For the latter part, consider the lattice of figure 3 and the ideals I_1 and I_2, namely the principal ideals generated by a and b. Then each of $\theta(I_1)$ and $\theta(I_2)$ is the universal congruence on L and hence is their intersection; while $\theta(I_1 \wedge I_2)$ is the null congruence on L. Thus g does not preserve products.

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On the elementary theory of linear order *

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A. Ehrenfeucht proved in [1] that the elementary theory T of linear order, even with a finite number of unary predicates adjoined (then denoted by T_1), is decidable. Here we establish the same result by a different method. We think that our proof is of interest on its own, because it gives further insight as to how much can be expressed in the first-order language of linear order (see Theorem 2 below) (1). It is only for the sake of a lighter notation that we restrict our proof to the theory T; there is an obvious generalization to a proof for T_1 (see also remark 3 below).

1. Let L be the first-order language with identity and one binary predicate $<$. T is the theory over L of the axioms

(a) $\neg u < u$, (b) $u < v \wedge v < w \rightarrow u < w$, (c) $u = v \vee u < v \vee v < u$.

We use $x < y \,(\mathrm{mod}\, A)$ to denote the order relation of an ordered set A, $|A|$ to denote the field of A. B is said to be a *segment* of A, if B is a submodel of A and if $x < y \,(\mathrm{mod}\, B)$ and $x < z < y \,(\mathrm{mod}\, A)$ implies $z \in |B|$. An ordered set $\mathcal{A}$ is said to be a *splitting* of A, if $|\mathcal{A}|$ is a set of (non-empty) segments of A, which partitions $|A|$, and if $B < C \,(\mathrm{mod}\, \mathcal{A})$ iff $x < y \,(\mathrm{mod}\, A)$ for all $x \in |B|$, $y \in |C|$. The elements of $|\mathcal{A}|$ are called the *parts* of the splitting.

ω, ω^*, η respectively denote the order type of the positive integers, negative integers, rationals. $+$ and $\cdot$ denote sum and product of order types (or ordered sets). Given a finite non-empty set F of order types, σF denotes the type which is characterized as follows ("σ" for "shuffling"): An ordered set A is of type σF iff there exists a splitting of A of order type η such that every part has a type from F and between any two different parts there are parts of each type from F (2).

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(1) A similar proof has been found independently by F. Galvin.

(2) In [8], p. 32, Th. Skolem proves a theorem which grants the uniqueness of shuffling. Mrs. A. Morel makes extensive use of shuffling in [6].

(Example: $\sigma\{1\} = \eta$, $\sigma\{\alpha\} = \alpha\cdot\eta$; the four types

$$\sigma\{1,2,3\}, \quad \sigma\{1,\sigma\{2,3\}\}, \quad \sigma\{2,\sigma\{3,1\}\}, \quad \sigma\{3,\sigma\{1,2\}\}$$

are pairwise distinct.)

Let M be the smallest class of order types which contains 1 and is closed under the operations $\alpha+\beta$, $\alpha\cdot\omega$, $\alpha\cdot\omega^*$, σF (F finite). We write $\alpha \models \Phi$ to express that the formula Φ holds for ordered sets of type α.

The results are as follows.

THEOREM 1. *The proposition "$\alpha\models\Phi$" is decidable for the class of pairs $\langle\alpha,\Phi\rangle \in M\times L$* (³).

THEOREM 2. *For every sentence $\Phi\in L$ which is consistent with T, there exists an $\alpha\in M$ such that $\alpha\models\Phi$.*

COROLLARY. *T is decidable.*

Proof of the Corollary. Theorems 1 and 2 imply that the set of consistent sentences is recursively enumerable. Therefore, so is the set of refutable sentences. Since the valid sentences of T are recursively enumerable, it follows that T is decidable.

Remarks. 1. Theorem 1 is slightly stronger than the following: *The theory of α is decidable for every $\alpha\in M$.* On the other hand it is clear that there are order types whose theory is undecidable. For instance, take $\alpha = \Sigma(\eta+a_n)$, where $\{a_n\}_{n\in\omega}$ is a non-recursive sequence of 2's and 3's. The theory of α is seen to be undecidable because the statement "$a_n = 2$" can be expressed by a single formula for each n.

