Extending continuous functions on $X \times Y$ to subsets of $\beta X \times \beta Y$

by

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The Stone-Čech compactification βX of the completely regular Hausdorff space X is that unique compactification of X to which each bounded continuous real-valued function on X admits a continuous extension. It is a remarkable and beautiful theorem of Glicksberg [4], reproved elegantly by Frolík in [2], that for infinite spaces X and Y the identity $\beta(X \times Y) = \beta X \times \beta Y$ holds precisely when every real-valued continuous function on $X \times Y$ is bounded.

Glicksberg's theorem, which gives a necessary and sufficient condition that each bounded continuous real-valued function on $X \times Y$ extends continuously to $\beta X \times \beta Y$, suggests the following two questions:

(1) Must a bounded function on $X \times Y$ which extends continuously to $\beta X \times Y$ and to $X \times \beta Y$ extend continuously to $\beta X \times \beta Y$?

(2) Suppose that every bounded real-valued continuous function on $X \times Y$ extends continuously to $\beta X \times Y$ and to $X \times \beta Y$. Does it follow that $\beta(X \times Y) = \beta X \times \beta Y$?

It is easy to reply in the negative to these questions as just posed by choosing for both X and Y the countably infinite discrete space N. For surely any bounded function on $N \times N$ extends continuously to $\beta N \times N$ and to $N \times \beta N$, but many bounded functions—the Kronecker delta function, for example—do not extend continuously to $\beta N \times \beta N$. One purpose of this paper is to answer (2) in the negative (and hence (1) also) by an example which is nontrivial in the sense that neither X nor Y has isolated points. Specifically, we show that a space concocted and studied by one of us and K. Ross in another connection in [1] has the required properties. The full results of that investigation will not be reproduced here, but we cite in detail from [1] those facts which we

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now need: there exists a nondiscrete topological Hausdorff group G which is a Lindelöf space (so that, roughly speaking, open sets in G are pretty large) and in which every G_δ set is open (so that, roughly speaking, there are some pretty small open sets). From the latter property it follows easily that G admits an unbounded continuous real-valued function (see [1] or 4K.2 of [3] or § 1 below) so that the relation $\beta(G\times G) = \beta G\times\beta G$ fails; but we give in § 3 below a very general theorem which shows that nevertheless each bounded real-valued continuous function on $G\times G$ extends continuously to $\beta G\times G$ and to $G\times\beta G$. In the course of developing this theorem we obtain (see especially 2.3 and 2.9) a couple of simple results relating the extendability to $\beta X\times\beta Y$ of real-valued functions on $X\times Y$ with the extendability to $\beta X\times\beta Y$ of real-valued functions on spaces of the form $A\times B$ where $X\subset A\subset\beta X$ and $Y\subset B\subset\beta Y$. The construction depends upon the following fact, which one of us will strengthen and generalize in a later communication: barring the existence of measurable cardinals, the relation $v(X\times\beta Y) = vX\times\beta Y$ holds for every pair of spaces (X, Y). Our remarks on the relation $v(X\times Y) = vX\times vY$ are summarized in 5.3.

We wish finally to acknowledge the friendly generosity of A. W. Hager, who in letters of February 15 and February 25, 1965 discussed our results and their relation to his work (not yet published) with S. Mrówka. The Hager-Mrówka theorems on functional extendability run generally in a direction different from ours, but our works overlap in theorem 3.1 below; in their more extensive treatment, this result will appear as one of several parallel theorems.

1. Definitions and results from the literature. Except for definition 1.1 below, each definition and theorem quoted in this section appears in the Gillman-Jerison text [3], where it is studied in detail and related to neighboring concepts. Many of the concepts we use, and the crucial theorems connecting them, originated with Hewitt in [5]. As in [3], we consider in this paper only completely regular Hausdorff spaces.

The ring of continuous real-valued functions on X is denoted by $C(X)$, and the set of bounded functions in $C(X)$ is denoted by $C^*(X)$. The space X is said to be *pseudocompact* if $C(X) = C^*(X)$. There is a pseudocompact space whose product with itself is not pseudocompact, but the product of any pseudocompact space with a compact space is pseudocompact.

If $X\subset Y$ and each function in $C(X)$ is the restriction to X of some function in $C(Y)$, then X is said to be *C-embedded* in Y. The expression "X is C^*-embedded in Y" is defined analogously, so that (according to the theorem quoted in the first sentence of this paper) the space βX is that unique compact space in which X is dense and C^*-embedded. If $X\subset A\subset\beta X$, then $\beta A = \beta X$.

