On types of lattices*

by

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Lattices can be classified in many ways. The most natural way is with the help of equalities. The most familiar lattices of this type are the distributive lattices and the modular lattices. The question which naturally arises in this connection is: Are there types of lattices other than the modular and distributive lattices satisfying equalities? The answer to this is shown to be in the affirmative. As a matter of fact it is shown that there are infinitely many types of lattices satisfying equalities. A set of these lattices satisfy equalities stronger than the modular law but weaker than the distributive law. They are termed supermodular lattices. Thus we define supermodularity in a lattice.

DEFINITION 1. A lattice which satisfies the equality

$$\text{(I)} \qquad \prod_{i=1}^{n}(x+y_i) = x+\sum_{i=1}^{n}\Big(\prod_{j=1}^{n} z_{ij}\Big),$$

where $z_{ij} = x+y_j$ if $i = j$ and $z_{ij} = y_j$ if $i \neq j$ for all elements $x, y_1, y_2, \dots, y_n$ in L is called a *supermodular lattice of order* n $(n \geqslant 3)$.

Supermodular lattices for $n < 3$ are not defined. Those lattices which satisfy the equality dual to that of (I) are called *dually supermodular lattices of order* n $(n \geqslant 3)$.

As n increases these laws tend more and more away from modularity and tend more and more towards distributivity. These imply modularity if we put $y_2 = y_3 = \dots = y_n$.

Another set of lattices satisfy equalities feebler than the modular law. They are called slightly modular lattices. Thus we define slightly modularity in a lattice.

DEFINITION 2. A lattice which satisfies the equality

$$\text{(II)} \qquad \prod_{i=1}^{n}(x+y_i) = x+\sum_{i=1}^{n}\Big(\prod_{j=1}^{n} z_{ij}\Big)$$

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where $z_{ij} = y_j$ if $j = i$ and $z_{ij} = x + y_j$ if $i \neq j$, for all elements $x, y_1, y_2, \ldots, y_n$ in L is called a *slightly modular lattice of order* n $(n \geqslant 2)$.

Slightly modular lattices for $n < 2$ are not defined. Those lattices which satisfy the equality dual to that of (II) are called *dually slightly modular lattices of order* n $(n \geqslant 2)$.

As n increases these tend more and more towards the modular law. The fact that the modular law implies these laws is easily seen. Further one can easily show that any sublattice, homomorphic image and product of slightly modular / dually slightly modular / supermodular / dually supermodular lattices of any order is a slightly modular / dually slightly modular / supermodular / dually supermodular lattice of the same order. We can also show that the lattice of ideals of a slightly modular / dually slightly modular / supermodular / dually supermodular lattice of any order is a slightly modular / dually slightly modular / supermodular / dually supermodular lattice of the same order.

The next way of classification of lattices is done with the help of complementation. In this connection the well-known lattices are the Boolean algebras, the complemented modular lattices, the relatively complemented lattices and not too well-known weakly complemented lattices. (A lattice is called *weakly complemented* if every interval of the type $(x, 0)$ is complemented.)

Another method of classification is done with the help of chains. A lattice is called *finite* if it has a finite number of finite chains. It is called *discrete* if all chains in it are finite and *semi-discrete* if between every pair of comparable elements there exists a finite maximal chain. The well-known results in this connection are:

(i) *Any weakly complemented, modular lattice is relatively complemented and is further complemented modular if it has the unit element* (cf. [1], p. 114).

(ii) *Any semi-discrete modular lattice is discrete* (cf. [1], p. 67).

Yet another way of classification of lattices is done with the help of projectivity and lattice translation (cf. [4]). Clearly unlike projectivity, lattice translation is not a symmetric relation. Thus we define the concept of an effective interval.

DEFINITION 3. An interval I is called *effective* if whenever I is a lattice translate of J then there exists a nontrivial subinterval J_1 (other than a point) of J such that J_1 is a lattice translate of I.

Any interval which is not effective is called an *ineffective interval.* One can easily see that any subinterval of an ineffective interval is an ineffective interval (cf. [4]).