2. Theorem 2 says that the set M is, from the point of view of elementary properties, dense in the class of all order types. It is interesting to compare this with the following result of P. Erdös and A. Hajnal ([4]): If our operations $\alpha\cdot\omega$, $\alpha\cdot\omega^*$, σF are replaced by the operations of forming arbitrary sums of type ω, ω^*, η, respectively, then the class of all denumerable order types is generated.

3. Concerning T_1: Theorems 1 and 2 hold in reference to the set M_1 which contains all possible one-element types and is generated by the operations $\alpha+\beta$, $\alpha\cdot\omega$, $\alpha\cdot\omega^*$, σF (which have an obvious meaning for types of models of T_1).

4. The set M is not minimal with respect to Theorem 2: M contains distinct types which are elementarily equivalent; also types like $\omega+\omega^*$ could be thrown out.

(³) It is understood that the elements of M are given as terms built by means of the operation symbols introduced above. We do not try to solve the word problem for the class of these terms.

Question: Is the subset $M_f\subset M$, which contains all finitely axiomatizable types from M, good for Theorem 2 (in other words, is the Boolean Algebra of sentences modulo T atomic)?

2. The Fraïssé method applied to ordered sets. Given an ordered set A and $a\in|A|$, $A^{<a}$ ($A^{>a}$) will denote the initial (terminal) segment of A determined by a.

DEFINITION. The *n-equivalence* ($\equiv_n$) between two ordered sets A_1, A_2 is recursively defined by

(a) $A_1 \equiv_0 A_2$ always,

(b) $A_1 \equiv_{n+1} A_2$ iff for every $a_1\in|A_1|$ there exists $a_2\in|A_2|$, and for every $a_2\in|A_2|$ there exists $a_1\in|A_1|$, such that $A_1^{<a_1}\equiv_n A_2^{<a_2}$ and $A_1^{>a_1}\equiv_n A_2^{>a_2}$.

Let P_n denote the set of all sentences from L which are in prenex form with a prefix of length n. "$\equiv$" denotes elementary equivalence.

LEMMA 1. (i) *$A_1\equiv A_2$ iff $A_1\equiv_n A_2$ for every n,*

(ii) *$A_1\equiv_n A_2$ implies that for all $\Phi\in P_n$, $A_1\models\Phi$ iff $A_2\models\Phi$.*

For the proof, see A. Ehrenfeucht [2], E. Engeler [3] or R. Fraïssé [5]. Our version of the definition of n-equivalence is easily seen to be equivalent to the definition given in [2], say, when restricted to ordered sets (see also [2], Theorem 11).

The relation $\equiv_n$ is an equivalence relation. The equivalence classes will be called n-types.

LEMMA 2. *For each n, there are only finitely many distinct n-types.*

The proof is immediate from our definition of n-equivalence.

LEMMA 3. *If ordered sets A_1, A_2 admit order isomorphic splittings such that corresponding parts are n-equivalent, then $A_1\equiv_n A_2$.*

Proof. The Lemma is trivial for $n = 0$. Assume then that it is true for n and that we are given isomorphic splittings $\mathcal{A}_1$, $\mathcal{A}_2$ of A_1, A_2 with corresponding parts $(n+1)$-equivalent. Let $a_1\in|A_1|$, i.e. $a_1\in|B_1|$ for some $B_1\in|\mathcal{A}_1|$. For the corresponding part $B_2\in|\mathcal{A}_2|$ we have $B_1\equiv_{n+1} B_2$ and $\mathcal{A}_1^{<B_1}\cong\mathcal{A}_2^{<B_2}$, $\mathcal{A}_1^{>B_1}\cong\mathcal{A}_2^{>B_2}$. Let $a_2\in|B_2|$ such that $B_1^{<a_1}\equiv_n B_2^{<a_2}$ and $B_1^{>a_1}\equiv_n B_2^{>a_2}$. The ordered sets $A_1^{<a_1}$, $A_2^{<a_2}$ admit order isomorphic splittings with n-equivalent last parts $B_1^{<a_1}$ and $B_2^{<a_2}$, and with all remaining corresponding parts $(n+1)$-equivalent. Clearly, $(n+1)$-equivalence implies n-equivalence. Therefore, by induction hypothesis, $A_1^{<a_1}\equiv_n A_2^{<a_2}$. Correspondingly, $A_1^{>a_1}\equiv_n A_1^{>a_2}$. Therefore $A_1\equiv_{n+1} A_2$.

