Combinatorial series and recursive equivalence types

by

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1. Summary and notation. The theory of recursive equivalence types is an effective analogue of the theory of cardinal numbers. It originated with Dekker [1] and has been developed at least by Dekker, Myhill, and the author ([2], [3], [4], [6], [7], [8], [9], [10], [11], [12], [13], [14]). Let $E = \{0, 1, 2, \ldots\}$ be the natural numbers. If $\alpha, \beta \subseteq E$, call α *recursively equivalent to* β if there exists a 1-1 partial recursive function p whose domain contains α, and which is such that $p(\alpha) = \beta$. The equivalence class $\langle\alpha\rangle$ of α under recursive equivalence is called a *recursive equivalence type* (RET). The set of RET's is denoted by Ω. With each n in E is associated the RET of all n-element subsets of E. If n is identified with this associated RET, then E becomes a subset of Ω. For $\alpha, \beta \subseteq E$, define $\alpha \oplus \beta = [2x|\ x \in \alpha] \cup [2x+1|\ x \in \beta]$ and $\alpha \otimes \beta = [2^x 3^y|\ x \in \alpha \text{ and } y \in \beta]$. Define addition and multiplication in Ω by $\langle\alpha\rangle + \langle\beta\rangle = \langle\alpha \oplus \beta\rangle$, $\langle\alpha\rangle \times \langle\beta\rangle = \langle\alpha \otimes \beta\rangle$. The set Λ of isols consists of all those $x \in \Omega$ such that $x+1 \neq x$. Then $E \subseteq \Lambda \subseteq \Omega$. The arithmetic of Λ is fairly well understood ([2], [3], [4], [6], [7], [8], [12], [13]). A fundamental tool for the analysis of Λ was the notion of a recursive combinatorial function $f\colon \times^n E \to E$ and its induced normal function $f_\Omega\colon \times^n \Omega \to \Omega$. This was introduced by Myhill [8] as follows. Any function $f\colon \times^n E \to E$ has a uniquely determined expansion $f(x_1, \ldots, x_n) =$

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$\sum c(i_1, \ldots, i_n)\binom{x_1}{i_1}\ldots\binom{x_n}{i_n}$. Here $c\colon \times^n E \to E^*$, where $E^* = \{0, \pm1, \pm2, \ldots\}$ is the rational integers; and $\binom{x}{i}$ is the number of i element subsets of an x element set. An $f\colon \times^n E \to E$ is called *combinatorial* if all $c(i_1, \ldots, i_n) \geqslant 0$. Let $\mathfrak{I}(E)$ be the class of all subset of E, let $\mathfrak{I}_{\mathrm{Fin}}(E)$ be the class of all finite subsets of E. Let $j\colon (\times^n \mathfrak{I}_{\mathrm{Fin}}(E)) \times E \to E$ be a fully effective 1-1 onto map. For any set Λ, if $\alpha \in \times^n \Lambda$, write α_i for the ith coordinate of α. For $\alpha, \beta \in \times^n \mathfrak{I}(E)$, write $\alpha \leqslant \beta$ if $\alpha_i \subseteq \beta_i$ for all i. Write $\langle\alpha\rangle$ for $(\langle\alpha_1\rangle, \ldots, \langle\alpha_n\rangle)$. Then Myhill's normal function $f_\Omega\colon \times^n \Omega \to \Omega$ is defined as follows. First define $\varphi^e\colon \times^n \mathfrak{I}_{\mathrm{Fin}}(E) \to \mathfrak{I}_{\mathrm{Fin}}(E)$ by

$$(1.1)\qquad \varphi^e(\alpha) = [j(\alpha, x)|\ 0 \leqslant x < c(\langle\alpha\rangle)] \quad \text{for} \quad \alpha \in \times^n \mathfrak{I}_{\mathrm{fin}}(E).$$

Then define $\varphi\colon \times^n \mathfrak{I}(E) \to \mathfrak{I}(E)$ by

$$(1.2)\qquad \varphi(\alpha) = \bigcup_{\beta \leqslant \alpha, \beta \in \times^n \mathfrak{I}_{\mathrm{fin}}(E)} \varphi^e(\beta), \quad \alpha \in \times^n \mathfrak{I}(E).$$

Finally, define $f_\Omega\colon \times^n \Omega \to \Omega$ by

$$(1.3)\qquad f_\Omega(\langle\alpha\rangle) = \langle\varphi(\alpha)\rangle, \quad \alpha \in \times^n \mathfrak{I}(E).$$

When f is recursive combinatorial—i.e., both recursive and combinatorial—this normal function is well-behaved and very useful for investigating Λ. Among the recursive combinatorial functions are $x+y$, xy, $(x+1)^y$, $x!$, $p(x_1, \ldots, x_n) = x_i$, and constant functions with values in E.

As Myhill observed [10], recursive combinatorial functions degenerate rather badly on $\Omega - \Lambda$ and are for that reason less useful to investigate Ω. The principal contribution of the present paper is the introduction and development of a formal generalization of the notion of combinatorial function which is designed for analysis of $\Omega - \Lambda$. This is the notion of a combinatorial series. A combinatorial series is a formal series $f = \sum c(i_1, \ldots, i_n)\binom{v_1}{i_1}\ldots\binom{v_n}{i_n}$ where

$$c\colon \times^n E \to E \cup \{\aleph_0\}.$$

Thus the generalization amounts to allowing $\aleph_0$ as a value for the coefficients as well as values in E. Corresponding to each such series f is a normal function $f_\Omega\colon \times^n \Omega \to \Omega$ given by (1.1), (1.2), (1.3). The only change is to allow $c(\langle\alpha\rangle)$ the value $\aleph_0$.

There will be two principal propositions, of which all other results are corollaries. The first concerns identities. A basic fact about recursive combinatorial functions is that they are closed under composition and any identity between them true in E yields an identity in Ω between corresponding normal functions. There are two obstacles to extending this result to combinatorial series. One obstacle is that combinatorial series are purely formal, and are not given as maps. How is formal composition to be defined? That is, if $f(u_1, \ldots, u_n), g_1(u_1, \ldots, u_k), \ldots, g_n(u_1, \ldots, u_k)$ are combinatorial series, then what series should the formal composition $f(g_1 \ldots g_n) = f(g_1(u_1, \ldots, u_k), \ldots, g_n(u_1, \ldots, u_k))$ be? This is answered in § 2 so as to extend the notion for combinatorial functions. The second obstacle concerns recursiveness of the series. A combinatorial function $f\colon \times^n E \to E$, $f = \sum c(i_1, \ldots, i_n)\binom{x_1}{i_1}\ldots\binom{x_n}{i_n}$, is recursive if and only if $c\colon \times^n E \to E$ is recursive. It is natural to call a combinatorial series