The space X is said to be *realcompact* if it is homeomorphic with a closed subset of a product of real lines; equivalently, if the quotient field of $C(X)$ by one of its maximal ideals M is the real field only when (for some x in X) M has the form

$$M = \{f\in C(X):\ f(x) = 0\}.$$

For each space X there is a unique realcompact space, denoted vX and called the *Hewitt realcompactification* or the *Nachbin completion* of X, in which X is dense and C-embedded. If $X\subset A\subset vX$, then $vA = vX$. The relation $X\subset vX\subset\beta X$ is valid, and in fact vX is the smallest realcompact subspace of βX which contains X. If $f\in C(\beta X)$ then f assumes on X every value which it assumes on vX, and this theorem characterizes vX as a subset of βX in the following sense: if $p\in\beta X\backslash vX$, then there is a function in $C(\beta X)$ which assumes value 0 at p but which is positive on X.

A space in which each G_δ is open is a *P-space*. Every subspace of a P-space is a P-space, and vX is a P-space if and only if X is a P-space. Each compact P-space is finite, and hence (since $vX = \beta X$ whenever X is pseudocompact) each pseudocompact P-space is finite.

1.1. Definition. A pair of spaces (X, Y) is called a *C^*-pair* if $X\times Y$ is C^*-embedded in $\beta X\times Y$ and in $X\times\beta Y$. The C^*-pair (X, Y) is a proper C^*-pair if $X\times Y$ is not C^*-embedded in $\beta X\times\beta Y$.

By way of illustration, we observe that any pair (X, Y) of discrete spaces is a C^*-pair; and according to Glicksberg's theorem a pair (X, Y) cannot be a proper C^*-pair if $X\times Y$ is pseudocompact. The following proposition will also help to fix ideas, but it is logically inessential to all that comes later.

1.2. Proposition. *In order that (X, Y) be a C^*-pair it is necessary and sufficient that $X\times Y$ be C^*-embedded in the space $(\beta X\times Y)\cup(X\times\beta Y)$.*

Proof. Sufficiency is clear, and necessity follows from this amusing bit of folklore, reported in 6H of [3]. In order that a function φ, mapping a space T to a space R, be continuous on T, it suffices that there exists a dense subset S of T for which the restriction of φ to $S\cup\{p\}$ is continuous for each p in T.

2. Theorems on proper C^*-pairs.

2.1. Theorem. *If $X\times Y$ is C^*-embedded in $\beta X\times Y$, then either X is pseudocompact or Y is a P-space.*

Proof. If not, then (because X is not pseudocompact) there is a sequence U_n of nonvoid open subsets of X such that each point in X admits a neighborhood meeting at most one of the sets U_n; and (because Y is not a P-space) there is a sequence Y_n of closed subsets of Y and a point $y_0 \notin \bigcup_n Y_n$ such that $y_0 \in \mathrm{cl}(\bigcup_n Y_n)$. Selecting in U_n a point x_n, we let f_n be a continuous function on X satisfying these conditions:

$$\text{(a) } 0 \leqslant f_n \leqslant 1; \quad \text{(b) } f_n(x_n) = 1; \quad \text{(c) } f_n \equiv 0 \text{ off } U_n.$$

Using the fact that $\{x_n\} \times Y_n$ is a closed subset of the (completely regular Hausdorff) space $U_n \times Y$, we find for each n a continuous function g_n on $U_n \times Y$ satisfying these conditions:

$$\text{(a) } 0 \leqslant g_n \leqslant 1; \quad \text{(b) } g_n(x_n, y_0) = 1; \quad \text{(c) } g_n \equiv 0 \text{ on } \{x_n\} \times Y_n.$$

Now define h_n on $X \times Y$ by the rule

$$h_n(x, y) = \begin{cases} f_n(x) \cdot g_n(x, y) & \text{if } x \in U_n, \\ 0 & \text{if } x \notin U_n, \end{cases}$$

and set $h = \sum_n h_n$. Each point in $X \times Y$ admits a neighborhood (of the form $U \times Y$) throughout which the bounded function h agrees with one of the continuous functions h_n. Hence $h \in C^*(X \times Y)$. To see that h admits no continuous extension to $\beta X \times Y$, use the fact that the noncompact set $\{x_n\}_{n=1}^{\infty}$ is closed in X to find a point p in $\beta X \setminus X$ with $p \in \mathrm{cl}_{\beta X}\{x_n\}$. Each neighborhood in $\beta X \times Y$ of the point (p, y_0) contains points of the form (x_n, y_0) and meets sets of the form $\{x_n\} \times Y_n$, so that any fu continuous at (p, y_0) and agreeing with h on $X \times Y$ must assume taneously the values 1 and 0 at (p, y_0).

