$a+x_1 \neq b+x_1$). Next put $b_i = a_1 x_2$; then $(a+x_1)x_2 = b_0 \geqslant b_1 \geqslant \ldots \geqslant b_n = (b+x_1)x_2$. Thus, for each i, (b_{i-1}, b_i) is a multiplicative translate of a prime interval (a_{i-1}, a_i). Hence every prime interval of (b_{i-1}, b_i) is projective to (a_{i-1}, a_i) and so in turn projective to (a, b) for $i = 1, 2, \ldots, n$. Thus there exists a maximal chain joining $\big((a+x_1)x_2, (b+x_1)x_2\big)$ such that each prime interval of it is projective to (a, b).

By a finite iteration of this process we see that there exists a finite maximal chain joining (c, d) such that each prime interval of it is projective to (a, b). But $c \gtrsim d$. Hence (c, d) is projective to (a, b). Thus the equivalence of the two definitions is established.

Note. It is worthwhile noting that we have given an answer to problem 10 of [3] in the first part of this paper. Problem 10 of [3] is: Give types of weakly modular lattices which are defined by identical relations and are different from the following three classes of lattices: (a) the class consisting of only the lattice of one element, (b) the class of distributive lattices, and (c) the class of modular lattices.

We see that the supermodular lattice of any order satisfies the requirements for an answer to the above problem.

In conclusion, my thanks are due to Professor V. S. Krishnan for his constant help during the preparation of this paper.

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On a Galois correspondence between the lattice of ideals and the lattice of congruences on a lattice L *

by

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This paper deals with a natural Galois correspondence between the lattice of congruence on L (denotes by $\theta(L)$) and the lattice of ideals of L (denoted by $I(L)$). To any congruence θ on L there corresponds an ideal of L, i.e., the zero class under the congruence θ on L, denoted by $I(\theta)$. Let f denote the mapping $\theta \to I(\theta)$, i.e., a mapping from $\theta(L)$ into $I(L)$. Conversely, starting with an ideal I of a lattice L there corresponds a congruence $\theta(I)$ on L, namely the congruence generated by the ideal I of L. Let g denote the mapping $I \to \theta(I)$; i.e., a mapping from $I(L)$ into $\theta(L)$. The mappings f and g define a Galois correspondence G between the lattice of ideals of L (under the usual ordering $<$) and the lattice of congruences on L (under the ordering $<'$, dual to the usual ordering $<$), since:

(i) If $\theta <' \varphi$ then $I(\theta) > I(\varphi)$.

(ii) If $I < J$ then $\theta(I) >' \theta(J)$.

(iii) For any I, $I\big(\theta(I)\big) \geqslant I$, and for any θ, $\theta\big(I(\theta)\big) \geqslant' \theta$.

Now in any Galois correspondence the closed elements of either are the image elements of the other. The closed elements of $I(L)$ (elements I such that $I\big(\theta(I)\big) = I$) and closed elements of $\theta(L)$ (elements θ such that $\theta\big(I(\theta)\big) = \theta$) are termed *congruence ideals* and *zero congruences* respectively. Hence we have:

Definition 1. An ideal I is a *congruence ideal* of L if and only if any lattice translate of any interval of I belongs to I or lies completely outside I.

An interval J is a *lattice translate* of an interval I of L if elements $x_1, x_2, \ldots, x_n$ can be found such that

$$J = \Big(\big((I+x_1)\cdot x_2\big)+x_3 \ldots\Big)\, x_n$$

* This paper forms a part of the Doctoral thesis submitted to the University of Madras in January 1964.

or

$$J = \Big(\big((I\cdot x_1)+x_2\big)\cdot x_3 \ldots\Big)x_n;$$

where n is finite and $+$, $\cdot$ occur alternately (cf. [2]).

DEFINITION 2. A congruence θ on L is a *zero congruence on* L if and only if any interval annulled by θ is a finite sum of lattice translates of intervals in the kernel of θ.

The smallest congruence θ on L under which a set S in L belongs to a single class under the congruence θ on L is called the *congruence generated by the set* S and is denoted by θ_S. Further any interval annulled by θ_S consists of a finite sum of lattice translates of intervals of S (cf. [2]).