$$\sum c(i_1, \ldots, i_n)\binom{v_1}{i_1}\ldots\binom{v_n}{i_n}$$

recursive if in the natural sense $c\colon \times^n E \to E \cup \{\aleph_0\}$ is recursive. (More precisely, let $h\colon E \cup \{\aleph_0\} \to E$ be defined by $h(\aleph_0) = 0$, $h(x) = x+1$ for $x \in E$. Then $c\colon \times^n E \to E \cup \{\aleph_0\}$ is called recursive if $h(c)\colon \times^n E \to E$ is recursive in the usual sense.) A counterexample in § 3 shows that such recursive series are not closed under composition. This obstacle is bypassed by considering a wider class of series than the recursive series defined as follows. Call $c\colon \times^n E \to E \cup \{\aleph_0\}$ an $\mathrm{R}\uparrow$ *function* (or a *limit of a monotone increasing recursive sequence of recursive functions*) if there exists a recursive sequence of recursive functions $f_n\colon \times^n E \to E$ such that for all $i \in \times^n E$ (1) $f_0(i) \leqslant f_1(i) \leqslant \ldots$; (2) $c(i)$ is the least upper bound in $E \cup \{\aleph_0\}$ of the values of $f_0(i), f_1(i), \ldots$ Call a combinatorial series $f = \sum c(i_1, \ldots, i_n)\binom{v_1}{i_1}\ldots\binom{x_n}{i_n}$ an $\mathrm{R}\uparrow$ *series* if $c\colon \times^n E \to E \cup \{\aleph_0\}$ is an $\mathrm{R}\uparrow$ function. Then $\mathrm{R}\uparrow$ series obviously include recursive series and we have

PROPOSITION I (§ 3). *The $\mathrm{R}\uparrow$ combinatorial series are closed under composition. If $f(v_1, \ldots, v_n), g^1(v_1, \ldots, v_k), \ldots, g^n(v_1, \ldots, v_k)$ are $\mathrm{R}\uparrow$ combinatorial series, then for $x \in \times^k \Omega$*

$$f_\Omega(g^1_\Omega(x), \ldots, g^n_\Omega(x)) = (f \circ (g^1, \ldots, g^n))_\Omega(x).$$

Consequently any formal identity between $\mathrm{R}\uparrow$ series yields an identity in Ω between corresponding normal functions.

The second principal proposition concerns inequalities. It is used to produce RET's satisfying specified systems of equations and inequations. It can be used to classify the possible algebraic relations between RET's. This will not be carried out here since it depends on more of the theory of addition of RET's than it is convenient to produce here. However, in § 5 an application is given to prove the following result of Myhill [10]. If $A \in \Omega$, the idemmultiplicity of A is the least $m \geqslant 1$ such that $A^m = 2A^m$ (or is ∞ if no such finite m exists). The idempotence of A is the least $n \geqslant 1$ such that $A^n = A^{n+1}$ (or is ∞ if no such n exists). The result is that for any m, n with $1 \leqslant m \leqslant n \leqslant \infty$, there exist $A \in \Omega - \Lambda$ of idemmultiplicity m, idempotence n.

The inequality theorem is as follows. Say that almost all x in Ω have property P if there exists an n in E such that P holds for all $x \in \Omega - \{0, 1, \ldots, n\}$. Also for $x, y \in \Omega$, write $x \leqslant y$ if there exists a z in Ω with $x+z = y$.

PROPOSITION II (§ 4). *Let* $f(u) = \sum c(i)\binom{u}{i}$ *and* $g(u) = \sum d(i)\binom{u}{i}$ *be* R$\uparrow$ *combinatorial series. Then the following three conditions are equivalent.*

(i) $f_\Omega(x) \leqslant g_\Omega(x)$ *for almost all* x *in* Ω.

(ii) *For some* $k \in E$, *there exists a recursive* R$\uparrow$ *series* $h(u)$ *such that* $f(u+k)+h(u) = g(u+k)$.

(iii) *For some* $k \in E$ *there exists an* R$\uparrow$ *function* $e\colon E \to E$ *such that for all* $i \in E$

$$c(i)\binom{k}{0}+c(i+1)\binom{k}{1}+\ldots+c(i+k)\binom{k}{k}+e(i)$$
$$= d(i)\binom{k}{0}+d(i+1)\binom{k}{1}+\ldots+d(i+k)\binom{k}{k}.$$

Moreover, there are c isols x such that for any pair of R$\uparrow$ *series* f, g *violating* (iii), $f_\Omega(x) \not\leqslant g_\Omega(x)$.

This proposition and (11.3) of [12] coincide when applied to recursive combinatorial functions of one variable. Proposition II can be generalized to functions of several variables. The main content is that algebraic properties of values of f_Ω are correlated with arithmetical properties of the coefficients $c(i)$.

Proposition II can be applied (§ 4) to show that Proposition I fails with the weaker hypothesis that the series have coefficient functions $c(i_1, \ldots, i_n)$ which are merely limits of recursive sequences of recursive functions rather than monotone limits.

Finally, Proposition I allows the resolution (§ 6) of some problems about the isolic integers raised in [13]. Adopt the terminology of [13]. Suppose that φ is a quantified conjunction of atomic formulas. Suppose that φ is true in E^* and in E^* φ has Skolem functions in both two number quantifier forms in the Kleene-Mostowski hierarchy. The main result is that then φ is true in Λ^*.

2. Combinatorial series and operators. A generalization of combinatorial operators ([8], [12]) suitable for combinatorial series is needed. Construe $\times^k\mathcal{T}(E)$ as a boolean algebra by defining for $\alpha, \beta \in \times^k\mathcal{T}(E)$, $i = 1, \ldots, k$, $(\alpha \vee \beta)_i = \alpha_i \cup \beta_i$, $(\alpha \wedge \beta)_i = \alpha_i \cap \beta_i$, $(\alpha - \beta)_i = \alpha_i - \beta_i$.

A precombinatorial operator is a $\psi\colon \times^k\mathcal{T}_{\text{fin}}(E) \to \mathcal{T}(E)$ such that (2.1) and (2.2) hold for $\alpha, \beta \in \times^k\mathcal{T}_{\text{fin}}(E)$.

(2.1) $\qquad \alpha \neq \beta \quad$ implies $\quad \psi(\alpha) \cap \psi(\beta) = \emptyset$

(where $\emptyset$ denotes the null set).

(2.2) $\langle\alpha\rangle = \langle\beta\rangle$ implies that $\psi(\alpha)$ and $\psi(\beta)$ have the same number of elements. (N.B. $\psi(\alpha)$ may be infinite.)