2.2. Theorem. *If (X, Y) is a proper C^*-pair, then both X and Y are P-spaces.*

Proof. If X, say, were not a P-space, then X is pseudocompact by theorem 2.1. But then by Glicksberg's theorem the pseudocompact space $X \times \beta Y$ is C^*-embedded in $\beta X \times \beta Y$, so that $\beta(X \times Y) = \beta X \times \beta Y$ and the C^*-pair (X, Y) is not proper.

2.3. Corollary. *Let $X \subset A \subset \beta X$ and suppose that (A, B) is a proper C^*-pair for some space B. Then $A \subset vX$.*

Proof. If the inclusion fails, then there is a point p in $A \setminus vX$. There is a function f in $C(\beta X)$ which is positive on X and which assumes value 0 at p, and so by 2.2 the set $f^{-1}(0) \cap A$, which is surely a G_δ in A, is a neighborhood in A of p which misses the dense subset X of A.

In 2.9 below we present the converse to a weak form of 2.3 but first we take a moment to emphasize a couple of special cases of 2.3. We recall from Chapter 12 of [3] that a cardinal number $\mathfrak{n}$ is said to be *measurable* if the discrete space of cardinality $\mathfrak{n}$ supports a countably additive measure assuming the values 0 and 1 (and only these values) and assigning measure 0 to each point; an equivalent condition is that the discrete space of cardinality $\mathfrak{n}$ fails to be realcompact. It is consistent with the usual axioms of set theory to assume that every cardinal is nonmeasurable. According to a recent communication from S. Tennenbaum, it is not known "whether the existence of measurable cardinals is consistent with set theory... their existence implies a number of esoteric but plausible propositions."

2.4. Corollary. *Let X be realcompact and let $X \subset A \subset \beta X$. Suppose that (A, B) is a proper C^*-pair for some space B. Then $A = X$.*

2.5. Corollary. *Let X be σ-compact (or even σ-pseudocompact) and let $X \subset A \subset \beta X$. Suppose that (A, B) is a proper C^*-pair for some space B. Then $A = X = N$.*

Proof. The space X is a countable union of pseudocompact P-spaces, hence is itself countable. X is infinite because the C^*-pair (A, B) is proper; hence X is (homeomorphic with) the realcompact space N, so that $N = X \subset A \subset vX = N$.

The primary interest of 2.6 below is that (see 4.4) its analogue for measurable cardinals fails.

2.6. Corollary. *Let $D \subset A \subset \beta D$, where D is a discrete space of nonmeasurable cardinal, and suppose that (A, B) is a proper C^*-pair for some space B. Then $A = D$.*

2.7. Discussion. The assertion that every metric space is realcompact is shown in [3] to be equivalent to the assertion that no measurable cardinals exist. Our next result, which serves as a lemma for 2.9 but which we believe is of interest in its own right, asserts that for practical purposes the relation $v(X \times K) = vX \times K$ holds for each space X and each compact space K. The proof that the relation does fail (for appropriately chosen X and K) in case a measurable cardinal does exist is postponed to 4.8 and 4.9.

For each space Y, the ring $C^*(\beta Y)$, which is ring-isomorphic to $C^*(Y)$, is a metric space relative to the metric induced by the norm

$$\|f\| = \sup\{|f(y)|: y \in \beta Y\} = \sup\{|f(y)|: y \in Y\}.$$

According to the discussion above, the hypothesis that the metric space $C^*(Y)$ is realcompact is essential in the following theorem, but its failure involves a pathology not encountered in everyday analysis. The reader familiar with lemma 1 of Glicksberg's [4] will recognize our debt to that source. The technique by which our function is extended, however, differs (of necessity) from that of Glicksberg's lemma 2.

2.8. Theorem. *If the metric space $C^*(Y)$ is realcompact, then the identity*

$$v(X \times \beta Y) = vX \times \beta Y$$

holds for each space X.

Proof. We may clearly suppose that Y is compact, so that $Y = \beta Y$. Given f in $C(X \times Y)$, we are to produce a function g continuous on the realcompact space $vX \times Y$ which agrees with f on $X \times Y$.

