sional space over the ring of integers). The quotient space $Q/M(E)$ is
isomorphic to the cartesian direct product of the abelian groups $G_i\otimes\mathbb{Z}_{+1}$ (if $i \neq j$), where $G_i \otimes N$ for $i = 1, ..., c$. The finiteness of the space $Q/M(E)$ is
equivalent to the rank of $M(E)$ being $t$ ($t = c$ and $\alpha_i \neq 0$ for $i = 1, ..., t$).

The effectiveness of the construction of the matrix $M(E)$ (cf. 3.2
and 4.1) gives an algorithm to decide whether or not the rank of $M(E)$
is equal to $t$. This gives an algorithm to decide for any presentation for
which the set $X$ of generators and the set $E$ of relations is finite, whether
or not the group so presented is finite. If it is decided that the nilpotency
of the group equals to or less than a given number $c$.

4.5. Final remarks. This final section is a continuation of the
remarks contained in section 4.1.

I. The algorithm for deciding the inclusion problem and the word
problem relative to the class of all finite presentations of nilpotent groups
of a given nil (described in 4.2-4.3).

II. The algorithm for deciding the finiteness problem relative to
the same class as in I (described in 4.4).

The base, for both I and II, was the algorithm of constructing
a normal base for a subgroup of a nilpotent free group of a given nil.
The algorithm was described in section 3.2, where the subgroup theorem
was proved by giving the explicit method of the construction.
A possibility of programming this algorithm for a computer was dis-
cussed in section 4.1.

The author believes that algorithms I and II can also be program-
med for a computer.

The author does not know how deep is the interest of topology and
other branches, in the practical possibility of deciding the word
problem and the finiteness problem in such a narrow class of presentations
as that for which the algorithms I and II are applicable. But he is glad
that he has been able to construct algorithms, practical as he hopes, for
a larger class of groups than the class of Abelian groups.

References

[2] P. Hall, Nilpotent groups, Canadian Mathematical Congress 1957, Litographed
notes.
[3] A. W. Mostowski, On the decidability of some problems in special classes of
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On topologies for $F^t$

by

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The terminology and propositions referred to by number are those
of [1].

Let $X$ be a space with geometry $G$ of length $m-1 > 0$. The purpose
of this paper is to investigate possible topologies on $F^t$, the set of $t$-flats
of $G$. Two possible topologies of $F^t$ are defined as follows:

I. Let $(f_i^t)_{i \in \mathbb{N}}$ be a net of $t$-flats. Define $\lim f^t = \{z \mid z$ is a limit point
for some net $(z_i)_{i \in \mathbb{N}}$, $z_i \in f_i^t \}$ and $\lim f^t = \{z \mid \text{there is a net $(z_i)_{i \in \mathbb{N}}$, $z_i \in f_i^t$,}
with $z_i \to z$. We say that $f^t \to f$ if $f$ is an $t$-flat and $\lim f^t = \lim f = f$.

II. Define $L_d(X) \subset X^t$ by $L_d(X) = \{(z_0, ..., z_t) \mid \{z_0, ..., z_t\}$ is linearly independent in $X\}$. If $z = (z_0, ..., z_t) \in L_d(X)$, let $z^*_t$ denote
$(z_t, ..., z_0)$. For $w \in L_d(X)$, define $w \sim f \text{ if } f(w^t) = f(z^*_t), \sim$ is an equiv-
lence relation. Let $\chi(L_d(X))$ with the quotient topology. There
is a natural map $p: X \to L_d(X)$ defined by $p(y) = f(y^*)$. $p$ is obviously 1-1
and onto, hence topologize $F^t$ so as to make $p$ a homeomorphism.

II is clearly equivalent to

II'. Let $(f_i^t)_{i \in \mathbb{N}}$ be a net of $t$-flats. Then $f^t \to f$ iff there is a basis
$(z_i, ..., z_t)$ for each $f_i^t$ such that $(z_i, ..., z_t) \to (z_0, ..., z_t) = z$ in $L_d(X)$
and $z^*_t$ is a basis for $f$.

That topology I is not necessarily the same as topology II is shown
by the following example:

Example 1. Let $X = \{(x, y) \in F^t \mid x^2 + y^2 < 1 \}$ with $(x, y) \mid 1 \leq x, y \leq 2, y = 0$.
and let $X$ have geometry $G_x$ induce from $F^t$ (with the usual Euclidean
geometry). Consider the sequence of 1-flats $(f_i^t)_{i \in \mathbb{N}}$ where $f_i^t = \{(x, y) \mid y
= \frac{1}{i^2} \} \cap X$. Then in topology II, this sequence fails to converge, but in
topology I, $f^t \to f = \{(x, y) \mid y = 0 \} \cap X$.