Further we know that the closed elements of any lattice under any Galois correspondence form a complete lattice which is a subsystem of the original with respect to products. Also the complete lattice of closed elements of either lattice is anti-isomorphic to the complete lattice of closed elements of the other in any Galois correspondence. Interpreting these results in the usual orderings $(I(L), <)$ and $(\theta(L), <)$ we have

THEOREM 1. *The congruence ideals on any lattice* L *form a complete lattice* $C(L)$ *closed for ideal products, but not for ideal sums.*

The fact that the family of congruence ideals are not closed for ideal sums follow from figure 1 below, where the ideal I = the principal ideal generated by a and J = the principal ideal generated by b are congruence ideals, while the ideal sum of I and J, i.e. the principal ideal generated by c is not a congruence ideal. Because the congruence generated by $I+J$ is the universal congruence on L.

LEMMA 1. *The zero congruences on any lattice* L *form a complete lattice* $Z(L)$.

LEMMA 2. $(C(L), <)$ *is isomorphic to* $(Z(L), <)$.

Thus we have the following theorem.

THEOREM 2. *The sum of a family of zero congruences* (θ_i) *is a zero congruence on* L *determined by the sum of the congruence ideals* $I_i = I(\theta_i)$ *in* $C(L)$; *while the product of even two zero congruences is not in general a zero congruence on* L.

Proof. The former part follows from lemmas 1 and 2. For the latter we make use of an example.

Consider the lattice of figure 2. The congruence generated by the principal ideals $\mu(b)$ and $\mu(c)$ are zero congruences. But their intersection is the congruence generated by the interval (a, b) and hence is not a zero congruence on L.

THEOREM 3. *Every ideal of a lattice* L *is closed under the Galois correspondence* G *if and only if* L *is a distributive lattice.*

Proof. Let L be a distributive lattice. It is well known (and can be easily verified) that any ideal of L is a congruence ideal and hence is closed under the Galois correspondence G.

Conversely, let every ideal of L be closed under G. Let, if possible, L be non-distributive; then L contains a sublattice of one of the two types given in figures 2 and 3. In either case the principal ideal generated by a is not a congruence ideal and hence is not closed under the Galois correspondence G; a contradiction to our assumption. Thus L is a distributive lattice.

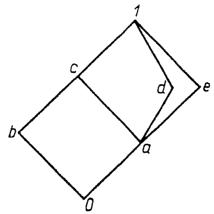

Fig. 1

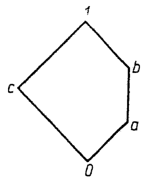

Fig. 2

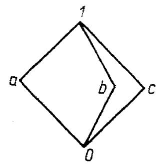

Fig. 3

THEOREM 4. *Every congruence on a lattice* L *with zero is closed under* G *if and only if for any interval* $p = (x, y)$ *of* L, *there exists another interval* $q = (z, 0)$ *such that* $\theta_p = \theta_q$ (θ_p, θ_q *being the congruences generated by the intervals* p *and* q, *respectively*).

Proof. Let every congruence on L be closed and let $p = (x, y)$ be any interval of L. Consider θ_p, the congruence generated by p. Now θ_p should be closed under G. Thus $\theta_p = \theta\big(I(\theta_p)\big)$, whence θ_p is generated by $I(\theta_p)$. Thus p is annulled by $\theta\big(I(\theta_p)\big)$, which implies that there exists a finite chain $x = x_0 \geqslant x_1 \geqslant \ldots \geqslant x_n = y$ such that (x_{i-1}, x_i) is a lattice translate of some interval (a_i, b_i) of $I(\theta_p)$. As $I(\theta_p)$ is an ideal containing a_i it contains the interval $(a_i, 0)$. Thus, for each i, (x_{i-1}, x_i) is a lattice translate of some interval $(a_i, 0)$. But $\bigcup_{i=1}^{n} a_i$ is in $I(\theta_p)$; as $I(\theta_p)$ is an ideal containing a_i for each i and n is finite. Put $z = \bigcup_{i=1}^{n} a_i$; then (x_{i-1}, x_i) is a lattice translate of $(z, 0)$ for each i, and hence p is annulled by θ_q, where $q = (z, 0)$. Also θ_p annuls q as q belongs to $I(\theta_p)$. Hence $\theta_p = \theta_q$. Thus the necessity of the condition is established.