COROLLARY 3.1. *If ordered sets A_1, A_2 admit order isomorphic splittings such that corresponding parts are elementarily equivalent, then $A_1\equiv A_2$.*

Proof by Lemma 1, (i) and Lemma 3.

Since isomorphy implies elementary equivalence, which in turn implies n-equivalence, $\equiv$ and $\equiv_n$ can be considered as relations between order types.

COROLLARY 3.2. *The relations* $\equiv$ *and* $\equiv_n$ *are compatible* (4) *with the operations* $\alpha+\beta$, $\alpha\cdot\omega$, $\alpha\cdot\omega^*$, σF.

Proof by Lemma 3.

LEMMA 4. *If ordered sets* A_1, A_2 *admit* n*-equivalent* [*elementary equivalent*] *splittings* $\mathcal{A}_1, \mathcal{A}_2$ *such that* $B_1 \equiv_n B_2$ $[B_1 \equiv B_2]$ *for all* $B_1 \in |\mathcal{A}_1|$, $B_2 \in |\mathcal{A}_2|$, *then* $A_1 \equiv_n A_2$ $[A_1 \equiv A_2]$.

Proof. Almost a repetition of the proof of Lemma 3.

3. Proof of Theorem 1. In this section, Greek letters $\alpha, \beta, \ldots$ will denote *terms* which represent the elements of M. $\beta <\cdot\, \alpha$ denotes that α is one of the terms $\beta+\gamma$, $\gamma+\beta$, $\beta\cdot\omega$, $\beta\cdot\omega^*$, σF with $\beta \in F$. The set $\{\beta\colon \beta <\cdot\, \alpha\}$ is finite for each $\alpha \in M$. $L(\alpha)$ denotes the language which contains in addition to the constants of L a binary predicate $\sim$, and, corresponding to each $\beta <\cdot\, \alpha$, a unary predicate Q_β.

Σ_α denotes the following set of sentences from $L(\alpha)$:

$$\Sigma_1 = \{\forall u, v(u = v \wedge u \sim u \wedge \neg\, u < u)\}.$$

If $\alpha \neq 1$ then Σ_α contains:

1) axioms for linear order,

2) if $\alpha = \beta+\gamma$ $[\alpha = \sigma F]$:

"$\sim$ is an equivalence relation which determines a splitting of order type 2 $[\eta]$",

if $\alpha = \beta\cdot\omega$ $[\alpha = \beta\cdot\omega^*]$:

"$\sim$ is an equivalence relation which determines a splitting of the elementary type of ω $[\omega^*]$",

3) "every element belongs to just one predicate Q_β, $\beta <\cdot\, \alpha$; if $u \sim v$ then u and v belong to the same Q_β",

4) if $\alpha = \beta+\gamma$:

$$\forall u, v\big(u < v \wedge \neg\, u \sim v \rightarrow Q_\beta(u) \wedge Q_\gamma(v)\big),$$

if $\alpha = \sigma F$:

$$\forall u, v\Big[u < v \wedge \neg\, u \sim v \rightarrow \bigwedge_{\beta \in F} \exists w\big(u < w < v \wedge Q_\beta(w)\big)\Big].$$

For $\Phi \in L$ and $\beta \in M$, let $\Phi(\beta)$ denote the formula

$$\forall v\big(Q_\beta(v) \rightarrow \Phi(\sim v)\big),$$

where v is a variable not occuring in Φ and $\Phi(\sim v)$ is obtained from Φ by restricting all quantifiers to the predicate $\lambda u(u \sim v)$.

(4) An equivalence relation $\sim$ is said to be *compatible* with an operation $\alpha\circ\beta$, if $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ implies $\alpha_1\circ\beta_1 \sim \alpha_2\circ\beta_2$.

Let

$$\Gamma_\alpha = \{\Phi(\beta)\colon \beta <\cdot\, \alpha \text{ and } \Phi \in L \text{ and } \beta \models \Phi\}.$$

$T(\alpha)$ will denote the set of all consequences from the axioms $\Sigma_\alpha \cup \Gamma_\alpha$.

Given a model $\mathcal{M}$ of $L(\alpha)$, its natural reduction (5) to L will be denoted by $\mathcal{M}'$.