DEFINITION 4. A lattice is defined to be *weakly modular* if all intervals in it are effective; and a lattice L is called *submodular* if every

lattice translate of any interval I of L is a finite sum of intervals projective with I or with subintervals of I.

It is known that the weakly modular lattice is a generalization of the modular and relatively complemented types (cf. [2]). In what follows we show that the submodular lattice is an intermediate generalization of the modular and relatively complemented types.

THEOREM 1. *Any relatively complemented lattice is submodular but not conversely.*

Proof. Let L be a relatively complemented lattice. Let $I = (x, y)$ be an interval of L. Consider the interval $J = (x+z, y+z)$ of L. Let (u, v) be a subinterval of J, i.e. $x+z \geqslant u > v \geqslant y+z$. Let t be a relative complement of u in $(x+z, v)$; then $x > xt \geqslant y$. Now $x+z \geqslant x+t \geqslant x+y+z \geqslant x+z$. Hence $x+t = x+z$. Thus (u, v), $(x+z, t)$ and (x, xt) are successive transposes. Hence (u, v) is projective to a subinterval of I. Similarly we can prove that any subinterval of (xz, yz) is also projective to a subinterval of I.

Let $K = (c, d)$ be a lattice translate of I. Then $c = f(x, x_1, \ldots, x_n)$ and $d = f(y, x_1, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n$ are in L and f is a finite lattice polynomial. As f is finitary, K is projective to a subinterval of I, by repeating the argument in the above para a finite number of times. Thus any lattice translate of I is projective to a subinterval of I. Thus any relatively complemented lattice is submodular.

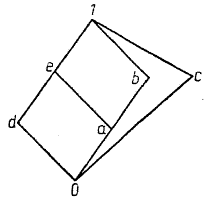

Fig. 1

For the converse, consider the lattice of figure 1. It is submodular as all prime intervals are projective to each other. But L is not relatively complemented as the element a has no complement in the interval $(b, 0)$.

THEOREM 2. *Any modular lattice is submodular but not conversely.*

Proof. We shall prove that any subinterval of $(x+z, y+z)$ is projective to a subinterval of (x, y) for all x, y, z in L. Then the proof follows on similar lines as in theorem 1.

Let $x+z \geqslant u > v \geqslant y+z$. Now $x+z \geqslant x+u \geqslant x+v \geqslant x+y+z = x+z$. Thus $x+u = x+v$. Therefore $xv \lneqq xu$ (L is modular). Also

$x > xu > xv \geqslant y$ and $v + xu = u(v+x) = u$. Thus (xu, xv) is projective to (u, v). Thus any subinterval of $(x+z, y+z)$ is projective to a subinterval of (x, y) and hence the theorem.

For the converse, it will suffice to note that the lattice of figure 1 contains the nonmodular five-element lattice as a sublattice; namely the sublattice consisting of the elements $(0, a, b, c, 1)$.

THEOREM 3. *Any submodular lattice is weakly modular but not conversely.*

Proof. That any submodular lattice is weakly modular follows from the definition. For the converse, consider the lattice of figure 2. This lattice is simple and hence is weakly modular. But it is not submodular as $(a, 0)$ and $(1, b)$ are lattice translates of each other and they are prime, but they are not projective.

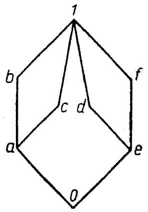

Fig. 2

Remark. Any simple lattice is weakly modular but not necessarily submodular.

LEMMA 1. *In a submodular lattice all prime lattice translates of any prime interval I are projective with I. Conversely, if this property holds in a lattice L and L is semidiscrete, then L is submodular.*

Proof. Let L be a submodular lattice. Let I be any prime interval of L and J a prime interval which is a lattice translate of I. As J is prime and L is submodular, J is projective with a subinterval of I. But as I is prime any subinterval of I is itself. Hence I is projective with J.