Conversely let L be a lattice satisfying the condition. Let θ be any congruence on L. We have $x \equiv y(\theta)$ if and only if $xy \equiv x+y(\theta)$. Let

p be the interval $(x+y, xy)$ of L. Let $q = (z, 0)$ be the interval corresponding to p such that $\theta_p = \theta_q$. Then $x \equiv y(\theta)$ implies $xy \equiv x+y(\theta)$ implies $0 \equiv z(\theta)$ (as $\theta_q = \theta_p \subset \theta$) implies $q \subset I(\theta)$ implies $\theta_q \subset \theta(I(\theta))$ implies $\theta_p \subset \theta(I(\theta))$ (as $\theta_p = \theta_q$) implies p is annulled by $\theta(I(\theta))$ implies $xy \equiv x+y(\theta(I(\theta)))$ implies $x \equiv y(\theta(I(\theta)))$.

Thus $\theta \subset \theta(I(\theta))$. But $\theta \supset \theta(I(\theta))$ for any θ on any lattice L, whence $\theta = \theta(I(\theta))$, i.e. every congruence on L is closed under the Galois correspondence G.

THEOREM 5. *Every ideal of L and every congruence on L is closed under G if and only if L is a relatively complemented, distributive lattice.*

Proof. Let L be a relatively complemented distributive lattice; then it satisfies the conditions of theorems 3 and 4; hence every ideal of L and every congruence on L is closed under G. (For $p = (x, y)$ $(x > y)$ choose $q = (z, 0)$ where z is the complement of y in the interval $(x, 0)$.)

Conversely let L be a lattice such that every ideal of L and every congruence on L is closed under G. So L satisfies the conditions of theorems 3 and 4 and hence it is a distributive lattice. To prove that it is relatively complemented it will suffice to prove that every interval of the type $(x, 0)$ is complemented (as L is distributive).

Now any lattice translate of (a, b) in a distributive lattice L can be written as $(ax+y, bx+y)$ for some x, y in L (cf. [2]). Therefore any lattice translate of $(z, 0)$ is of the type $(zx+y, y)$. Thus, if $q = (z, 0)$ and $p = (v, u)$ is a lattice translate of q, then $v = u+zx$ for some x in L. Consider any interval $p = (c, b)$ of L. Let q be the interval corresponding to p such that $\theta_p = \theta_q$, by Theorem 4. Now p is annulled by θ_q; hence it is a finite sum of lattice translates of q; say $\sum(b_i, b_{i-1})$, where $b = b_0 \leqslant b_1 \leqslant \ldots \leqslant b_n = c$. Now (b_i, b_{i-1}) is a lattice translate of $q = (z, 0)$ for each i. Hence, for each i, $b_i = b_{i-1}+zx_i$ for some x_i in L. Therefore $b_n = b_0 + \sum zx_i$, i.e. $c = b+d$, where $d = \sum zx_i$.

Next consider bd. Let, if possible, bd be other than zero. Now $(bd, 0)$ is annulled by θ_p and the interval $(c, 0)$ being a sublattice of a distributive lattice is itself distributive and so is its dual. Thus $0 = bdt$ for some t in $(c, 0)$ as c becomes the zero of the lattice $(c, 0)$ under its dual ordering. But $bdt = bd = 0$. Thus d is the complement of b in $(c, 0)$. Hence every interval of the type $(c, 0)$ of L is complemented and so L is relatively complemented.

THEOREM 6. *The mapping f: $\theta \to I(\theta)$ preserves all products but does not preserve sums in general in $I(L)$, not even in $C(L)$.*

Proof. Let (θ_i) be a family of congruences on L. Let I_i be the zero class under θ_i for each i; and let $\theta = \bigcap_i \theta_i$ and let I be the zero class under θ.

Required to prove $I = \prod_i I_i$. Now x is in I if and only if $x \equiv 0(\bigcap_i \theta_i)$ if and only if $x \equiv 0(\theta_i)$ for each i, if and only if x is in I_i for each i, if and only if x is in $\prod_i I_i$. Thus $I = \prod_i I_i$ and hence the mapping f preserves all products.

For the other part, we give an example. Consider the lattice of figure 4 and the congruences θ_1, θ_2 of L. It is easily seen that the zero class I under $\theta_1+\theta_2$ is neither I_1+I_2 nor the sum of I_1 and I_2 in $C(L)$.