With each x in X we associate a real-valued function $\bar{f}x$ on Y by the rule $(\bar{f}x)(y) = f(x, y)$. Then $\bar{f}x \in C(Y) = C^*(Y)$, and the mapping $\bar{f}$ from X into $C^*(Y)$ is easily checked to be continuous: Given $x \in X$ and $\varepsilon > 0$ there exist for each y in Y neighbourhoods U_y of x and V_y of y such that

$$|f(x, y) - f(x', y')| < \varepsilon/2 \quad \text{whenever} \quad (x', y') \in U_y \times V_y .$$

If $\{V_{y_1}, \ldots, V_{y_n}\}$ covers Y and we set $U = \bigcap_{k=1}^{n} U_{y_k}$, then

$$|f(x, y) - f(x', y)| < \varepsilon \quad \text{whenever} \quad (x', y) \in U \times Y ,$$

so that $\|\bar{f}x - \bar{f}x'\| < \varepsilon$ whenever $x' \in U$.

Since $\bar{f}$ is continuous from X into the realcompact space $C^*(Y)$, there is (see 8.7 of [3], for example) a continuous extension $\bar{g}$ of $\bar{f}$ mapping vX into $C^*(Y)$. The desired function g on $vX \times Y$ is now defined as follows:

$$g(p, y) = (\bar{g}p)(y) .$$

To see that g is continuous at a point (p, y) in $vX \times Y$, choose $\varepsilon > 0$ and find a neighborhood U of p such that

$$\|\bar{g}p - \bar{g}p'\| < \varepsilon/2 \quad \text{whenever} \quad p' \in U .$$

If the neighborhood V of y in Y is chosen so that

$$|(\bar{g}p)(y) - (\bar{g}p)(y')| < \varepsilon/2 \quad \text{whenever} \quad y' \in V,$$

then $U \times V$ is a neighborhood of (p, y) with the property that

$$|g(p, y) - g(p', y')| < \varepsilon \quad \text{whenever} \quad (p', y') \in U \times V .$$

Since g agrees with f on $X \times Y$, our proof is complete.

Our next result is the promised partial converse to 2.3.

2.9. Theorem. *Let $X \subset A \subset vX$ and suppose that (X, Y) is a proper C^*-pair for some space Y for which the metric space $C^*(Y)$ is realcompact. Then (A, Y) is a proper C^*-pair.*

Proof. We must show that the space $A \times Y$ has the following properties:

(a) *it is C^*-embedded in $\beta A \times Y$;*

(b) *it is C^*-embedded in $A \times \beta Y$;*

(c) *it is not C^*-embedded in $\beta A \times \beta Y$.*

To prove (a), we associate with a given f in $C^*(A \times Y)$ its restriction to $X \times Y$, called g. By hypothesis, g admits a continuous extension $\bar{g}$ to $\beta X \times Y = \beta A \times Y$, and the restriction of $\bar{g}$ to $A \times Y$ is clearly f.

To prove (b), let $f \in C^*(A \times Y)$ and let g denote the restriction of f to $X \times Y$. By hypothesis there is a continuous function $\bar{g}$ on $X \times \beta Y$ whose restriction to $X \times Y$ agrees with f there, and by 2.8 the function $\bar{g}$ admits a continuous extension $\bar{h}$ to $vX \times \beta Y$. The restriction of $\bar{h}$ to $A \times \beta Y$ is the desired extension of f.

For (c), note that the space $X \times Y$ is surely C^*-embedded in $A \times Y$. If the latter space were C^*-embedded in $\beta X \times \beta Y$ then the former would be also, contrary to the hypothesis that the C^*-pair (X, Y) is proper.

2.10. Corollary. *Let (X, Y) be a proper C^*-pair and suppose that* $\operatorname{card} X$ *and* $\operatorname{card} Y$ *are nonmeasurable cardinals. If $X \subset A \subset vX$ and $Y \subset B \subset vY$, then (A, B) is a proper C^*-pair.*

Proof. It is shown in 15.24 of [3] that a metric space of nonmeasurable cardinal is realcompact. Since the class of nonmeasurable cardinals is closed under the usual operations of cardinal arithmetic (see 12.5 of [3]) and contains $\mathfrak{c}$, we see that $C^*(Y)$, whose cardinality surely does not exceed $\mathfrak{c}^{\operatorname{card} Y}$, is realcompact. Thus (A, Y) is a proper C^*-pair by 2.9. A similar argument gives the desired conclusion on (A, B), once it is noted that $\operatorname{card} C^*(A) = \operatorname{card} C^*(X)$.

3. A proper C^*-pair without isolated points. In this section we show that the nondiscrete topological group G constructed and studied in [1] has the property that (G, G) is a proper C^*-pair. For our first theorem, which we believe duplicates a small fraction of the Hager–Mrówka work referred to in the introduction, the reader should recall the following definition: a mapping π is said to be *closed* if πK is closed whenever K is closed.