Conversely let L be a semi-discrete lattice such that all prime lattice translates of any prime interval I are projective with I. Let (c, d) be any interval of L and let (a, b) be a lattice translate of (c, d). Let $c = c_0 \succ c_1 \succ \ldots \succ c_n = d$ be a finite maximal chain joining c to d (such a chain exists as L is semi-discrete). As (a, b) is a lattice translate of (c, d) there exist elements $x_1, x_2, \ldots, x_n$ of L such that $a = f(c, x_1, x_2, \ldots, x_n)$ and $b = f(d, x_1, x_2, \ldots, x_n)$ where f is a finite lattice polynomial. Let $a_i = f(c_i, x_1, \ldots, x_n)$ for $i = 2, 3, \ldots, n$. Then $a = a_0 \geqslant a_1 \geqslant \ldots \geqslant a_n = b$ and (a_{i-1}, a_i) is a lattice translate of (c_{i-1}, c_i). Let $a_{i-1} = b_0 \succ b_1 \succ \ldots \succ b_p = a_i$ be a finite maximal chain joining a_{i-1} to a. Then (b_{j-i}, b_j) is a lattice translate of (c_{i-1}, c_i) for $j = 1, 2, \ldots, p$. This is true for $i = 1, 2, \ldots, n$. Hence (a, b) consists of a finite sum of intervals projective to subintervals of (c, d). Thus L is submodular.

The word "submodular lattice". is due to O. Tamaschke (cf. [5]), who defines submodularity for lattices of finite length. He makes use of the concept of a prime interval—a property of finite character—hence the very same definition cannot be adopted to define submodularity for general lattices. We give his definition in this section and show that for all semi-discrete lattices both the definitions are equivalent.

DEFINITION 5. O. Tamaschke defines a lattice to be *submodular* if it satisfies the following conditions:

$(\underline{\mathrm{S}})$ *If $x+y \succ y$ ($a \succ b$ means a covers b), then for any prime interval (p, q) such that $x \geqslant p \succ q \geqslant xy$; (p, q) is projective to $(x+y, y)$.*

$(\bar{\mathrm{S}})$ *If $x \succ xy$, then for any prime interval (p, q) such that $x+y = p \succ q \geqslant y$: (p, q) is projective to (x, xy).*

THEOREM 4. *For a semi-discrete lattice L, the definitions 4 and 5 of submodularity are equivalent.*

Proof. First we prove that definition 4 of submodularity implies definition 5. Let $x+y \succ y$ and (p, q) be any prime interval such that $x \geqslant p \succ q \geqslant xy$; then (p, q) can be written as $(p, q) = \Big(\big((x+y, y) \cdot x\big) + q\Big) \cdot p$ and hence (p, q) is a lattice translate of $(x+y, y)$. Both the intervals $(x+y, y)$ and (p, q) being prime, they are projective by lemma 1. Thus L satisfies $(\underline{\mathrm{S}})$. Similarly we can prove that L satisfies $(\bar{\mathrm{S}})$ and hence L is submodular in the sense of O. Tamaschke.

Conversely, let L be a semi-discrete lattice satisfying $(\underline{\mathrm{S}})$ and $(\bar{\mathrm{S}})$. Let (a, b) and (c, d) be a pair of prime intervals such that (c, d) is a lattice translate of (a, b). It will suffice to prove that (a, b) and (c, d) are projective intervals as the submodularity of L will then follow by Lemma 1.

Consider $\big((a, b)+x\big)$, a nontrivial additive translate of (a, b). Then $a \geqslant a(b+x) \geqslant b$. Now $a(b+x) \neq a$. Because if $a(b+x) = a$, then $(b+x) \geqslant a$ and so $a+x = a+b+x = b+x$; a contradiction as $a+x \neq b+x$. Thus $a(b+x) = b$ (as $a \succ b$). Now $a \succ b = a(b+x)$ and so

(α) *(a, b) is projective to every prime interval in $(a+x, b+x)$ as $a+x = a+x+b$ and L satisfies $(\bar{\mathrm{S}})$.*