Example 2 shows that the topology defined by II is not always
$T_1$ even when $X$ is.
Example 2. Let $X = E^6$ with geometry defined as follows: Define $q:E^6 \to E^6$ by
\[ q((x, y)) = (x, y), \quad (x, y) \in E^6 - \{(0, 0), (0, 1), (1, 0), (1, 1), (1, 1)\}, \]
\[ q((0, 0)) = (0, 1), \]
\[ q((0, 1)) = (0, 0), \]
\[ q((1, 0)) = (1, 1), \]
\[ q((1, 1)) = (1, 0). \]

If $G$ represents the usual Euclidean geometry on $E^6$, let $X$ have geometry $q(G)$. Then the sequence of $1$-flats $\{f^i\}_{i \in \mathbb{N}}$ such that $f^i(x, y) = (x, y)$ converges to both the images under $q$ of the lines whose equations are $y = 0$ and $y = 1$. If $f^i \to f$ in topology I, then $f^i \to f$ in topology II, hence if topology II is $T_4$, topology I also is.

Theorem 1. If $F^m = \{[x] \mid \exists x \in X\}$, the following statements are equivalent:

(a) $X$ is $T_4$ and $F^i$ with topology II is $T_4$, $0 \leq i \leq m$.
(b) If $x^j \to x^i$, $j = 0, \ldots, i$, and $(x^0, \ldots, x^i)$ is linearly independent, then there is some $y_x \in X$ such that $x \to y_x$ implies $(x^0, \ldots, x^i)$ is linearly independent, $0 \leq i \leq m$.
(c) If $(x^0, \ldots, x^n)$ is linearly independent, there is an open neighborhood of each $x^i$, $U_{x^i}$, $i = 0, \ldots, n$ such that for each set $S = (x^0, \ldots, x^n)$, $y_x \in U_{x^i}, S$ is linearly independent, $0 \leq i \leq m$.

Proof. (b) and (c) are clearly equivalent. Before completing the proof, we prove the following.

Lemma 1. If $\dim(f(y^0, \ldots, y^{n+1})) = k$, then $S = (y^0, \ldots, y^{n+1})$ contains at least two bases for $f(S)$.

Proof. If $k = 0$, then $y^0$ and $(y^0)$ are both bases for $f(S)$. Suppose that $S$ has been proved for $k - 1 \geq 0$. $S = (y^0, \ldots, y^{n+1})$ contains at least one basis for $f(S)$, i.e., a maximal linearly independent subset; hence we may suppose that $(y^0, \ldots, y^n)$ in such a basis. If $g(S - (y^0)) = k$, then $g(S - (y^0)) = f(S)$ and $S - (y^0)$ is another basis for $f(S)$. If $g(S - (y^0)) = k - 1$, then by the induction assumption, $S - (y^0)$ contains a basis $B$ for $g(S - (y^0))$ which includes $y^{n+1}$, whence $B \cup (y^0)$ is a basis for $f(S)$ distinct from $(y^0, \ldots, y^n)$. Proof of theorem 1 completed. (a) implies (b). If (b) does not hold, we can find, for each element $n$ of a directed set $X$ and some integer $k$, a set $S = (x^0, \ldots, x^n)$ such that $x^j \to x^i$, $j = 0, \ldots, k$, with $S = (x^0, \ldots, x^n)$ linearly independent, but $\dim(f((x^0, \ldots, x^n))) = k - 1$. By lemma 1, $(x^0, \ldots, x^n)$ contains two distinct bases for $f((x^0, \ldots, x^n))$ for each $r \in \mathbb{N}$; hence we can find a subset of $f((x^0, \ldots, x^n))$, say $(f^r)_{r \in \mathbb{N}}$, such that $\cap_r (x^0, \ldots, x^n)$ and $\cap_r (x^0, \ldots, x^n)$ are bases for each $f^r$ for fixed $p$ and $q$, $p \neq q$. Then $f^r \to f^r$ in topology II, hence $F^r$ could not be $T_4$.