3.1. Theorem. *If the projection π from $X \times Y$ to Y is closed, then $X \times Y$ is C^*-embedded in $\beta X \times Y$.*

Proof. Let $f \in C^*(X \times Y)$. The restriction of f to each set of the form $X \times \{y\}$ (with $y \in Y$) is continuous and bounded there, hence extends continuously to $\beta X \times \{y\}$. This observation furnishes us with a function $\bar{f}$ defined on $\beta X \times Y$ which extends f and which assumes only values in the real interval $[-\|f\|, \|f\|]$. It remains to show that $\bar{f}$ is continuous, and according to the lemma cited in the proof of 1.2 it is enough to show the following: for each point (p, y) in $\beta X \times Y$ and each $\varepsilon > 0$ there is a neighborhood W of (p, y) in $\beta X \times Y$ such that

$$|\bar{f}(x', y') - \bar{f}(p, y)| < \varepsilon \quad \text{whenever} \quad (x', y') \in W \cap (X \times Y) .$$

To achieve this, fix (p, y) and define

$$K = \{(x', y') \in X \times Y\colon |f(x', y') - \bar{f}(p, y)| \leqslant \varepsilon/2 \text{ and } |f(x', y') - \bar{f}(p, y)| \geqslant \varepsilon\}.$$

Then K is closed in $X \times Y$ and K contains no point of the form (x', y) with $x' \in X$. Denoting by $\bar{\pi}$ the projection from $\beta X \times Y$ onto Y, we consequently have: $\bar{\pi}^{-1}(\pi K)$ is a closed subset of $\beta X \times Y$ which misses $\bar{\pi}^{-1}(\{y\})$. For the desired neighborhood W in $\beta X \times Y$ of (p, y) we can take the set $W = U \times V$, where by definition

$$U = \{p' \in \beta X\colon |\bar{f}(p', y) - \bar{f}(p, y)| < \varepsilon/2\}$$

and

$$V = \bar{\pi}^{-1}(Y \backslash \pi K).$$

3.2. THEOREM. *If Y is a P-space and X is Lindelöf, then the projection π from $X \times Y$ onto Y is closed.*

Proof. If K is closed in $X \times Y$ and $y \in Y \backslash \pi K$, then for each x in X there is a neighborhood U_x of x and a neighborhood V_x of y such that $U_x \times V_x \cap K = \emptyset$. There is a sequence x_k of points in X such that $X = \bigcup_{k=1}^{\infty} U_{x_k}$, and defining $V = \bigcap_{k=1}^{\infty} V_{x_k}$ we see that V is a neighborhood of y which misses πK. Thus $y \notin \mathrm{cl}(\pi K)$ and π is closed.

3.3. EXAMPLE. *There is a nondiscrete Hausdorff topological group G for which (G, G) is a proper C^*-pair.*

Proof. The following descriptive definition of G is quoted from 3.2 of [1]: "Let A be an index set of cardinality $\aleph_1$ and let G consist of all elements x in $\prod_{\alpha \in A} \{-1, +1\}_\alpha$ such that $x_\alpha = 1$ for all but finitely many coordinates α. Let Ω be the first uncountable ordinal and well-order A according to the order-type Ω: $A = \{\alpha\colon \alpha < \Omega\}$. For $\alpha \in A$, let

$$H_\alpha = \{x \in G\colon x_\beta = 1 \text{ for all } \beta < \alpha\}.$$

We decree that the subgroups H_α and each of their translates be open and thereby obtain a basis for a topology under which G is a topological group. Clearly G is a P-space and G is not discrete."

The text in [1] continues with a proof that G is Lindelöf. According to 3.2 above, then, each projection from $G \times G$ to G is closed, so by 3.1 $G \times G$ is C^*-embedded in $\beta G \times G$ and in $G \times \beta G$. Since the P-space G is infinite, it is not pseudocompact, and consequently by Glicksberg's theorem the C^*-pair (G, G) is proper.

4. Counterexamples and corollaries.

We have seen in theorem 2.2 that every proper C^*-pair is a pair of P-spaces. It is a consequence of our next result, theorem 4.2, that the converse assertion fails. We begin with a proposition which is probably well known.

4.1. PROPOSITION. *Let D be a discrete space for which* $\mathrm{card}\, D > \aleph_0$. *Then some point in βD lies in the closure of no countable subset of D.*

Proof. Because D is discrete the points p in βD may be identified with ultrafilters A^p on D, the correspondence chosen so that for each $p \in \beta D$ and $S \subset D$ we have $S \in A^p$ if and only if $p \in \mathrm{cl}_{\beta D} S$. (See [3], especially 6.5, for details.) The family $\mathcal{F}$ of subsets of D whose complement in D is countable is a filter on D, and for the desired point we may choose any p for which $\mathcal{F} \subset A^p$.