A combinatorial operator is a $\varphi\colon \times^k\mathcal{T}(E) \to \mathcal{T}(E)$ such that there exists a precombinatorial operator $\psi\colon \times^k\mathcal{T}_{\text{fin}}(E) \to \mathcal{T}(E)$ satisfying (2.3) below for all $\alpha \in \times^k\mathcal{T}(E)$.

(2.3) $$\varphi(\alpha) = \bigcup_{\beta \leqslant \alpha, \beta \in \times^k\mathcal{T}_{\text{fin}}(E)} \psi(\beta).$$

Then ψ is uniquely determined by φ since by (2.3) and (2.1), $\psi(\beta) = \varphi(\beta) - \bigcup_{\beta' < \beta} \varphi(\beta')$ for all $\beta \in \times^k\mathcal{T}_{\text{fin}}(E)$. It is thus legitimate to write ψ as φ^e, and we have

(2.4) $$\varphi^e(\alpha) = \varphi(\alpha) - \bigcup_{\beta<\alpha} \varphi(\beta) \quad \text{for} \quad \alpha \in \times^k\mathcal{T}_{\text{fin}}(E).$$

For $x \in \varphi(\times^k E)$, define x_φ as the $\alpha \in \times^k\mathcal{T}_{\text{Fin}}(E)$ such that $x \in \varphi^e(\alpha)$. By (2.2) and (2.1) each combinatorial operator $\varphi\colon \times^k\mathcal{T}(E) \to \mathcal{T}(E)$ yields a function $c\colon \times^k E \to E \cup \{\aleph_0\}$ given for $i \in \times^k E$ by $c(i) =$ cardinality of $\varphi^e(\alpha)$ whenever $\langle\alpha\rangle = i$. We call $\sum c(i_1, \ldots, i_k)\binom{v_1}{i_1}\ldots\binom{v_k}{i_k}$ the combinatorial series $\varphi^\#$ induced by φ. Note that by (1.1), (1.2), (1.3), every combinatorial series is induced by at least one combinatorial operator.

THEOREM 2.1. *Combinatorial operators are closed under composition. That is, if* $\varphi\colon \times^n\mathcal{T}(E) \to \mathcal{T}(E)$, $\psi_1\colon \times^k\mathcal{T}(E) \to \mathcal{T}(E), \ldots, \psi_n\colon \times^k\mathcal{T}(E) \to \mathcal{T}(E)$ *are combinatorial operators, then* $\varphi \circ (\psi_1, \ldots, \psi_n)\colon \times^k\mathcal{T}(E) \to \mathcal{T}(E)$ *is also a combinatorial operator.* (*Here we put* $\big(\varphi \circ (\psi_1, \ldots, \psi_n)\big)(\alpha_1, \ldots, \alpha_k) = \varphi\big(\psi_1(\alpha_1, \ldots, \alpha_k), \ldots, \psi_n(\alpha_1, \ldots, \alpha_k)\big)$.) *Further, suppose* $\varphi'\colon \times^n\mathcal{T}(E) \to \mathcal{T}(E)$, $\psi_1'\colon \times^k\mathcal{T}(E) \to (E), \ldots, \psi_n'\colon \times^k\mathcal{T}(E) \to \mathcal{T}(E)$ *are combinatorial operators such that* $\varphi'^\# = \varphi^\#$, $(\psi_1')^\# = \psi_1^\#, \ldots, (\psi_n')^\# = \psi_n^\#$. *Then* $\big(\varphi' \circ (\psi_1', \ldots, \psi_n')\big)^\# = \big(\varphi \circ (\psi_1, \ldots, \psi_n)\big)^\#$.

Proof. Define $\theta\colon \times^k\mathcal{T}_{\text{fin}}(E) \to E$ for $\alpha \in \times^k\mathcal{T}_{\text{fin}}(E)$ by

(2.5) $x \in \theta(\alpha)$ if and only if: (i) x_φ exists; (ii) u_{ψ_i} exists for all $u \in (x_\varphi)_i$, $i = 1, \ldots, n$; (iii) $\alpha = \bigvee_{i=1}^{n} \bigvee_{u \in (x_\varphi)_i} u_{\psi_i}$.

It will eventually be shown that θ is precombinatorial (Lemma 2.3), that θ yields a combinatorial operator which is $\varphi \circ (\psi_1, \ldots, \psi_n)$ (Lemma 2.4), and that the coefficient function for $\big(\varphi \circ (\psi_1, \ldots, \psi_n)\big)^\#$ is determined by the coefficient functions for $\varphi^\#, \psi_1^\#, \ldots, \psi_n^\#$ (Lemma 2.4). This will prove the theorem.

For $\alpha \in \times^k\mathcal{T}(E)$, let $I(\alpha)$ consist of all $\beta \in \times^k\mathcal{T}(E)$ with $\beta \leqslant \alpha$. Call a $2n$-tuple $(F_1, \ldots, F_n, f_1, \ldots, f_n)$ *admissible* for α if:

(2.6) (i) $F_1, \ldots, F_n \subseteq I(\alpha)$; (ii) f_i is a function with domain F_i; (iii) for $\beta \in F_i$, $\emptyset \neq f_i(\beta) \subseteq \psi_i^e(\beta)$; (iv) $\alpha = \bigvee_{i=1}^{n} \bigvee_{\beta \in F_i} \beta$.

LEMMA 2.2. *$x \in \theta(\alpha)$ if and only if there exists an $(F_1, \ldots, F_n, f_1, \ldots, f_n)$ admissible for α such that*

$$x \in \varphi^e\Big(\bigcup_{\beta \in F_1} f_1(\beta), \ldots, \bigcup_{\beta \in F_n} f_n(\beta)\Big).$$

Proof. Suppose $x \in \theta(\alpha)$. Define F_i to be the set of u_{ψ_i} such that $u \in (x_\varphi)_i$, $i = 1, \ldots, n$. For $\beta \in F_i$, define $f_i(\beta) = (x_\varphi)_i \cap \psi_i^e(\beta)$. We prove that $(F_1, \ldots, F_n, f_1, \ldots, f_n)$ is admissible for α. Note that (2.6) (ii) is clear, while (2.5) (iii) implies (2.6) (i), (iv). As for (2.6) (iii), suppose $\beta \in F_i$. By definition $\beta = u_{\psi_i}$ for some $u \in (x_\varphi)_i$, so $u \in \psi_i^e(\beta)$. Thus $u \in (x_\varphi)_i \cap \psi_i^e(\beta) = f_i(\beta)$. Therefore $\emptyset \neq f_i(\beta) \subseteq \psi_i^e(\beta)$. Last, we show that $x \in \varphi^e\big(\bigcup_{\beta \in F_1} f_1(\beta), \ldots, \bigcup_{\beta \in F_n} f_n(\beta)\big)$. As above, for $u \in (x_\varphi)_i$ we have $u \in f_i(\beta)$ where $\beta = u_{\psi_i}$. Thus $(x_\varphi)_i \subseteq \bigcup_{\beta \in F_i} f_i(\beta)$. The converse inclusion is evident, so $x_\varphi = \big(\bigcup_{\beta \in F_1} f_1(\beta), \ldots, \bigcup_{\beta \in F_n} f_n(\beta)\big)$, and the desired conclusion follows.

