For the latter part, consider the lattice of figure 3 and the ideals $I_1$ and $I_2$, namely the principal ideals generated by $a$ and $b$. Then each of $\theta(I_1)$ and $\theta(I_2)$ is the universal congruence on $L$ and hence is their intersection; while $\theta(I_1 \land I_2)$ is the null congruence on $L$. Thus $g$ does not preserve products.

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On the elementary theory of linear order

by

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A. Ehrenfeucht proved in [1] that the elementary theory $T$ of linear order, even with a finite number of unary predicates adjoined (then denoted by $T_1$), is decidable. Here we establish the same result by a different method. We think that our proof is of interest on its own, because it gives further insight as to how much can be expressed in the first-order language of linear order (see Theorem 2 below) (*). It is only for the sake of a lighter notation that we restrict our proof to the theory $T_1$; there is an obvious generalization to a proof for $T_1$ (see also remark 3 below).

1. Let $L$ be the first-order language with identity and one binary predicate $\prec$. $T$ is the theory over $L$ of the axioms

(a) $\neg u \prec u$,  
(b) $u \prec v \land v \prec w \rightarrow u \prec w$,  
(c) $u \equiv v \lor u \prec v \lor v \prec u$.

We use $x \prec y (mod A)$ to denote the order relation of an ordered set $\mathcal{A}$, $|A|$ to denote the field of $\mathcal{A}$. If $B$ is said to be a segment of $\mathcal{A}$ if $B$ is a submodel of $\mathcal{A}$ and if $x \prec y (mod B)$ and $x \prec y (mod A)$ implies $x \prec y (mod A)$. An ordered set $\mathcal{A}$ is said to be a splitting of $\mathcal{A}$ if $|\mathcal{A}|$ is a set of (non-empty) segments of $\mathcal{A}$, which partitions $|\mathcal{A}|$, and if $B \equiv C (mod A)$ implies $x \equiv y (mod A)$ for all $x \in B$, $y \in C$. The elements of $|\mathcal{A}|$ are called the parts of the splitting.

$a$, $a^*$, $\eta$ respectively denote the order type of the positive integers, negative integers, rationals. $+$ and $\cdot$ denote sum and product of order types (or ordered sets). Given a finite non-empty set $F$ of order types, $eF$ denotes the type which is characterized as follows ("$\eta$

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(*) A similar proof has been found independently by F. Galvin.

(1) In [8], p. 32, Th. Stolzen proves a theorem which grants the uniqueness of shuffling. Mrs. A. Morde makes extensive use of shuffling in [9].
(Example: \( \sigma(1) = \eta \), \( \sigma(2) = \sigma(3) \); the four types
\[ \sigma(1,2,3), \sigma(1, \sigma(2,3)), \sigma(2, \sigma(3,1)), \sigma(3, \sigma(1,2)) \]
are pairwise distinct.)

Let \( M \) be the smallest class of order types which contains \( 1 \) and
is closed under the operations \( \alpha + \beta, \alpha \cdot \omega, \alpha \cdot \omega^* \), \( F \) (finite). We
write \( \sigma \vdash \Phi \) to express that the formula \( \Phi \) holds for ordered sets of
type \( \alpha \).

The results are as follows.

**Theorem 1.** The proposition \( \sigma \vdash \Phi \) is decidable for the class of pairs \( \langle \sigma, \Phi \rangle \in M \times L \).

**Theorem 2.** For every sentence \( \Phi \in L \) which is consistent with \( T \),
there exists an \( \sigma \in M \) such that \( \sigma \vdash \Phi \).

**Corollary.** \( T \) is decidable.

Proof of the Corollary. Theorems 1 and 2 imply that the set of consistent
sentences is recursively enumerable. Therefore, so is the set of refutable sentences. Since the valid sentences of \( T \) are recursively enumerable, it follows that \( T \) is decidable.

Remarks. 1. Theorem 1 is slightly stronger than the following:
The theory of \( \alpha \) is decidable for every \( \alpha \in M \). On the other hand it is
clear that there are order types whose theory is undecidable. For instance,
take \( \alpha = \omega \cdot \omega^* \), where \( \langle \alpha, \omega^* \rangle \) is a non-recursive sequence of \( 2 \)'s and
\( 3 \)'s. The theory of \( \alpha \) is seen to be undecidable because the statement
\( \alpha = \omega^* \) cannot be expressed by a single formula for each \( n \).

2. Theorem 2 says that the set \( M \) is, from the point of view of
elementary properties, dense in the class of all order types. It is interesting
to compare this with the following result of P. Erdős and A. Hajnal ([4]): If our operations \( \alpha \cdot \omega, \alpha \cdot \omega^*, \omega^* \) are replaced by the operations of forming arbitrary sums of type \( \alpha, \omega^*, \eta \), respectively, then the class of all denumerable order types is generated.