4.2. THEOREM. *For each discrete space D, the following assertions are equivalent*: (a) $\mathrm{card}\, D = \aleph_0$;
(b) (D, Y) *is a proper C^*-pair for each infinite P-space Y.*

Proof. (a)$\Rightarrow$(b). Surely $D \times Y$ is C^*-embedded in $D \times \beta Y$, and when $\mathrm{card}\, D = \aleph_0$ then 3.1 applies to show that $D \times Y$ is C^*-embedded in $\beta D \times Y$. Because both D and Y are infinite and $D \times Y$ is not pseudocompact, however, this latter space is not C^*-embedded in $\beta D \times \beta Y$.

(b)$\Rightarrow$(a). The condition $\mathrm{card}\, D < \aleph_0$ is incompatible with (b), so our proof by contradiction will begin with the assumption that there exists $E \subset D$ with $\mathrm{card}\, E = \aleph_1$. For Y we choose the space $Y = E \cup \{q\}$, where q is a point in $Y \backslash E$ and neighborhoods of q are by definition those subsets V of Y for which $q \in V$ and $\mathrm{card}(Y \backslash V) \leqslant \aleph_0$. We well-order E according to the smallest ordinal number of cardinality $\aleph_1$ and we define a function f on $D \times Y$ as follows:

$$f(x, y) = \begin{cases} 0 & \text{if } x \in D \backslash E; \\ 0 & \text{if } (x, y) = (x_\alpha, x_\gamma) \in E \times E \text{ and } \alpha > \gamma; \\ 1 & \text{if } (x, y) = (x_\alpha, x_\gamma) \in E \times E \text{ and } \alpha \leqslant \gamma; \\ 1 & \text{if } y = q \text{ and } x \in E. \end{cases}$$

Now f is clearly continuous at any point not in $D \times \{q\}$; and f is also continuous at points in $D \times \{q\}$ because for each α the set

$$\{(x_\alpha, q)\} \cup \{(x_\alpha, x_\gamma)\colon \gamma > \alpha\}$$

is a neighborhood of (x_α, q) throughout which f assumes only the value 1.

According to 4.1 there exists $p \in \beta E \backslash E \subset \beta D \backslash D$ such that p is in the closure (in βE, say) of no countable subset of E. There is by hypothesis (b) a continuous function $\bar{f}$ on $(D \cup \{p\}) \times Y$ which agrees with f on $D \times Y$. Since $p \in \mathrm{cl}_{\beta E}\{x_\alpha\colon \alpha > \gamma\}$ for each γ, we have $\bar{f}(p, x_\gamma) = 0$ for each $x_\gamma \in E \subset Y$ so that $\bar{f}(p, q) = 0$. But $\bar{f}(x_\alpha, q) = 1$ for each $x_\alpha \in E$, so that $\bar{f}(p, q) = 1$.

4.3. Discussion. We have been unable to prove the following conjecture [1], a strengthened form of 4.2: *If (X, Y) is a proper C^*-pair for each infinite P-space Y, then $X = N$.*

[1] Added in proof: A. W. Hager has established this conjecture.

As it stands, however, 4.2 is strong enough to show the failure of the measurable analogue of 2.6.

4.4. Example. *If D is a discrete space of measurable cardinal, then there is a proper C^*-pair (A, B) for which $D \subset A \subset \beta D$ and $A \neq D$.*

Proof. Choose $p \in vD \backslash D$, set $A = D \cup \{p\}$, let B be the countably infinite discrete space and apply 4.2.

4.5. Example. *There exist P-spaces X and Y such that (X, Y) is not a proper C^*-pair.*

Proof. The pair (E, Y) of 4.2 is such a pair.

Our next example shows that the converse to 2.1 fails.

4.6. Example. *There is a pseudocompact space X and a P-space Y such that $X \times Y$ is not C^*-embedded in $\beta X \times Y$.*

Proof. We take for Y the space $Y = E \cup \{q\}$ of 4.2, and (adopting notation from [3]) we denote by W that ordinal space according to which E was well-ordered. Then E is discrete and dense in Y and W with its interval topology is a pseudocompact space whose Stone-Čech compactification βW is homeomorphic with the space $W^* = W \cup \{W\}$ of ordinals less than or equal to the first uncountable ordinal. An argument very similar to that given in (b)$\Rightarrow$(a) of 4.2 now shows that the continuous function f, defined on $W \times Y$ by the rule

$$f(a, y) = \begin{cases} 1 & \text{if} \quad y = q; \\ 1 & \text{if} \quad y = x_\gamma \text{ with } \gamma \geqslant a; \\ 0 & \text{otherwise} \end{cases}$$

admits no continuous extension to the point $(\{W\}, q) \in \beta W \times Y$.