Conversely, let $(F_1, \ldots, F_n, f_1, \ldots, f_n)$ be admissible for α with $x \in \varphi^e\big(\bigcup_{\beta \in F_1} f_1(\beta), \ldots, \bigcup_{\beta \in F_n} f_n(\beta)\big)$. Then x_φ exists by definition, verifying (2.5) (i). Certainly also $x_\varphi \leqslant \big(\bigcup_{\beta \in F_1} f_1(\beta), \ldots, \bigcup_{\beta \in F_n} f_n(\beta)\big)$. Thus $(x_\varphi)_i \subseteq \bigcup_{\beta \in F_i} f_i(\beta)$. If $u \in (x_\varphi)_i$, this implies that $u \in f_i(\beta)$ for a $\beta \in F_i$. By (2.6) (iii), $f_i(\beta) \subseteq \psi_i^e(\beta)$, so u_{ψ_i} exists and $u_{\psi_i} \leqslant \beta$. Thus (2.5) (ii) holds. Further, by (2.6) (iv), $\beta \leqslant \alpha$, so $u_{\psi_i} \leqslant \alpha$. Thus $\bigvee_{i=1}^{n} \bigvee_{u \in (x_\varphi)_i} u_{\psi_i} \leqslant \alpha$. To get the converse inequality, observe that by (2.6) (iv) it suffices to show that for $\beta \in F_i$, we have $\beta \leqslant \bigvee_{i=1}^{n} \bigvee_{u \in (x_\varphi)_i} u_{\psi_i}$. By (2.6) (iii), $f_i(\beta)$ is non-empty. Let $u \in f_i(\beta)$. By (2.6) (iii), $u \in \psi_i^e(\beta)$, so $u_{\psi_i} = \beta$.

LEMMA 2.3. *θ is a precombinatorial operator.*

Proof. To verify (2.1), simply note that $x \in \theta(\alpha)$ implies that α is determined by x, since $\alpha = \bigvee_{i=1}^{n} \bigvee_{u \in (x_\varphi)_i} u_{\psi_i}$. Thus $x \in \theta(\alpha) \cap \theta(\beta)$ implies $\alpha = \beta$. To verify (2.2), suppose that $\alpha, \alpha' \in \times^k \mathcal{S}_{\mathrm{fin}}(E)$ and $\langle\alpha\rangle = \langle\alpha'\rangle$. Then there exist 1-1 onto $p_i\colon \alpha_i \to \alpha_i'$, $i = 1, \ldots, k$. Let $p\colon I(\alpha) \to I(\alpha')$ be the 1-1 onto map such that for $\beta \in I(\alpha)$, $p\beta = (p_1\beta_1, \ldots, p_k\beta_k)$. For $i = 1, \ldots, n$, define a 1-1 onto $q_i\colon \bigcup_{\beta \in I(\alpha)} \psi_i^e(\beta) \to \bigcup_{\beta' \in I(\alpha')} \psi_i^e(\beta')$ such that if $\beta' = p\beta$, then $q_i(\psi_i^e(\beta)) = \psi_i^e(\beta')$. (Such maps q_i exist since $\langle\beta\rangle = \langle\beta'\rangle$ and ψ_i satisfies (2.1) and (2.2).)

There is a 1-1 correspondence between admissible $2n$-tuples for α and admissible $2n$-tuples for α' given by

$$(F_1, \ldots, F_n, f_1, \ldots, f_n) \leftrightarrow (F_1', \ldots, F_n', f_1', \ldots, f_n'),$$

where $F_i' = [p\beta \mid \beta \in F_i]$ and $f_i'(\beta') = q_i(f_i(\beta))$ for $\beta \in F_i$ and $\beta' = p\beta$. Then since q_i is 1-1 onto, $f_i(\beta)$ has the same number of elements as $f_i'(\beta')$. Due to (2.1), for fixed i and distinct β_1, β_2 we have that $f_i(\beta_1)$ is disjoint from $f_i(\beta_2)$. Thus $\bigcup_{\beta \in F_i} f_i(\beta)$ has the same number of elements as $\bigcup_{\beta' \in F_i'} f_i'(\beta')$. Therefore (2.2) for φ^e yields that $\varphi^e\big(\bigcup_{\beta \in F_1} f_1(\beta), \ldots, \bigcup_{\beta \in F_n} f_n(\beta)\big)$ and $\varphi^e\big(\bigcup_{\beta' \in F_1'} f_1'(\beta'), \ldots, \bigcup_{\beta' \in F_n'} f_n'(\beta')\big)$ have the same number of elements. Further, if $(F_1, \ldots, F_n, f_1, \ldots, f_n)$ and $(G_1, \ldots, G_n, g_1, \ldots, g_n)$ are admissible for α, we claim that $\varphi^e\big(\bigcup_{\beta \in F_1} f_1(\beta), \ldots, \bigcup_{\beta \in F_n} f_n(\beta)\big)$ and $\varphi^e\big(\bigcup_{\beta \in G_1} g_1(\beta), \ldots, \bigcup_{\beta \in G_n} g_n(\beta)\big)$ are disjoint. Otherwise, there is an x in both these sets. Then $(x_\varphi)_i = \bigcup_{\beta \in F_i} f_i(\beta)$. Since $f_i(\beta) \subseteq \psi_i^e(\beta)$ and (2.1) holds for ψ_i^e, it follows that $f_i(\beta) = (x_\varphi)_i \cap \psi_i^e(\beta)$. Then $F_i = [\beta \mid (x_\varphi)_i \cap \psi_i^e(\beta) \neq \emptyset]$. Since the same argument applies with G_i replacing F_i and g_i replacing f_i, we conclude that

$$(F_1, \ldots, F_n, f_1, \ldots, f_n) = (G_1, \ldots, G_n, g_1, \ldots, g_n).$$

Taking a union over all $2n$-tuples admissible for α in one case and for α' in the other shows that

$$\theta(\alpha) = \bigcup \varphi^e\Big(\bigcup_{\beta \in F_1} f_1(\beta), \ldots, \bigcup_{\beta \in F_n} f_n(\beta)\Big)$$

and

$$\theta(\alpha') = \bigcup \varphi^e\Big(\bigcup_{\beta' \in F_1'} f_1'(\beta'), \ldots, \bigcup_{\beta' \in F_n'} f_n'(\beta')\Big)$$

have the same number of elements.