LEMMA 5. (i) *If* $\mathcal{M}$ *is a model of* $T(\alpha)$ *and* $\Phi \in L$, *then* $\mathcal{M} \models \Phi$ *iff* $\mathcal{M}' \models \Phi$.

(ii) *For every ordered set* A *of type* α *there is a model* $\mathcal{M}$ *of* $T(\alpha)$ *such that* $\mathcal{M}' = A$.

(iii) $\mathcal{M}_1' \equiv \mathcal{M}_2'$ *for any two models* $\mathcal{M}_1, \mathcal{M}_2$ *of* $T(\alpha)$ (6).

(i) is immediate.

Proof of (ii). Immediate if A is a one-element set. If not, there exists a splitting of A in accordance with the structure of term α, such that the parts are of types β, $\beta <\cdot\, \alpha$. We interprete $\sim$ as an equivalence relation on $|A|$ which determines such a splitting, and we let Q_β mark the β-parts. The model $\mathcal{M}$ thus obtained satisfies $\Sigma_\alpha \cup \Gamma_\alpha$.

Proof of (iii). Immediate for $\alpha = 1$. For composite α, inspection of $\Sigma_\alpha \cup \Gamma_\alpha$ shows that $\mathcal{M}_1'$ and $\mathcal{M}_2'$ admit splittings which satisfy the hypotheses of Corollary 3.1 in the cases $\alpha = \beta+\gamma$, $\alpha = \sigma F$, and the hypothesis of Lemma 4 in the cases $\alpha = \beta\cdot\omega$, $\alpha = \beta\cdot\omega^*$. Thus $\mathcal{M}_1' \equiv \mathcal{M}_2'$.

COROLLARY 5.1. *If* $\alpha \in M$ *and* $\Phi \in L$, *then*

$$\Phi \in T(\alpha) \text{ iff } \alpha \models \Phi.$$

Proof. Let $\Phi \in T(\alpha) \cap L$. Let A be an ordered set of type α. Let $\mathcal{M}$ be a model of $T(\alpha)$ such that $\mathcal{M}' = A$ (Lemma 5 (ii)). Then $\mathcal{M} \models \Phi$. Therefore $A \models \Phi$ (Lemma 5 (i)). On the other hand, Lemma 5 implies that $T(\alpha)$ is complete with respect to sentences in L. Therefore the converse implication.

A finite sequence of pairs

$$\langle \alpha_1, \Phi_1\rangle, \ldots, \langle \alpha_p, \Phi_p\rangle$$

such that $\alpha_i \in M$ and $\Phi_i \in L(\alpha_i)$, $i = 1, \ldots, p$, will be called a *proof sequence*, if for all $i = 1, \ldots, p$:

Either Φ_i follows from formulas Φ_j with $j < i$ and $\alpha_j = \alpha_i$ by means of the usual rules of inference,

or Φ_i is a logical axiom,

or $\Phi_i \in \Sigma_{\alpha_i}$,

or $\Phi_i = \Phi_j(\alpha_j)$ for some $j < i$ with $\alpha_j <\cdot\, \alpha_i$ and $\Phi_j \in L$.

(5) Thus $\mathcal{M}'$ arises from $\mathcal{M}$ by dropping interpretations of terms which do not belong to L.

(6) In fact, $T(\alpha)$ is complete. But our version of n-equivalence is not adequate for the proof of this.

LEMMA 6. (i) *The set of all proof sequences is recursively enumerable,*

(ii) *if $a \in M$ and $\Phi \in L(a)$, then $\Phi \in T(a)$ iff the pair $\langle a, \Phi\rangle$ occurs in some proof sequence.*

Proof. (i) is immediate and (ii) follows from Corollary 5.1 by a straight foreward induction.

Proof of Theorem 1. Corollary 5.1 and Lemma 6 together imply that the set of pairs $\langle a, \Phi\rangle \in M \times L$ with $a \models \Phi$ is recursively enumerable. Since the theory of a is complete (and $a \models \Phi$ iff $a \models \Phi^*$, Φ^* universal closure of Φ), the proposition "$a \models \Phi$" is decidable.

4. Proof of Theorem 2. n will be fixed throughout this section. An ordered set A [order type a] is said to be *good* if $A[a]$ is n-equivalent to some $\beta \in M$.