In 2.8 we showed in effect that the relation $v(X \times K) = vX \times K$ holds for each compact space K for which the metric space $C^*(K)$ is realcompact. In case a measurable cardinal exists, however, then (as shown below) there a compact space K for which $C^*(K)$ is not realcompact, and in our final example we show that for at least one such space K the relation $v(X \times K) = vX \times K$ does fail.

4.7. Proposition. *Let D be a discrete space of measurable cardinal and let $K = \beta D$. Then $C^*(K)$ is not realcompact.*

Proof. With each x in D we associate that continuous function f_x defined on K as follows:

$$f_x(y) = \begin{cases} 1 & \text{if} \quad x = y; \\ 0 & \text{if} \quad x \neq y. \end{cases}$$

The family $\mathcal{F} = \{f_x : x \in D\}$ is a discrete set of measurable cardinality, hence is not realcompact. But $\mathcal{F}$ is closed in the metric space $C^*(K)$ and therefore (since every closed subspace of a realcompact space is realcompact) the space $C^*(K)$ is not realcompact.

4.8. Example. *Let $\mathfrak{n}$ be the smallest measurable cardinal, and let D be the discrete space of cardinality $\mathfrak{n}$. Then the relation $v(D \times \beta D) = vD \times \beta D$ fails.*

Proof. Well-order D according to the smallest ordinal number Ω of cardinality $\mathfrak{n}$, so that for each ordinal $a < \Omega$ the set D_a, defined by the identity

$$D_a = \{d_\gamma : \gamma < a\},$$

has cardinality less than $\mathfrak{n}$.

Since the discrete space D is not realcompact, there is a point p in $vD \backslash D$. It is a consequence of 12.3 (a) in [3] that $p \notin \mathrm{cl}_{vD} E$ whenever $E \subset D$ and $\operatorname{card} E < \mathfrak{n}$. Since $D = E \cup (D \backslash E)$ for each $E \subset D$, then, we have

$$(*) \qquad p \in \mathrm{cl}_{vD} E \quad \text{whenever} \quad E \subset D \text{ and } \operatorname{card}(D \backslash E) < \operatorname{card} D.$$

Now define f on $D \times D$ as follows:

$$f(x_a, x_\gamma) = \begin{cases} 0 & \text{if} \quad a > \gamma; \\ 1 & \text{if} \quad a \leqslant \gamma, \end{cases}$$

and let $\bar{f}$ denote the continuous extension of f to $D \times \beta D$. If g extends $\bar{f}$ continuously to $(D \cup \{p\}) \times \beta D$, then from $(*)$ we must have $g(p, x_\gamma) = 0$ for each γ, so that $g(p, p) = 0$. But also from $(*)$ we have $g(x_a, p) = 1$ for each a, so that $g(p, p) = 1$.

4.9. Theorem. *The following are equivalent:*

(a) *$v(X \times K) = vX \times K$ for each space X and each compact space K;*

(b) *there are no measurable cardinals.*

Proof. (a)$\Rightarrow$(b). If (b) fails, then so must (a) by 4.8.

(b)$\Rightarrow$(a). According to 2.8 we need only show that $C^*(K)$ is realcompact for each compact space K, and in the presence of (b) this follows from 15.24 of [3], according to which each metric space of non-measurable cardinal is realcompact.

5. Concerning the relation $v(X \times Y) = vX \times vY$. It follows readily from Glicksberg's theorem that the relation $v(X \times Y) = vX \times vY$ holds whenever $X \times Y$ is pseudocompact, but according to 2.8 above the converse implication fails. Lacking a simple topological condition equivalent to the condition $v(X \times Y) = vX \times vY$, we content ourselves with summarizing (and slightly generalizing) the theorems above. The following definition is handy: A subset X of a topological space $\tilde{X}$ is said to be *G_δ-dense* in X if X meets each nonvoid G_δ subset of $\tilde{X}$.

