A rigid sphere *

by

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The goal of this paper is to answer a question raised by B. J. Ball in [3]. An embedding of a 2-sphere S in S^3 will be constructed with the property that any homeomorphism of S^3 onto itself, which is invariant on S, in pointwise fixed on S. S will appear as the image of a tame 2-sphere under an upper semicontinuous decomposition of S^3. The non-degenerate elements of this decomposition space will be arcs of the type constructed by Alford and Ball in [2].

The terminology of [2] will be used. Suppose that A is an arc in E^3, p is an endpoint of A, and A is locally polyhedral except at p. Then the *penetration index* of A at p is the smallest cardinal number n such that there are arbitrarily small 2-spheres enclosing p and containing no more than n points of A. In [2] a sequence of arcs $A_1, A_2, \ldots$ is constructed such that each A_i is locally polyhedral except at an endpoint p_i, and the penetration index of A_i at p_i is $2i+1$. If A is an arc in E^3, then A *is of type* i if and only if the embeddings of A and A_i are equivalent. Let E^3_+ denote the set of points in E^3 each of whose third coordinates is non-negative.

LEMMA 1. *Let pq be an arc of type r in E^3_+ such that $q \in \mathrm{Bd}(E^3_+)$, $pq-\{q\} \subset \mathrm{Int}(E^3_+)$, and pq has penetration index $2r+1$ at p. Then there exists an open subset U of E^3 such that $pq \subset U$, and if S is a polyhedral 2-sphere in U such that* (i) $pq \subset \mathrm{Int}(S)$, *and* (ii) *$S$ is in general position with respect to* $\mathrm{Bd}(E^3_+)$, *then $S \cap \mathrm{Bd}(E^3_+)$ contains at least $2r+1$ mutually disjoint simple closed curves each of which contains q on its interior with respect to* $\mathrm{Bd}(E^3_+)$.

Proof. Let pq be an arc satisfying the hypothesis of Lemma 1. Let K be a polyhedral 2-sphere such that (i) $q \in \mathrm{Int} K$, (ii) K intersects pq in exactly $2r+1$ points, and (iii) if K' is a 2-sphere such that $q \in \mathrm{Int} K'$, and $K' \subset \mathrm{Int} K$, then $K' \cap pq$ contains at least $2r+1$ points. Let x be a point of E^3_-. Let xp denote the straight line interval from x to p and let y be a point of xp between x and p. Let r be the first point of $pq \cap K$ in the order from q to p, and let s be a point of rq between r and q.

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Now let h be a homeomorphism of E^3 onto itself which takes pq onto xq, takes p onto x, r onto y, and s onto p. Let U denote $h(\operatorname{Int} K)$. Then U satisfies the conclusion of Lemma 1.

Description of the example: In E^3, let X denote the plane $Z = 0$, and let $R = \{r_1, r_2, r_3, \ldots\}$ be the set of points in X, both of whose coordinates are rational numbers.

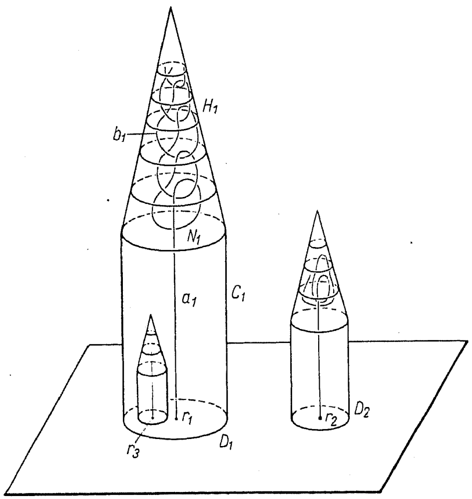

Fig. 1

Let D_1 be a circular disk in X, with center at r_1, such that (i) $\operatorname{diam} D_1 < 1$, and (ii) $\operatorname{Bd}(D_1) \cap R = \emptyset$. Let C_1 denote the solid cylinder of height 1 over D_1, N_1 denote the top face of C_1, and H_1 denote the solid cone over N_1 from the point $(r_1, 2)$.