Dully we can prove that

(β) *(a, b) is projective to every prime interval in (ax, bx) $(a \succ b)$.*

Next as (c, d) is a lattice translate of (a, b) there exist a finite number of elements $x_1, x_2, \ldots, x_n$ such that $(c, d) = \Big(\big((a, b)+x_1\big)\, x_2 \ldots\Big) x_n$. Let $a+x_1 = a_0 \succ a_1 \succ \ldots \succ a_m = b+x_1$ be a maximal chain joining $(a+x_1)$ and $(b+x_1)$; then (a_{i-1}, a_i) is projective to (a, b) for each i, by (α) (as

$a+x_1 \neq b+x_1$). Next put $b_i = a_1 x_2$; then $(a+x_1)x_2 = b_0 \geqslant b_1 \geqslant \ldots \geqslant b_n = (b+x_1)x_2$. Thus, for each i, (b_{i-1}, b_i) is a multiplicative translate of a prime interval (a_{i-1}, a_i). Hence every prime interval of (b_{i-1}, b_i) is projective to (a_{i-1}, a_i) and so in turn projective to (a, b) for $i = 1, 2, \ldots, n$. Thus there exists a maximal chain joining $\big((a+x_1)x_2, (b+x_1)x_2\big)$ such that each prime interval of it is projective to (a, b).

By a finite iteration of this process we see that there exists a finite maximal chain joining (c, d) such that each prime interval of it is projective to (a, b). But $c \gtrsim d$. Hence (c, d) is projective to (a, b). Thus the equivalence of the two definitions is established.

Note. It is worthwhile noting that we have given an answer to problem 10 of [3] in the first part of this paper. Problem 10 of [3] is: Give types of weakly modular lattices which are defined by identical relations and are different from the following three classes of lattices: (a) the class consisting of only the lattice of one element, (b) the class of distributive lattices, and (c) the class of modular lattices.

We see that the supermodular lattice of any order satisfies the requirements for an answer to the above problem.

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On a Galois correspondence between the lattice of ideals and the lattice of congruences on a lattice L *

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This paper deals with a natural Galois correspondence between the lattice of congruence on L (denotes by $\theta(L)$) and the lattice of ideals of L (denoted by $I(L)$). To any congruence θ on L there corresponds an ideal of L, i.e., the zero class under the congruence θ on L, denoted by $I(\theta)$. Let f denote the mapping $\theta \to I(\theta)$, i.e., a mapping from $\theta(L)$ into $I(L)$. Conversely, starting with an ideal I of a lattice L there corresponds a congruence $\theta(I)$ on L, namely the congruence generated by the ideal I of L. Let g denote the mapping $I \to \theta(I)$; i.e., a mapping from $I(L)$ into $\theta(L)$. The mappings f and g define a Galois correspondence G between the lattice of ideals of L (under the usual ordering $<$) and the lattice of congruences on L (under the ordering $<'$, dual to the usual ordering $<$), since:

(i) If $\theta <' \varphi$ then $I(\theta) > I(\varphi)$.

(ii) If $I < J$ then $\theta(I) >' \theta(J)$.

(iii) For any I, $I\big(\theta(I)\big) \geqslant I$, and for any θ, $\theta\big(I(\theta)\big) \geqslant' \theta$.

Now in any Galois correspondence the closed elements of either are the image elements of the other. The closed elements of $I(L)$ (elements I such that $I\big(\theta(I)\big) = I$) and closed elements of $\theta(L)$ (elements θ such that $\theta\big(I(\theta)\big) = \theta$) are termed *congruence ideals* and *zero congruences* respectively. Hence we have:

Definition 1. An ideal I is a *congruence ideal* of L if and only if any lattice translate of any interval of I belongs to I or lies completely outside I.

An interval J is a *lattice translate* of an interval I of L if elements $x_1, x_2, \ldots, x_n$ can be found such that

$$J = \Big(\big((I+x_1)\cdot x_2\big)+x_3 \ldots\Big)\, x_n$$

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