3. Concerning \( T \): Theorems 1 and 2 hold in reference to the set \( M \),
which contains all possible one-element types and is generated by the
operations \( \alpha + \beta, \alpha \cdot \omega, \alpha \cdot \omega^*, F \) (which have an obvious meaning for
types of models of \( T \)).

4. The set \( M \) is not minimal with respect to Theorem 2: \( M \) con-
tains distinct types which are elementarily equivalent; also types like
\( \omega + \omega^* \) could be thrown out.

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**Question:** Is the subset \( M \subset M \), which contains all finitely axiomatizable types from \( M \), good for Theorem 2 (in other words, is the
Boolean Algebra of sentences modulo \( T \) atomic)?

**2. The Fraïssé method applied to ordered sets.** Given an
ordered set \( A \) and \( a \in |A| \), \( A^{<a} \) (\( A^{<a} \)) will denote the initial (terminal)
segment of \( A \) determined by \( a \).

**Definition.** The \( n \)-equivalence \( (=_n) \) between two ordered sets \( A_1, A_2 \)
is recursively defined by

(a) \( A_1 =_1 A_2 \) always,

(b) \( A_1 =_{n+1} A_2 \) iff for every \( a_1 \in |A_1| \) there exists \( a_2 \in |A_2| \), and for every
\( a_2 \in |A_2| \) there exists \( a_1 \in |A_1| \), such that \( A_1^{(a_1)} = A_2^{(a_2)} \) and \( A_2^{(a_2)} = A_1^{(a_1)} \).

Let \( P_n \) denote the set of all sentences from \( L \) which are in prenex
form with a prefix of length \( n \). \( =^{\infty} \) denotes elementary equivalence.

**Lemma 1.** (i) \( A_1 =_1 A_2 \) iff \( A_1 =_n A_2 \) for every \( n \),

(ii) \( A_1 =_1 A_2 \) implies that for all \( \Phi \in P_n \), \( \Phi \vdash \Phi \).

For the proof, see A. Ehrenfeucht ([2]), B. Engeler ([3]) or R. Fraïssé ([5]).

Our version of the definition of \( n \)-equivalence is evidently seen to be equivalent
to the definition given in [2], say, when restricted to ordered sets
(see also [2], Theorem 11).

The relation \( =_n \) is an equivalence relation. The equivalence classes
will be called \( n \)-types.

**Lemma 2.** For each \( n \), there are only finitely many distinct \( n \)-types.

The proof is immediate from our definition of \( n \)-equivalence.

**Lemma 3.** If ordered sets \( A_1, A_2 \) admit order isomorphic splittings
such that corresponding parts are \( n \)-equivalent, then \( A_1 =_n A_2 \).

**Proof.** The Lemma is trivial for \( n = 0 \). Assume that \( =_1 \) is true
and that we are given isomorphic splittings \( A_1, A_2 \) of \( A_1, A_2 \), with
corresponding parts \( (n-1) \)-equivalent. Let \( a \in |A_1| \), i.e., \( a \in |B_1| \)
for some \( B_1 \subset |A_1| \). For the corresponding part \( B_1 \subset |A_1| \) we have \( B_1^{(a)} = B_2^{(a)} \),
and \( A_1^{(a)} \approx A_2^{(a)} \). Let \( a \in |B_1| \) such that \( B_1^{(a)} = B_2^{(a)} \)
and \( B_2^{(a)} = B_1^{(a)} \). The ordered sets \( A_1^{(a)} \), \( A_2^{(a)} \) admit order isomorphic splittings with \( n \)-equivalent last parts \( B_1^{(a)} \) and \( B_2^{(a)} \), and with all re-
mainding corresponding parts \( (n-1) \)-equivalent. Clearly, \( (n+1) \)-equiva-
lence implies \( n \)-equivalence. Therefore, by induction hypothesis,
\( A_1^{(a)} =_n A_2^{(a)} \). Correspondingly, \( A_1^{(a)} =_n A_2^{(a)} \). Therefore \( A_1 =_n A_2 \).

**Corollary 3.1.** If ordered sets \( A_1, A_2 \) admit order isomorphic splittings
such that corresponding parts are elementarily equivalent, then \( A_1 =_1 A_2 \).

**Proof by Lemma 1.** (i) and Lemma 3.

Since isomorphism implies elementary equivalence, which in turn
implies \( n \)-equivalence, \( =_n \) and \( =_0 \) can be considered as relations between order types.
COROLLARY 3.2. The relations \( = \) and \( \equiv \) are compatible \(^{(1)}\) with the operations \( a \cdot \beta, a \cdot \omega, a \cdot \alpha \cdot \omega, a \cdot F \).

