interior point. It follows then that $x_1 \in B$, and since B is convex, $F^0 C(S) \subseteq B$, therefore $f \cap F^0 C(S) = \emptyset$. This completes the proof of (a).

(b) is true for $m = 1$. Assume that (b) has been proved for $m-1 \geqslant 0$. $f \cap (\bigcup_{i=2}^{m} \overline{x_0 x_i}) = T$ contains $m-1$ points since f intersects each segment in an interior point. By the induction assumption, $f \cup (\bigcup_{i=2}^{m} \overline{x_0 x_i}) = h \cap$ $\cap (\bigcup_{i=2}^{m} \overline{x_0 x_i})$ is a linearly independent set, call it T', of $m-2$ points. Since $f \cap \overline{x_0 x_1} \cap f_{m-2}(T') = \emptyset$, T is linearly independent.

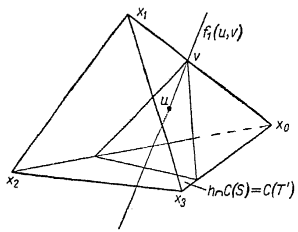

Fig. 2

Certainly $C(T) \subseteq f \cap C(S)$. By the induction assumption, $C(T') = f \cap F^1 C(S) = h \cap F^1 C(S)$. Suppose $u \in f \cap C(S)$. Set $\{v\} = f \cap \overline{x_1 x_0}$. Then $f_1(u, v)$ intersects $F^1 C(S)$ at a point of h, whence $u \in C(T)$ (3.6).

THEOREM 2. *Suppose that X and G form an m-arrangement. Let $S = \{x_0, \ldots, x_m\}$ be a linearly independent subset of X and select $y \in \operatorname{Int} C(S)$. Let f^j_{m-1} be an $m-1$-flat which contains y and does not intersect $F^j C(S)$, $j = 0, \ldots, m$. f^j_{m-1} disconnects X into two convex open sets A_j and B_j, $F^j C(S) \subseteq A_j$ and $x_j \in B_j$. Set $U(x_j) = \bigcap_{i \neq j} A_i$, $j = 0, \ldots, m$. $U(x_j)$ is a convex open neighborhood of x_j. Then if $T = \{w_0, \ldots, w_m\}$ where $w_j \in U(x_j)$, $j = 0, \ldots, m$, T is linearly independent.*

Proof. Theorem 2 is true for $m = 1$. Assume it has been proved for $m - 1 \geqslant 0$. $f^0_{m-1} \cap C(S) = C(Q)$ where $Q = f^0_{m-1} \cap (\bigcup_{i=1}^{m} \overline{x_0 x_i})$. For $j = 1, \ldots, m$, $f^j_{m-1} \cap f^0_{m-1}$ is an $m-2$-flat contained in f^0_{m-1} which contains y (which is in $\operatorname{Int} C(T)$) and such that $(f^j_{m-1} \cap f^0_{m-1}) \cap F^j C(Q) = \emptyset$. Set $\{z_j\} = f^0_{m-1} \cap \overline{x_0 x_j}$, thus $Q = \{z_1, \ldots, z_m\}$. Set $V(z_j) = \bigcap_{\substack{k=1 \\ k \neq j}}^{m} A_k \cap f^0_{m-1}$.

By the above observations and the induction hypothesis, if $\{u_1, \ldots, u_m\}$ is a set such that $u_j \in V(z_j)$, $j = 1, \ldots, m$, then $\{u_1, \ldots, u_m\}$ is linearly independent. Set $\{u_j\} = \overline{w_0 w_j} \cap f^0_{m-1}$, $j = 1, \ldots, m$. This intersection is

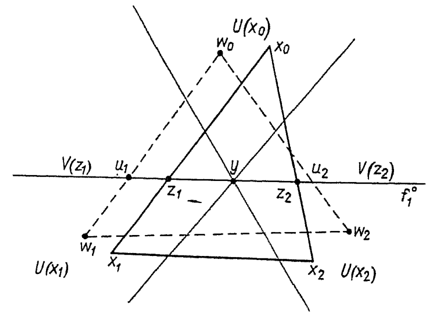

Fig. 3

non-empty for each j since w_0 and w_j are in different components of $X - f^0_{m-1}$. $u_j \in V(z_j)$ since both w_0 and w_j are in $\bigcap_{\substack{k=1 \\ k \neq j}}^{m} A_k$ a convex set, hence $\overline{w_0 w_j} \subseteq \bigcap_{\substack{k=1 \\ k \neq j}}^{m} A_k$. Hence if $\{w_0, \ldots, w_m\}$ is linearly dependent, it must be contained in an $m-2$-flat for which $\{u_1, \ldots, u_m\}$ is a basis; hence that flat must be f^0_{m-1}. But similar reasoning shows that it must also be f^j_{m-1}, $j = 1, \ldots, m$, a contradiction.

COROLLARY. *If X and G form an m-arrangement, then F^i with topology* II *is T_2, $0 \leqslant i \leqslant m$.*

Proof: Theorem 1 (c).

In the light of the previous results we may define a "meaningful" derivative in the situation where X and geometry G form an m-arrangement: Suppose $Y \subseteq X$. Then a k-flat f, $k \geqslant 1$, is said to be *tangent* to Y at $y \in Y$ if given any directed set N and for any $\nu \in N$ any linearly independent set of points $\{y^\nu_0, \ldots, y^\nu_k\} \subseteq Y$ such that $y^\nu_j \to y$, $j = 0, \ldots, k$, then $f_k(\{y^\nu_0, \ldots, y^\nu_k\}) \to f$ in topology II. It is easily seen that if there is any flat tangent to Y at y, such a flat is unique.

Reference

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