LEMMA 2.4. *The combinatorial operator induced by the precombinatorial operator θ is $\varphi \circ (\psi_1, \ldots, \psi_n)$. Moreover, the coefficient function for $(\varphi \circ (\psi_1, \ldots, \psi_n))^\#$ is determined by the coefficient functions for $\varphi^\#, \psi_1^\#, \ldots, \psi_n^\#$.*

Proof. $x \in \varphi(\psi_1(\alpha), \ldots, \psi_n(\alpha))$ is equivalent to the assertion that x_φ exists and that $(x_\varphi)_i \subseteq \psi_i(\alpha)$ for $i = 1, \ldots, n$. In turn this is equivalent to the assertion that x_φ exists and that for $u \in (x_\varphi)_i$, u_{ψ_i} exists and $u_{\psi_i} \leqslant \alpha$. If $\beta = \bigvee_{i=1}^{n} \bigvee_{u \in (x_\varphi)_i} u_{\psi_i}$, this means that $\beta \leqslant \alpha$. Applying the definition of θ, it is clear that $x \in \varphi(\psi_1(\alpha), \ldots, \psi_n(\alpha))$ is equivalent to the existence of a (finite) β such that $x \in \theta(\beta)$ and $\beta \leqslant \alpha$. Thus θ induces $\varphi \circ (\psi_1, \ldots, \psi_n)$ according to (2.3). Finally, note that for $\alpha \in \times^k \mathcal{S}_{\mathrm{fin}}(E)$, the proof of Lemma 2.3 calculates the cardinality of $\theta(\alpha)$ from the cardinalities of various $\varphi^e\big(\bigcup_{\beta \in F_1} f_1(\beta_1), \ldots, \bigcup_{\beta \in F_n} f_n(\beta)\big)$, the latter depending only on $\varphi^\#, \psi_1^\#, \ldots, \psi_n^\#$. Thus $\big(\varphi(\psi_1(\alpha), \ldots, \psi_n(\alpha))\big)^\#$ depends only on $\varphi^\#, \psi_1^\#, \ldots, \psi_n^\#$.

From Theorem 2.1 it follows that there is an operation on combinatorial series induced by composition of combinatorial operators. It is appropriate to characterize the induced operation on series directly. Define the sum and scaler multiple of combinatorial series as follows. Let

$$f = \sum c(i_1, \ldots, i_n) \binom{v_1}{i_1} \cdots \binom{v_n}{i_n},$$

$$g = \sum d(i_1, \ldots, i_n) \binom{v_1}{i_1} \cdots \binom{v_n}{i_n},$$

let $x \in E \cup \{\aleph_0\}$. Then

$$f+g = \sum \big(c(i_1, \ldots, i_n) + d(i_1, \ldots, i_n)\big) \binom{v_1}{i_1} \cdots \binom{v_n}{i_n},$$

$$xf = \sum xc(i_1, \ldots, i_n) \binom{v_1}{i_1} \cdots \binom{v_n}{i_n}.$$

(All operations on coefficients are in the sense of cardinal arithmetic in $E \cup \{\aleph_0\}$ throughout.) Let 0 be the series which is the identity with respect to addition—i.e., which has an identically zero coefficient function.

Since combinatorial series f with finite coefficients are in a 1-1 correspondence with combinatorial functions f^{-} of natural numbers, and combinatorial functions of natural numbers are closed under composition, there is an operation on combinatorial series with finite coefficients which corresponds to composition of functions. As temporary notation, if $f(v_1, \ldots, v_n)$, $g^1(v_1, \ldots, v_k), \ldots, g^n(v_1, \ldots, v_k)$ are combinatorial series with finite coefficients, let $f * (g1, \ldots, gn)$ denote the combinatorial series h (with finite coefficients) such that for $(x_1, \ldots, x_k) \in \times^k E$, $h^{-}(x_1, \ldots, x_k) = f^{-}\big(g^{1-}(x_1, \ldots, x_k), \ldots, g^{n-}(x_1, \ldots, x_k)\big)$. If

$$F = \sum c(i_1, \ldots, i_k) \binom{v_1}{i_1} \cdots \binom{v_k}{i_k}$$

is a combinatorial series, define combinatorial series

$$F_e = \sum d(i_1, \ldots, i_k) \binom{v_1}{i_1} \cdots \binom{v_k}{i_k}, \qquad F_R = \sum e(i_1, \ldots, i_k) \binom{v_1}{i_1} \cdots \binom{v_k}{i_k}$$

as follows. For $c(i_1, \ldots, i_k) = \aleph_0$, let $d(i_1, \ldots, i_k) = 0$ and $e(i_1, \ldots, i_k) = 1$. For $c(i_1, \ldots, i_k) < \aleph_0$, let $d(i_1, \ldots, i_k) = c(i_1, \ldots, i_k)$ and $e(i_1, \ldots, i_k) = 0$. It follows immediately that $F = F_e + \aleph_0 F_R$, and obviously F_e, F_R have finite coefficients. Next suppose given a combinatorial series $F = \sum c(i_1, \ldots, i_k) \binom{v_1}{i_1} \cdots \binom{v_k}{i_k}$ with finite coefficients. Define a combinatorial series with finite coefficients

$$F_\infty(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{2k}) = \sum d(i_1, \ldots, i_{2k}) \binom{v_1}{i_1} \cdots \binom{v_{2k}}{i_{2k}}$$

as follows. Let $d(i_1, \ldots, i_{2k})$ have as values only 0, 1, and let $d(i_1, \ldots, i_{2k}) = 1$ if and only if there exists a $c(j_1, \ldots, j_k) > 0$ such that: (i) $i_1 \leqslant j_1, \ldots, i_k \leqslant j_k$, with some inequality strict; (ii) $i_1 + i_{k+1} \leqslant j_1, \ldots, i_k + i_{2k} \leqslant j_k$; (iii) whenever $i_{k+t} = 0$, we have $i_t = j_{t'}$, $t = 1, \ldots, k$.

The formal composition of combinatorial series can be defined as follows using these definitions.