Proof by Lemma 3.

LEMMA 4. If ordered sets \( A_2 \) and \( A_3 \) admit \( n \)-equivalent [elementary equivalent] splittings \( A_2, A_3 \) such that \( B_i = B_i \) for all \( B_i \in [A_i], B_i \in [A_i], \) then \( A_2 = A_3 \).

Proof. Almost a repetition of the proof of Lemma 3.

3. PROOF OF THEOREM 1. In this section, Greek letters \( \alpha, \beta, \ldots \) will denote terms which represent the elements of \( M. \beta < \alpha \) denotes that \( \beta \) is one of the terms \( \beta + \gamma, \gamma + \beta, \beta \cdot \omega, \beta \cdot \alpha, \beta \cdot \alpha \cdot \omega, \beta \cdot F \) with \( \beta \in F \). The set \( \{ \beta \mid \beta < \alpha \} \) is finite for each \( \alpha \in M. \) \( L(a) \) denotes the language which contains in addition to the constants of \( L \) a binary predicate \( \sim \), and, corresponding to each \( \beta < \cdot \alpha, \) a unary predicate \( Q_\beta \).

\[ \Sigma_a = \{ \forall u, u < v \wedge u \sim u \wedge \neg u < u \} \]

If \( a \neq 1 \) then \( \Sigma_a \) contains:

1) axioms for linear order,
2) if \( a = \beta + \gamma \) [\( a = \alpha F \)]:
   - \( \sim \) is an equivalence relation which determines a splitting of order type \( \gamma \) \( \gamma \) (4),
   - \( \sim \) is an equivalence relation which determines a splitting of the elementary type of \( \omega \) \( \omega \) (4),
3) every element belongs to just one predicate \( Q_\beta, \beta < \alpha \) if \( u \sim v \) then \( u \) and \( v \) belong to the same \( Q_\beta \),
4) if \( a = \beta + \gamma: \)
   - \( \forall u, v, u < v \wedge u \sim v \rightarrow Q_\beta(u) \wedge Q_\beta(v) \),
   - \( \forall u, v, u < v \wedge u \sim v \rightarrow \forall \in \Delta \{ u < w < v \wedge Q_\beta(u) \} \).

For \( \Phi \in L \) and \( \beta \in M \), let \( \Phi(\beta) \) denote the formula

\[ \forall \in \{ (\sim \circ Q_\beta(v) \rightarrow \Phi(\sim v)) \} \]

where \( \circ \) is a variable not occurring in \( \Phi \) and \( \Phi(\sim v) \) is obtained from \( \Phi \) by restricting all quantifiers to the predicate \( \sim u \).

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\(^{(1)}\) An equivalence relation \( \sim \) is said to be compatible with an operation \( a \cdot \beta. \)

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Let

\[ \Gamma_\beta = \{ \Phi(\beta) \mid \beta < \alpha \text{ and } \Phi \in L \text{ and } \beta \vdash \Phi \} \]

\( T(a) \) will denote the set of all consequences from the axioms \( \Sigma_a \wedge \Gamma_\beta \).

Given a model \( M \) of \( L(a) \), its natural reduction \(^{(2)}\) to \( L \) will be denoted by \( M' \).

LEMMA 5. (i) If \( M \) is a model of \( T(a) \) and \( \Phi \in L \), then \( M \models \Phi \iff M' \models \Phi \).

(ii) For every ordered set \( A \) of type \( a \) there is a model \( M \) of \( T(a) \) such that \( M' = A \).

(iii) \( M_1 = M_2 \) for any two models \( M_1, M_2 \) of \( T(a) \).

(i) is immediate.

Proof of (ii). Immediate if \( A \) is a one-element set. If not, there exists a splitting of \( A \) in accordance with the structure of term \( a, \) such that the parts are of types \( \beta, \beta < \cdot a. \)

We interpret \( \sim \) as an equivalence relation on \( [A] \) which determines such a splitting, and we let \( Q_\beta \) mark the \( \beta \)-parts. The model \( M \) thus obtained satisfies \( \Sigma_a \wedge \Gamma_\beta \).

Proof of (iii). Immediate for \( a = 1. \) For composite \( a, \) inspection of \( \Sigma_a \wedge \Gamma_\beta \) shows that \( M_1 \) and \( M_2 \) admit splittings which satisfy the hypotheses of Corollary 3.1 in the cases \( a = \beta + \gamma, \alpha = \alpha F. \) and the hypothesis of Lemma 4 in the cases \( a = \beta \cdot \omega, \alpha = \alpha \cdot \omega. \) Thus \( M_1 \equiv M_2 \).