As a consequence of Corollary 3.2, we have

LEMMA 7. *The class of good order types is closed under the operations* $a+\beta$, $a\cdot\omega$, $a\cdot\omega^*$, σF.

B is said to be a *bounded* segment of an ordered set A, if B is a segment of some closed segment $[x, y]$ of A.

LEMMA 8. *If every bounded segment of a denumerable ordered set A is good, then A is good.*

Proof. We assume that A has a first, but no last, element (the proof is similar in the other cases).(7) Let S be a subset of A of order type ω which is cofinal in A (A denumerable). We partition the set of all unordered pairs $\{x, y\} \subseteq S$ according to the n-type of the half-open segment $[x, y)$ of A (if $x < y$. $[y, x)$ otherwise). By Lemma 2, there is only a finite number of n-types. Thus, by Ramsey's Theorem (see [7]), there is an infinite subset $S_1 \subseteq S$ such that any two segments $[x, y)$ with x, y in S_1 and $x < y$ are n-equivalent. As an infinite subset of an ordered set of type ω, S_1 is cofinal in S. Therefore, S_1 is cofinal in A. By Lemma 3,

$$A \equiv_n A^{<x} + [x, y)\cdot\omega \quad \text{for any} \quad x, y \in S_1,\ x < y\,.$$

The segments $A^{<x}$ and $[x, y)$ are bounded, hence good. Therefore, by Lemma 7, A is good.

LEMMA 9. *Every ordered set is good.*

Proof. By the Skolem-Löwenheim Theorem and Lemma 1 (i), we may assume that the given ordered set A is denumerable.

Let $x \sim y$, $x, y \in |A|$, denote that every segment of the closed segment $[x, y]$ is good.

(7) If A has no first and no last element then $A = B+C+D$ where B has a last element, D has a first element, and C is good since bounded.

1. *$\sim$ is an equivalence relation with the splitting property.*

Proof immediate by Lemma 7 (in reference to addition).

2. *Every equivalence class C has the property that each segment of C is good* (8).

Proof immediate by Lemma 8.

3. *The splitting $\mathcal{A}$ determined by $\sim$ is of a dense order type* (9).

Proof immediate by 2 above and Lemma 7.

4. *There is only one equivalence class:* $|\mathcal{A}| = \{A\}$.

Proof. By 2 above and Lemma 2, there exists a finite subset $F' \subseteq M$ and a function h from $|\mathcal{A}|$ onto F' such that $hC \equiv_n C$, all $C \in |\mathcal{A}|$. For $C < C' (\mathrm{mod}\, \mathcal{A})$, let $h(C, C')$ denote the range of the function h restricted to the open segment (C, C') of $\mathcal{A}$. Supposing that there are different equivalence classes, $h(C_1, C_2)$ assumes a minimal subset F of F' for some $C_1 < C_2 (\mathrm{mod}\, \mathcal{A})$ (F' finite!). Let $x \in |C_1|$, $y \in |C_2|$ and consider a segment B of the segment $[x, y]$ of A. We prove that B is good. This is true if B is segment of some $C \in |\mathcal{A}|$. Otherwise, by 3 above, B is of the form $B_1 + \bigcup \mathcal{B} + B_2$, where $\mathcal{B}$ is some segment of (C_1, C_2) of order type η, $\bigcup \mathcal{B}$ denotes the corresponding segment of A, and B_1, B_2 are (possibly empty) segments of some classes $C_1', C_2' \in |\mathcal{A}|$. By minimality of F, $h(C, C') = F$ for all $C, C' \in |\mathcal{B}|$ with $C < C' (\mathrm{mod}\, \mathcal{A})$. Therefore, by Lemma 3, $\bigcup \mathcal{B} \equiv_n \sigma F$. B_1, B_2 are good by 2 above. Hence B is good. Therefore $x \sim y$, which contradicts the assumption $C_1 < C_2$.

The Lemma now follows from 2 and 4.

Proof of Theorem 2. Let Φ be a sentence which is consistent with T. We may assume that Φ is in prenex form. Let A be a model of Φ, n the length of the prefix of Φ. By Lemma 9, $A \equiv_n a$ for some $a \in M$, and by Lemma 1 (ii), $a \models \Phi$.

(8) In particular, C is good.

(9) That is, one of the types 1, η, $\eta+1$, $1+\eta$, $1+\eta+1$.

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