Definition. Suppose that $F(v_1, \ldots, v_n)$, $G^1(v_1, \ldots, v_k), \ldots, G^n(v_1, \ldots, v_k)$ are combinatorial series. Then $F \circ (G^1, \ldots, G^n)$ is the combinatorial series

$$F_e * (G_e^1, \ldots, G_e^n) + \aleph_0\big((F_e)_\infty\big) * (G_e^1, \ldots, G_e^n, G_R^1, \ldots, G_R^n)$$
$$+ \aleph_0 F_R * (G_e^1, \ldots, G_e^n) + \aleph_0\big((F_R)_\infty\big) * (G_e^1, \ldots, G_e^n, G_R^1, \ldots, G_R^n).$$

(2.7) *Suppose that $F, G^1, \ldots, G^n$ have finite coefficients. Then* $F \circ (G^1, \ldots, G^n) = F * (G^1, \ldots, G^n)$.

Proof. $G_e^i = G^i$, $F_e = F$, $G_R^i = 0$, $F_R = 0$, $(F_R)_\infty = 0$.

From now on, write $\circ$ uniformly, dropping $*$.

(2.8) *Suppose that F has finite coefficients. Then* $F \circ (G^1, \ldots, G^n) = F \circ (G_e^1, \ldots, G_e^n) + \aleph_0 F_\infty(G_e^1, \ldots, G_e^n, G_R^1, \ldots, G_R^n)$.

Proof. Apply $F_R = 0$, $(F_R)_\infty = 0$, and (2.7) above. We require lemmas connecting combinatorial operators φ and combinatorial series $\varphi^\#$ induced by φ. The first three are obvious

(2.9) *Let φ, ψ: $\times^k \mathfrak{I}(E) \to \mathfrak{I}(E)$ be combinatorial operators. Define $\varphi \oplus \psi$: $\times^k \mathfrak{I}(E) \to \mathfrak{I}(E)$* by $(\varphi \oplus \psi)(\alpha) = \varphi(\alpha) \oplus \psi(\alpha)$ *for $\alpha \in \times^k \mathfrak{I}(E)$. Then $\varphi \oplus \psi$ is a combinatorial operator and $\varphi \oplus \psi$ induces $\varphi^\# + \psi^\#$.*

(2.10) *Let φ: $\times^k \mathfrak{I}(E) \to \mathfrak{I}(E)$ be a combinatorial operator. Let X be an x element subset of E. Define ψ: $\times^k \mathfrak{I}(E) \to \mathfrak{I}(E)$ by $\psi(\alpha) = X \otimes \varphi(\alpha)$ for $\alpha \in \times^k \mathfrak{I}(E)$. Then ψ is a combinatorial operator and $\psi^\# = x\varphi^\#$.*

(2.11) *Let ψ: $\times^k \mathfrak{I}(E) \to \mathfrak{I}(E)$ be a combinatorial operator. Define combinatorial operators ψ_R, ψ_e: $\times^k \mathfrak{I}(E) \to \mathfrak{I}(E)$ by requiring for $\alpha \in \times^k \mathfrak{I}_{\text{fin}}(E)$: (i) If $\psi^e(\alpha)$ is finite, then $\psi_R^e(\alpha) = \emptyset$ and $\psi_e^e(\alpha) = \psi^e(\alpha)$. (ii) If $\psi^e(\alpha)$ is infinite, $\psi_R^e(\alpha) = \{x\}$* (where x is the least member of $\psi^e(\alpha)$) *and $\psi_e^e(\alpha) = \emptyset$. Then ψ_e, ψ_R are combinatorial operators and $(\psi^\#)_R = (\psi_R)^\#$, $(\psi^\#)_e = (\psi_e)^\#$.*

Lemma 2.5. *Let φ: $\times^n \mathfrak{I}(E) \to \mathfrak{I}(E)$ be a combinatorial operator with $\varphi^\# = \sum c(i_1, \ldots, i_n) \binom{v_1}{i_1} \cdots \binom{v_n}{i_n}$. Let ψ: $\times^{2n} \mathfrak{I}(E) \to \mathfrak{I}(E)$ be defined by*

$$\psi(\alpha_1, \ldots, \alpha_{2n}) = \varphi\big(\alpha_1 \oplus (E \otimes \alpha_{n+1}), \alpha_2 \oplus (E \otimes \alpha_{n+2}), \ldots, \alpha_n \oplus (E \otimes \alpha_{2n})\big).$$

Then ψ is a combinatorial operator and

$$\psi^\# = \varphi^\# + \aleph_0(\varphi^\#)_\infty.$$

(By convention $\varphi^\#$ is here regarded as a series in $v_1, \ldots, v_{2n}$.)

Proof. If $\gamma \subseteq E$, let $p_2(\gamma)$ consist of all $y \in E$ such that there exists an $x \in E$ with $\{x\} \otimes \{y\} \in \gamma$. Use Lemmas 2.2 and 2.4 or an easy direct argument to show that for finite $a_1, \ldots, a_{2n}$, $\psi^e(a_1, \ldots, a_{2n})$ is a disjoint union of all $\varphi^e(a_1 \oplus \gamma_1, \ldots, a_n \oplus \gamma_n)$ such that $\gamma_1, \ldots, \gamma_n$ are finite, and such that $\gamma_t \subseteq E \otimes a_{n+t}$ and $p_2(\gamma_t) = a_{n+t}$ for $t = 1, \ldots, n$. Of course if a_{n+t} is empty, the one and only choice for γ_t is $\emptyset$.

However if a_{n+t} is non-empty and has i_{n+t} elements and $j \geqslant i_{n+t}$, it is an elementary exercise that there are infinitely many $\gamma_t \subseteq E \otimes a_{n+t}$ such that γ_t has j elements and $p_2(\gamma_t) = a_{n+t}$. This observation will be applied to evaluate $\psi^{\#} = \sum d(i_1, \ldots, i_{2n}) \binom{v_1}{i_1} \ldots \binom{v_{2n}}{i_{2n}}$. Suppose that $i_1, \ldots, i_{2n}$ are given. Choose $a_t \subseteq E$ of cardinality i_t, $t = 1, \ldots, 2n$.

Case I. There is a $c(j_1, \ldots, j_n) > 0$ such that: (i) $i_1 \leqslant j_1, \ldots, i_n \leqslant j_n$, with some inequality strict; (ii) $i_1 + i_{n+1} \leqslant j_1, \ldots, i_n + i_{2n} \leqslant j_n$; (iii) whenever $i_{n+t} = 0$, then $i_t = j_t$ for $t = 1, \ldots, n$. Then certainly for some t among $1, \ldots, n$, we have $i_{n+t} \neq 0$. The remark above assures that there exist infinitely many distinct $(\gamma_1, \ldots, \gamma_n)$ with γ_t of cardinality $j_t - i_t$, $t = 1, \ldots, n$ and

$$\varphi^e(a_1 \oplus \gamma_1, \ldots, a_n \oplus \gamma_n) \subseteq \psi^e(a_1, \ldots, a_{2n}).$$

Since each $\varphi^e(a_1 \oplus \gamma_1, \ldots, a_n \oplus \gamma_n)$ has $c(j_1, \ldots, j_n) > 0$ elements and for distinct $(\gamma_1, \ldots, \gamma_n)$, $(\gamma_1', \ldots, \gamma_n')$ we have $\varphi^e(a_1 \oplus \gamma_1, \ldots, a_n \oplus \gamma_n)$ and $\varphi^e(a_1 \oplus \gamma_1', \ldots, a_n \oplus \gamma_n')$ are disjoint, obviously $\psi^e(a_1, \ldots, a_{2n})$ is infinite. Therefore in this case $d(i_1, \ldots, i_{2n}) = \aleph_0$.