Let D_2 be a circular disk in X, with center at r_2, such that (i) $\operatorname{diam} D_2 < 1/2$, (ii) $\operatorname{Bd}(D_2) \cap R = \emptyset$, (iii) $r_1 \notin D_2$, and (iv) either $D_2 \subset \operatorname{Int} D_1$, or $D_2 \cap D_1 = \emptyset$. Let C_2 denote the solid cylinder of height 1/2 over D_2, N_2 denote the top face of C_2, and H_2 denote the solid cone over N_2 from the point $(r_2, 1)$.

This process is continued. For each positive integer n, D_n is chosen so that (i) $\operatorname{diam} D_n < 1/n$, (ii) $\operatorname{Bd}(D_n) \cap R = \emptyset$, (iii) for each i, $i < n$, $r_i \notin D_n$, and (iv) either there exists an integer k, $k < n$, such that $D_n \subset \operatorname{Int}(D_k)$ or $D_n \cap [\bigcup_{i<n} D_i] = \emptyset$.

LEMMA 2. *Suppose that j is a positive integer, and ε is a positive number. Then there exists a disk D in X such that* (i) $r_j \in \operatorname{Int}(D)$, (ii) $\operatorname{diam} D < \varepsilon$, *and* (iii) $\operatorname{Bd}(D) \subset D_j - [\bigcup_{k>j} \operatorname{Int}(D_k)]$.

Proof. It follows from the construction that $D_j - [\bigcup_{k>j} \operatorname{Int}(D_k)]$ is a Sierpiński curve which has r_j as an inaccessible point. This implies the conclusion of Lemma 2.

For each positive integer j, let a_j be the straight line interval from r_j to the point $(r_j, 1/j)$, and let b_j be an arc of type j, which lies in H_j, has endpoints $(r_j, 1/j)$ and $(r_j, 2/j)$, and has penetration index $2j+1$ at $(r_j, 2/j)$. Let α_j be $a_j \cup b_j$. See Figure 1.

LEMMA 3. *Suppose that j is a positive integer and that U is an open subset of E^3 such that $\alpha_j \subset U$. Then there exists a 2-sphere S, such that* (i) $S \subset U$, (ii) $\alpha_j \subset \operatorname{Int}(S)$, (iii) *for each k, $S \cap a_k = \emptyset$, and* (iv) *$S \cap X$ consists of exactly $2j+1$ simple closed curves.*

Proof. Let j be a positive integer and let U be an open set containing α_j. Let V be a cylindrical neighborhood of a_j which lies in U. Now it follows from Lemma 2 that there exist mutually disjoint simple closed curves $\gamma_1, \gamma_2, \ldots, \gamma_{2j+1}$ in X such that for each i, $r_j \in \operatorname{Int}\gamma_i$, $\gamma_i \subset V$, and $\gamma_i \subset D_j - [\bigcup_{k>j} \operatorname{Int} D_k]$. For each i let F_i be the right circular cylinder from γ_i to N_j. Notice that $F_i \subset U$, and for each k, $F_i \cap a_k = \emptyset$. Let δ_i denote the boundary component of F_i on N_j.

Now there is a punctured disk L such that (i) L has boundary components $\delta_1, \delta_2, \ldots, \delta_{2j+1}$, (ii) L lies, except for its boundary, above the plane of N_j, (iii) for each k, $a_k \cap L = \emptyset$, and (iv) $L \subset V$. Now $L \cup [\bigcup_1^{2j+1} F_i]$ is a punctured disk which lies in U, has boundary components $\gamma_1, \gamma_2, \ldots, \gamma_{2j+1}$ and misses each a_k. Now the simple closed curves $\gamma_1, \gamma_2, \ldots, \gamma_{2j+1}$ may be capped off in U, below the plane X, to yield a 2-sphere S which satisfies the conclusion of the lemma.

The collection $\{\alpha_1, \alpha_2, \ldots\}$ is upper semi-continuous. This follows from the fact that for each positive number ε, there are only a finite number of elements in the collection which have diameter greater than ε.

We now consider S^3 as the one point compactification of E^3. Let G be the upper semi-continuous decomposition of S^3, whose only non-degenerate elements are $\alpha_1, \alpha_2, \ldots$ It follows from [4] that S^3/G is topologically equivalent to S^3. Let P be the projection mapping of S^3 onto S^3/G. Let S denote the one point compactification of X, S' denote $P(S)$, and for each j, let q_j denote $P(\alpha_j)$. Let Q denote the set $\{q_1, q_2, \ldots\}$.