Because we are dealing only with completely regular Hausdorff spaces, in which each nonvoid G_δ contains a nonvoid set of the form $f^{-1}(0)$ for some continuous real-valued function f, the following observation

observation is valid: *X is G_δ-dense in $\widetilde{X}$ if and only if X meets each non-void closed G_δ subset of $\widetilde{X}$.*

5.1. PROPOSITION. *If X is G_δ-dense in $\widetilde{X}$ and Y is G_δ-dense in $\widetilde{Y}$, then $X\times Y$ is G_δ-dense in $\widetilde{X}\times\widetilde{Y}$.*

5.2. THEOREM. *If $X\times Y$ is C^*-embedded in $vX\times vY$, then $X\times Y$ is C-embedded in $vX\times vY$.*

Proof. According to 1.18 of [3] it suffices to show that each non-void closed G_δ subset Z of $vX\times vY$ which misses $X\times Y$ satisfies a certain condition. Since X and Y are G_δ-dense in vX and vY respectively, there are by 5.1 no such sets Z.

5.3. THEOREM. *Let* card Y *be nonmeasurable and suppose that either*

(a) *Y is compact;*

or

(b) *the projection from $X\times Y$ to X is closed;*

or

(c) *$X\times Y$ is C^*-embedded in $X\times\beta Y$.*

Then $v(X\times Y)=vX\times vY$.

Proof. It suffices to deduce the desired conclusion from hypothesis (c), the implication (a)$\Rightarrow$(b) being well-known and the implication (b)$\Rightarrow$(c) being given by 3.1. According to 5.2 it is enough to show that each bounded continuous real-valued function on $X\times Y$ extends continuously to $vX\times vY$. By (c) we can extend to $X\times\beta Y$, and 2.8 takes us from there to the space $vX\times\beta Y$, which contains $vX\times vY$.

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On defining well-orderings

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0. Introduction. Let $L_{\alpha\omega}$ be the extension of the finitary first-order predicate logic obtained by allowing conjunctions and disjunctions of μ-sequences of formulas ($\mu<\alpha$). The purpose of this note is to show that *even allowing arbitrary many non-logical constants the notion of a well-ordered relation cannot be expressed by a set of sentences of the infinitary first-order language $L_{\alpha\omega}$* (i.e. that for all α, $\mathbf{W}\notin\mathbf{PC}_\Delta(L_{\alpha\omega})$ where $\mathbf{W}$ is the class of all non-empty well-orderings) (0).

The method used can be summarized as follows: First we determine an upper bound for the Hanf-number of $L_{\alpha\omega}$ (1). Then we show that if for some α, $\mathbf{W}\in\mathbf{PC}_\Delta(L_{\alpha\omega})$, then there would exist cardinals $\varkappa$, λ, and a sentence Φ of $L_{\varkappa\omega}$ such that (i) Φ does not have arbitrarily large models, and (ii) Φ has a model of cardinality λ and λ is equal to the upper bound previously obtained for the Hanf-number of $L_{\varkappa\omega}$.

I. The language $L_{\alpha\omega}$. It is convenient for our purposes to define the language $L_{\alpha\omega}$ in a slightly different (but clearly equivalent) way to that suggested in the introduction.

DEFINITION 1.1 (2).

(.1) T is a *pseudo-α-tree* if and only if T is a set of finite sequences of ordinals smaller than α such that (i) $0\in T$, (ii) if $\langle\mu_0,\ldots,\mu_{n-2},\mu_{n-1}\rangle\in T$, then $\langle\mu_0,\ldots,\mu_{n-2}\rangle\in T$ and for all $\delta<\mu_{n-1}$, $\langle\mu_0,\ldots,\mu_{n-2},\delta\rangle\in T$, and (iii) if $s\in T$, then $|\{\mu\colon s^\frown\langle\mu\rangle\in T\}|<\alpha$.

(0) This answers a problem raised by Professor Mostowski, namely, whether there existed an α such that $\mathbf{W}\in\mathbf{PC}_\Delta(L_{\alpha\omega})$.

(1) See Definition 4.1.

(2) Standard set-theoretical terminology will be used. In particular an *ordinal* is the set of smaller ordinals (small greek letters: μ, δ, ϱ, ξ, shall denote ordinals; ω is the smallest infinite ordinal). $\mu+\delta$ is the ordinal *sum* of μ and δ. A *cardinal* is an initial ordinal and if X is a set then $|X|$ is the cardinal of X. A function whose domain is an ordinal will also be called a *sequence*. If f and g are sequences, then $f^\frown g$ is the concatenation of f and g (i.e. the sequence h such if ξ, ζ are the domains of f and g respectively then $h=\{(\mu,f(\mu))\colon\mu\in\xi\}\cup\{(\xi+\mu,g(\mu))\colon\mu\in\zeta\}$.