Case II. Suppose that $c(j_1, \ldots, j_n) > 0$ is such that: (i) $i_1 + i_{n+1} \leqslant j_1, \ldots, i_n + i_{2n} \leqslant j_n$; (ii) $i_{n+t} = 0$ implies $i_t = j_t$, $t = 1, \ldots, n$. Then $i_1 = j_1, \ldots, i_n = j_n$.

In this case the only choice of $\gamma_1, \ldots, \gamma_n$ is $\gamma_1 = \ldots = \gamma_n = \emptyset$, so $\psi^e(a_1, \ldots, a_{2n})$ is $\varphi^e(a_1 \oplus \emptyset, \ldots, a_n \oplus \emptyset)$. By (2.2) for φ, the set $\varphi^e(a_1 \oplus \emptyset, \ldots, a_n \oplus \emptyset)$ has the same number of elements as $\varphi^e(a_1, \ldots, a_n)$. Therefore $d(i_1, \ldots, i_{2n}) = c(i_1, \ldots, i_n)$. Apply this calculation and the definition of $(\varphi^{\#})_\infty$ to get the conclusion.

LEMMA 2.6. *Let* $\varphi\colon \times^n \mathfrak{I}(E) \to \mathfrak{I}(E)$, $\psi^1\colon \times^k \mathfrak{I}(E) \to \mathfrak{I}(E), \ldots, \psi^n\colon \times^k \mathfrak{I}(E) \to \mathfrak{I}(E)$ *be combinatorial operators. Let* $\varphi^{\#} = F$, $\psi^{i\#} = G^i$. *Suppose that* $\varphi^{\#}$ *has finite coefficients. Then*

$$\big(\varphi(\psi^1, \ldots, \psi^n)\big)^{\#} = F(G_\varepsilon^1, \ldots, G_\varepsilon^n) + \aleph_0 F_\infty(G_\varepsilon^1, \ldots, G_\varepsilon^n, G_R^1, \ldots, G_R^n).$$

Proof. Let ψ be chosen as for Lemma 2.5. Let $\zeta\colon \times^{2n}\mathfrak{I}(E) \to \mathfrak{I}(E)$ be a combinatorial operator with $\zeta^{\#} = F_\infty$. Apply (2.9), (2.10), Lemma 2.5 to get that

$$\psi^{\#} = \varphi^{\#} + \aleph_0(\varphi^{\#})_\infty = \big(\varphi(a_1, \ldots, a_n) \oplus E \otimes \zeta(a_1, \ldots, a_{2n})\big)^{\#}.$$

Then the last assertion of Theorem 2.1 yields

$$(2.12)\qquad \big(\psi(\psi_\varepsilon^1, \ldots, \psi_\varepsilon^n, \psi_R^1, \ldots, \psi_R^n)\big)^{\#} = \big[\varphi(\psi_\varepsilon^1, \ldots, \psi_\varepsilon^n) \oplus \big(E \otimes \zeta(\psi_\varepsilon^1, \ldots, \psi_\varepsilon^n, \psi_R^1, \ldots, \psi_R^n)\big)\big]^{\#}.$$

Now apply (2.9), (2.10), (2.11) to get that $\psi^{i\#} = G^i = G_\varepsilon^i + \aleph_0 G_R^i = \big(\psi_\varepsilon^i \oplus (E \otimes \psi_R^i)\big)^{\#}$. Then an application of the last assertion of Theorem 2.1 yields

$$(2.13)\qquad \big(\varphi(\psi^1, \ldots, \psi^n)\big)^{\#} = \big[\varphi\big(\psi_\varepsilon^1 \oplus (E \otimes \psi_R^1), \ldots, \psi_\varepsilon^n \oplus (E \otimes \psi_R^n)\big)\big]^{\#}.$$

A look at the definition of ψ in the proof of Lemma 2.5 reveals $\psi(\psi_\varepsilon^1, \ldots, \psi_\varepsilon^n, \psi_R^1, \ldots, \psi_R^n)$ is $\varphi\big(\psi_\varepsilon^1 \oplus (E \otimes \psi_R^1), \ldots, \psi_\varepsilon^n \oplus (E \otimes \psi_R^n)\big)$. Combine (2.12) and (2.13) and this observation to get

$$\big(\varphi(\psi^1, \ldots, \psi^n)\big)^{\#} = \big[\varphi(\psi_\varepsilon^1, \ldots, \psi_\varepsilon^n) \oplus \big(E \otimes \zeta(\psi_\varepsilon^1, \ldots, \psi_\varepsilon^n, \psi_R^1, \ldots, \psi_R^n)\big)\big]^{\#}.$$

The right side can be rewritten using (2.9), (2.10) as

$$\big(\varphi(\psi_\varepsilon^1, \ldots, \psi_\varepsilon^n)\big)^{\#} + \aleph_0\big(\zeta(\psi_\varepsilon^1, \ldots, \psi_\varepsilon^n, \psi_R^1, \ldots, \psi_R^n)\big)^{\#}.$$

But $\varphi^{\#} = F$, $(\psi_\varepsilon^1)^{\#} = G_\varepsilon^1, \ldots, (\psi_\varepsilon^n)^{\#} = G_\varepsilon^n$, $(\psi_R^1)^{\#} = G_R^1, \ldots, (\psi_R^n)^{\#} = G_R^n$, $\zeta^{\#} = F_\infty$ all are series with finite coefficients, i.e., represent combinatorial functions on E. Composition for these series corresponds to composition of functions. Applying a known result for combinatorial functions [12] we get that

$$\big(\varphi(\psi_\varepsilon^1, \ldots, \psi_\varepsilon^n)\big)^{\#} = \varphi^{\#}\big((\psi_\varepsilon^1)^{\#}, \ldots, (\psi_\varepsilon^n)^{\#}\big)$$

and

$$\zeta(\psi_\varepsilon^1, \ldots, \psi_\varepsilon^n, \psi_R^1, \ldots, \psi_R^n)^{\#} = \zeta^{\#}\big((\psi_\varepsilon^1)^{\#}, \ldots, (\psi_\varepsilon^n)^{\#}, (\psi_R^1)^{\#}, \ldots, (\psi_R^n)^{\#}\big).$$

Apply (2.11) to get the conclusion of the lemma.