(b) implies (a). Suppose that $F^i$ is not $T_4$. Then we can find a net of $i$-flats $\{f^i\}_{i \in X}$ such that $f^i \to f$ and $f^i \to f^i$, $i \neq f^i$. Then for each $x \in X$ we have bases $(x^0, \ldots, x^n)$ and $(y^0, \ldots, y^n)$ of $x^i$ such that $x^i \to x^i$, $y^i \to y^i$, $i = 0, \ldots, n$, and $(x^0, \ldots, x^n)$ and $(y^0, \ldots, y^n)$ are bases of $f$ and $f^i$, respectively. We may suppose $y^i \to y^i$. Then $(x^0, \ldots, x^n, y^i)$ is linearly independent, but (b) is not satisfied.

If $F^m = \{[x] \mid \exists x \in X\}$, $F^m$ with topology II is clearly homeomorphic to $X$.

Lemma 2. Suppose that $X$ and $G$ form an $m$-arrangement, $S = (x^0, \ldots, x^n)$ is a linearly independent subset of $X$ and $y \in \text{Int}(O(S))$. (a) Then for any face of $O(S)$, $F^m(O(S))$, there is at least one $m-1$-flat $f$ which contains $y$ and which does not intersect $F^m(O(S))$. (b) Moreover, $f \cap O(S) = C(T)$, where $T = f \cap (\bigcup \mu x^0 x^i)$ is a linearly independent set of $m-1$ points.

Proof. Lemma 2 is trivially verified for $m = 1$. Assume that it has been proved for $m-1 \geq 0$. Let $S = (x^0, \ldots, x^n)$ be a linearly independent subset of $X$, and $y \in \text{Int}(O(S))$. We may suppose that $i = 0$ and $y \in \overline{S}$ for some $x \in O(S)$ (3.6). By the induction assumption, we have some $m-2$-flat $g \subset f_m(O(S))$ which contains $x$ but does not intersect $F^m(O(S))$. $g$ disconnects $O(S)$ into two components, one containing $x_0$ and the other containing $F^m(O(S))$ (4.13, 3.25, and 3.26); label this latter component $C$.

A simple argument shows that $f(S) \subset C \cap \text{Int}(O(S)) \neq 0$; hence choose $w$ in this intersection. Again using the induction assumption, there is an $m-2$-flat $h$ which contains $w$ and does not intersect $F^m(O(S))$. Let $f = f_m(h \cup (S))$. $f$ disconnects $X$ into two convex components $A$ and $B$. We may suppose $x_0 \in A$. For $i = 2, \ldots, n$, $f \cap \text{Int}(x^0 x^i) \neq \emptyset$; hence
By the above observations and the induction hypothesis, if \( \{u_1, ..., u_m\} \) is a set such that \( u_j \in V(x_j), \) \( j = 1, ..., m \), then \( \{u_1, ..., u_m\} \) is linearly independent. Set \( \{u_j\} = u_0 u_j \cap f_{m-j}, \) \( j = 1, ..., m. \) This intersection is non-empty for each \( j \) since \( u_0 \) and \( u_j \) are in different components of \( X \setminus f_{m-j} \), \( u_j \in V(x_j) \) since both \( u_0 \) and \( u_j \) are in \( \bigcap_{i=0}^{m} A_k \) a convex set, hence

\[
\bigcap_{k=0}^{m} A_k.
\]

Hence if \( \{u_1, ..., u_m\} \) is linearly dependent, it must be contained in an \( m-2 \)-flat for which \( \{u_1, ..., u_j\} \) is a basis; hence that flat must be \( f_{m-j} \). But similar reasoning shows that it must also be \( f_{m-j}, \) \( j = 1, ..., m, \) a contradiction.

**Corollary.** If \( X \) and \( G \) form an \( m \)-arrangement, then \( F \) with topology \( \Pi \) is \( T_\infty, 0 \leq i \leq m. \)

**Proof:** Theorem 1 (c).

In the light of the previous results we may define a “meaningful” derivative in the situation where \( X \) and geometry \( G \) form an \( m \)-arrangement: Suppose \( Y \subseteq X. \) Then a \( k \)-flat \( f, \) \( k \geq 1, \) is said to be tangent to \( Y \) at \( y \in Y \) if given any directed set \( N \) and for any \( v \in N \) any linearly independent set of points \( \{y_1, ..., y_k\} \subseteq Y \) such that \( y \rightarrow y_j, \) \( j = 0, ..., k, \) then \( f(y_0, ..., y_k) \rightarrow y \) in topology \( \Pi. \) It is easily seen that if there is any flat tangent to \( Y \) at \( y \), such a flat is unique.

**Reference**

[1] M. C. Gemignani, Topological geometry and a new characterization of \( R^n. \)

*Notes Dema Journ. of Formal Logic.* 1, VII (1966), pp. 57-100.

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