THEOREM. *Suppose that h is a homeomorphism of S^3/G onto S^3/G such that $h(S') = S'$. Then the restriction of h to S' is the identity.*

Proof. Suppose that h is a homeomorphism of S^3/G onto S^3/G, $h(S') = S'$, and j is a positive integer such that $h(q_j) \neq q_j$. It will be shown that this assumption leads to a contradiction.

Case 1. Suppose that $h(q_j)$ is not an element of Q. Let A be an arc on S' such that $h(q_j) \in A$, and $A \cap Q = \emptyset$. Then $P^{-1}(A)$ is an arc on S, and $P^{-1}(A) \cap R = \emptyset$. Then there exists a tame 2-sphere S'' such that $P^{-1}(A) \subset S''$, and for each positive integer k, $S'' \cap a_k = \emptyset$. It follows from Theorem 2 of [5] that $P(S'')$ is a tame 2-sphere in S^3/G and hence that A is a tame arc in S^3/G. Then, $h^{-1}(A)$ is a tame arc on S' and $q_j \in A$.

Let B be an arc on S' such that $B \cap Q = \{q_j\}$, and q_j is an endpoint of B. Now B is locally tame except possibly at q_j, and since q_j lies on the tame arc $h^{-1}(A)$, it follows from [6] that B is tame. Now $P^{-1}(B)$ is an arc which is the union of a_j and an arc which lies in S. Let x denote the endpoint of $P^{-1}(B)$ which lies in X.

Let U be an open set in S^3 such that (i) $a_j \subset U$, (ii) $x \notin U$, (iii) U is the union of elements of the decomposition G, and (iv) U satisfies the conclusion of Lemma 1 with respect to a_j. Now $P(U)$ is open in S^3/G and contains q_j. Since B is tame, there exists a tame 2-sphere L such that (i) $q_j \in \operatorname{Int}(L)$. (ii) $L \cap S$ is a single point, (iii) $L \subset P(U)$, and (iv) $L \cap Q = \emptyset$.

Now $P^{-1}(L)$ is a 2-sphere in S^3 such that (i) $a_j \subset \operatorname{Int}(P^{-1}(L))$, (ii) $P^{-1}(L) \subset U$, (iii) $P^{-1}(L) \cap P^{-1}(B)$ is a single point, and (iv) from [1], $P^{-1}(L)$ is tame. Now there exists a polyhedral 2-sphere M in S^3 such that (i) $a_j \subset \operatorname{Int}(M)$, (ii) $M \subset U$, (iii) $M \cap P^{-1}(B)$ is a single point, and (iv) is in general position with respect to X. But since $M \cap P^{-1}(B)$ is a single point, and $x \notin U$, M can contain at most one simple closed curve on X which contains r_j on its interior with respect to X. This contradicts Lemma 1.

Case 2. Suppose that there exists an integer k, $k \neq j$, such that $h(q_j) = q_k$. Let A be an arc on S such that $A \cap Q = \{q_j\}$, and q_j is an endpoint of A. Now it follows from Lemma 3 and the type of argument given in Case 1 that the penetration index of A at q_j is $2j+1$. Now $h(A)$ is an arc on S and $h(A) \cap Q = \{q_k\}$, for otherwise we are in Case 1. Repeating the argument again, we see that the penetration index of $h(A)$ at q_k is $2k+1$. This is a contradiction since the penetration index is an embedding invariant. Therefore, the theorem is established.

References

[1] S. Armentrout, *Upper semi-continuous decompositions of E^3 with at most countably many elements*, Annals of Math. 78 (1963), pp. 605-618.

[2] W. R. Alford and B. J. Ball, *Some almost polyhedral wild arcs*, Duke Jour. 30 (1963), pp. 33-38.

[3] B. J. Ball, *Penetration Indices and applications*, *Topology of 3-manifolds and related topics*, M. K. Fort, Jr., Editor, pp. 37-39, New Jersey, 1962.

[4] D. S. Gillman and J. M. Martin, *Countable decompositions of E^3 into points and pointlike arcs*, Notices Amer. Math. Soc. 10 (1963), p. 74.

[5] J. Hempel, *A surface in S^3 is tame if it can be deformed into each complementary domain*, Trans. Amer. Math. Soc. 111 (1964), pp. 273-287.

[6] E. E. Moise, *Affine structures in 3-manifolds. VIII. Invariance of the knot types; local tame embedding*, Annals. of Math. 59 (1954), pp. 159-170.

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