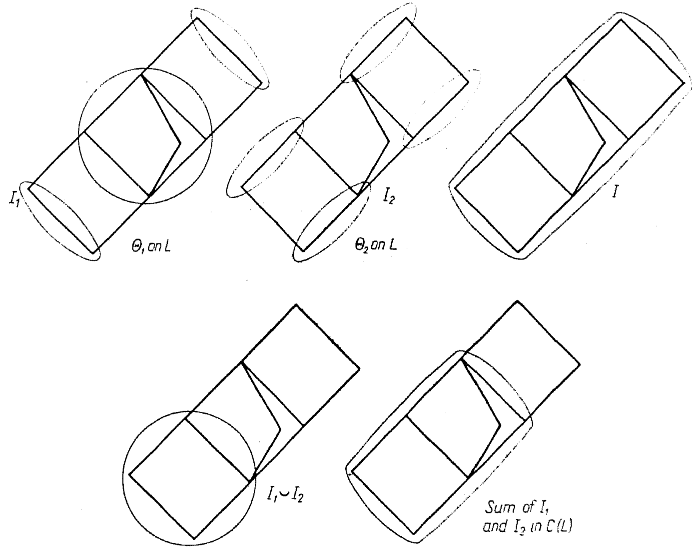

Fig. 4

THEOREM 7. *The mapping g: $I \to \theta(I)$ preserves all sums but does not preserve products in general.*

Proof. Let (I_i) be a family of ideals of L. Let $\theta(I_i)$ be the congruence generated by I_i. Then $\theta(\bigcup_i I_i) = \bigcup_i \theta(I_i)$. Because $\theta(I_i)$ annuls I_i for each i. Hence $\bigcup_i \theta(I_i)$ annuls $\bigcup_i I_i$ implies $\bigcup_i \theta(I_i) \supset \theta(\bigcup I_i)$. Also $\bigcup_i I_i \supset I_i$ for each i, and so $\theta(\bigcup_i I_i) \supset \theta(I_i)$ for each i, which implies $\theta(\bigcup_i I_i) \supset \bigcup_i \theta(I_i)$. Thus $\theta(\bigcup_i I_i) = \bigcup_i \theta(I_i)$. That is the correspondence g preserves all sums.

For the latter part, consider the lattice of figure 3 and the ideals I_1 and I_2, namely the principal ideals generated by a and b. Then each of $\theta(I_1)$ and $\theta(I_2)$ is the universal congruence on L and hence is their intersection; while $\theta(I_1 \wedge I_2)$ is the null congruence on L. Thus g does not preserve products.

My thanks are due to Professor V. S. Krishnan for his constant help and encouragement.

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On the elementary theory of linear order *

by

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A. Ehrenfeucht proved in [1] that the elementary theory T of linear order, even with a finite number of unary predicates adjoined (then denoted by T_1), is decidable. Here we establish the same result by a different method. We think that our proof is of interest on its own, because it gives further insight as to how much can be expressed in the first-order language of linear order (see Theorem 2 below) (1). It is only for the sake of a lighter notation that we restrict our proof to the theory T; there is an obvious generalization to a proof for T_1 (see also remark 3 below).

1. Let L be the first-order language with identity and one binary predicate $<$. T is the theory over L of the axioms

(a) $\neg u < u$, (b) $u < v \wedge v < w \rightarrow u < w$, (c) $u = v \vee u < v \vee v < u$.

We use $x < y (\mathrm{mod} A)$ to denote the order relation of an ordered set A, $|A|$ to denote the field of A. B is said to be a *segment* of A, if B is a submodel of A and if $x < y (\mathrm{mod} B)$ and $x < z < y (\mathrm{mod} A)$ implies $z \in |B|$. An ordered set $\mathcal{A}$ is said to be a *splitting* of A, if $|\mathcal{A}|$ is a set of (non-empty) segments of A, which partitions $|A|$, and if $B < C (\mathrm{mod} \mathcal{A})$ iff $x < y (\mathrm{mod} A)$ for all $x \in |B|$, $y \in |C|$. The elements of $|\mathcal{A}|$ are called the *parts* of the splitting.

ω, ω^*, η respectively denote the order type of the positive integers, negative integers, rationals. $+$ and $\cdot$ denote sum and product of order types (or ordered sets). Given a finite non-empty set F of order types, σF denotes the type which is characterized as follows ("σ" for "shuffling"): An ordered set A is of type σF iff there exists a splitting of A of order type η such that every part has a type from F and between any two different parts there are parts of each type from F (2).

* The author wish to thank A. Tarski for his valuable comments on this paper.

(1) A similar proof has been found independently by F. Galvin.

(2) In [8], p. 32, Th. Skolem proves a theorem which grants the uniqueness of shuffling. Mrs. A. Morel makes extensive use of shuffling in [6].