COROLLARY 5.1. If \( a \in M \) and \( \Phi \in L, \) then

\[ \Phi \in T(a) \iff M \models \Phi \]

Proof. Let \( \Phi \in T(a) \). Let \( A \) be an ordered set of type \( a. \) Let \( M \) be a model of \( T(a) \) such that \( M' = A \) (Lemma 5 (ii)). Then \( M \models \Phi. \) Therefore \( A \models \Phi \) (Lemma 5 (i)). On the other hand, Lemma 5 implies that \( T(a) \) is complete with respect to sentences in \( L. \) Therefore the converse implication.

A finite sequence of pairs

\[ \langle a_1, \Phi_1 \rangle, \ldots, \langle a_p, \Phi_p \rangle \]

such that \( a_i \in M \) and \( \Phi_i \in L(a_i), i = 1, \ldots, p, \) will be called a proof sequence, if for all \( i = 1, \ldots, p: \)

Either \( \Phi_i \) follows from formulas \( \Phi_j \) with \( j < i \) and \( a_j < a_i \) by means of the usual rules of inference,

or \( \Phi_i \) is a logical axiom,

or \( \Phi_i \in \Sigma_{a_j} \)

or \( \Phi_i = \Phi_i(a) \) for some \( j < i \) with \( a_j < a_i \) and \( \Phi_j \in L. \)

Thus \( M' \) arises from \( M \) by dropping interpretations of terms which do not belong to \( L. \)

In fact, \( T(a) \) is complete. But our version of \( n \)-equivalence is not adequate for the proof of this.

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Footnotes:

1. An equivalence relation \( \sim \) is said to be compatible with an operation \( a \cdot \beta. \)
2. Thus \( M' \) arises from \( M \) by dropping interpretations of terms which do not belong to \( L. \)
3. In fact, \( T(a) \) is complete. But our version of \( n \)-equivalence is not adequate for the proof of this.
LEMMA 6. (i) The set of all proof sequences is recursively enumerable,
(ii) if $a \in M$ and $\Phi \in L(a)$, then $\Phi \in \mathcal{L}(a)$ iff the pair $(a, \Phi)$ occurs in some proof sequence.

Proof. (i) is immediate and (ii) follows from Corollary 5.1 by a straight forward induction.

Proof of Theorem 1. Corollary 5.1 and Lemma 6 together imply that the set of pairs $(a, \Phi) \in M \times L$ with $a \models \Phi$ is recursively enumerable. Since the theory of $a$ is complete (and $a \models \Phi$ iff $a \models \Phi^*$, $\Phi^*$ universal closure of $\Phi$), the proposition "$a \models \Phi$" is decidable.

4. Proof of Theorem 2. $a$ will be fixed throughout this section.

An ordered set $A$ (order type $\alpha$) is said to be good if $A[a]$ is $\alpha$-equivalent to some $\beta \in M$.

As a consequence of Corollary 3.2, we have

LEMMA 7. The class of good order types is closed under the operations $a + \beta$, $a + \text{w} \alpha$, $a - \text{w} \alpha$, $a \Phi$.

$B$ is said to be a bounded segment of an ordered set $A$, if $B$ is a segment of some closed segment $[x, y]$ of $A$.

LEMMA 8. If every bounded segment of a denumerable ordered set $A$ is good, then $a \models \Phi$.

Proof. We assume that $\Phi$ has a first, but no last, element (the proof is similar in the other cases). (1) Let $S$ be a subset of $A \Phi$ of order type $\alpha$ which is cofinal in $A$ (denumerable). We partition the set of all unordered pairs $(x, y) \in S \times S$ according to the $\alpha$-type of the half-open segment $[y, x]$ of $A$ (if $x < y$, $(y, x)$ otherwise). By Lemma 2, there is only a finite number of $\alpha$-types. Thus, by Ramsey's Theorem (see [7]), there is an infinite subset $S_1 \subseteq S$ such that any two segments $[x, y]$ of $A$ with $x, y$ in $S_1$ and $x < y$ are $\alpha$-equivalent as an infinite subset of an ordered set of type $\alpha$, $S_1$ is cofinal in $S$. Therefore, $S_1$ is cofinal in $A$.

By Lemma 3,

$$A \models \alpha \Phi + [x, y] \cdot \alpha \text{ for any } x, y \in S_1, x < y.$$ 

The segments $A \Phi$ and $[x, y]$ are bounded, hence good. Therefore, by Lemma 7, $A \models \Phi$.

LEMMA 9. Every ordered set is good.

Proof. By the Skolem-Löwenheim Theorem and Lemma 1 (i), we may assume that the given ordered set $A$ is denumerable.

Let $x \models \Phi$, $x, y \in A$, denote that every segment of the closed segment $[x, y]$ is good.

$(\dagger)$ If $A$ has no first and no last element then $A = B + C - D$ where $B$ has a last element, $D$ has a first element, and $C$ is good since bounded.

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