THEOREM 2.2. *Let* $\varphi\colon \times^n \mathfrak{I}(E) \to \mathfrak{I}(E)$, $\psi^1\colon \times^k \mathfrak{I}(E) \to \mathfrak{I}(E), \ldots, \psi^n\colon \times^k \mathfrak{I}(E) \to \mathfrak{I}(E)$ *be combinatorial operators. Then*

$$\big(\varphi(\psi^1, \ldots, \psi^n)\big)^{\#} = \varphi^{\#} \circ (\psi^{1\#}, \ldots, \psi^{n\#}).$$

Proof. Due to (2.11), $\varphi^{\#} = \big(\varphi_\varepsilon \oplus (E \otimes \varphi_R)\big)^{\#}$. Thus the last assertion of Theorem 2.1 and (2.9), (2.10) yield

$$\big(\varphi(\psi^1, \ldots, \psi^n)\big)^{\#} = \big(\varphi_\varepsilon(\psi^1, \ldots, \psi^n) \oplus E \otimes \varphi_R(\psi^1, \ldots, \psi^n)\big)^{\#} = \big(\varphi_\varepsilon(\psi^1, \ldots, \psi^n)\big)^{\#} + \aleph_0\big(\varphi_R(\psi^1, \ldots, \psi^n)\big)^{\#}.$$

If $\varphi^{\#} = F$, $\psi^{i\#} = G^i$, then an application of Lemma 2.6 and also (2.11) yields immediately

$$\varphi(\psi^1, \ldots, \psi^n)^{\#} = [F_\varepsilon \circ (G_\varepsilon^1, \ldots, G_\varepsilon^n) + \aleph_0 F_{\varepsilon\infty} \circ (G_\varepsilon^1, \ldots, G_\varepsilon^n, G_R^1, \ldots, G_R^n)] + \aleph_0 [F_R \circ (G_\varepsilon^1, \ldots, G_\varepsilon^n) + \aleph_0 F_{R\infty} \circ (G_\varepsilon^1, \ldots, G_\varepsilon^n, G_R^1, \ldots, G_R^n)].$$

The right-hand side is by definition $F \circ (G^1, \ldots, G^n)$, i.e. it is $\varphi^{\#} \circ (\psi^{1\#}, \ldots, \psi^{n\#})$.

From now on write $f\circ(g^1,\ldots,g^n)$ as $f(g^1,\ldots,g^n)$ when no confusion is possible.

3. Partial recursive combinatorial operators and R↑ combinatorial functions. Let $\mathfrak{G}_k\colon \times^k\mathfrak{I}_{\text{fin}}(E)\to E$ be one of the usual fully effective 1-1 onto maps. A combinatorial operator $\varphi\colon \times^k\mathfrak{I}(E)\to\mathfrak{I}(E)$ is called *partial recursive* if

(3.1) The map on $\varphi(\times^k E)$ to E given by $x\to\mathfrak{G}_k(x_\varphi)$ is partial recursive.

LEMMA 3.1. *If $\varphi\colon \times^n\mathfrak{I}(E)\to\mathfrak{I}(E)$, $\psi^1\colon \times^k\mathfrak{I}(E)\to\mathfrak{I}(E),\ldots,\psi^n\colon \times^k\mathfrak{I}(E)\to\mathfrak{I}(E)$ are partial recursive combinatorial operators, then $\varphi(\psi^1,\ldots,\psi^n)$ is also a partial recursive combinatorial operator.*

Proof. Theorem 2.1 implies that $\zeta=\varphi(\psi^1,\ldots,\psi^n)$ is a combinatorial operator. We describe how to recursively enumerate all pairs $(x,\mathfrak{G}_k(x_\zeta))$. Effectively enumerate all $(x,\mathfrak{G}_n(x_\varphi))$ and all $(u,\mathfrak{G}_k(u_{\psi^1})),\ldots,(u,\mathfrak{G}_k(u_{\psi^n}))$. This is possible due to (3.1) for $\varphi,\psi^1,\ldots,\psi^n$. If at any stage an $(x,\mathfrak{G}_n(x_\varphi))$ has been generated such that for all $i=1,\ldots,n$ and all $u\in(x_\varphi)_i$, $(u,\mathfrak{G}_k(u_\varphi i))$ has been generated, then list $\big(x,\mathfrak{G}_k(\bigvee_{i=1}^{n}\bigvee_{u\in(x_\varphi)_i}u_{\psi^i})\big)$ as an $(x,\mathfrak{G}_k(x_\zeta))$. Then (2.5) assures that all and only the correct pairs are listed.

It is easy to see that for a combinatorial operator φ, φ is partial recursive in the sense of (3.1) if and only if φ is a partial recursive functional. From that point of view the immediately preceding argument can be omitted since partial recursive functionals are closed under composition.

Since the next few theorems concern R↑ functions, some remarks on these functions are in order. It is an easy exercise that the characteristic function of a subset α of E, 1 on α and 0 on $E-\alpha$, is the limit of a monotone increasing recursive sequence of recursive functions if and only if α is recursively enumerable. Similarly, the characteristic function of α is the limit of a monotone decreasing recursive sequence of recursive functions if and only if α is the complement of a recursively enumerable set. Thus the characteristic functions in R↑ are exactly the characteristic functions of recursively enumerable sets, while the characteristic function of the complement of a recursively enumerable but not recursive set is always outside R↑. (A function of the latter type can be expressed in both two number quantifier forms in the Kleene-Mostowski hierarchy.) A useful characterization of membership in R↑ is

(3.2) $c\colon \times^k E\to E\cup\{\aleph_0\}$ *is* R↑ *if and only if there exists a recursively enumerable family $\{\beta_i\}_{i\in\times^k E}$ of recursively enumerable sets such that for all i, β_i has $c(i)$ elements.*

Proof. Suppose that $f_n\colon \times^k E\to E$ is a monotone ↑ recursive sequence of recursive functions such that $c(i)=\lim_n f_n(i)$ for all $i\in\times^k E$. Define the recursively enumerable family $\{\beta_i\}_{i\in\times^k E}$ by letting β_i consist of all $n\in E$ such that for some j, $f_j(i)>n$. Obviously $c(i)=\lim_j f_j(i)$ implies β_i has $c(i)$ elements. Conversely, suppose that $\{\beta_i\}_{i\in\times^k E}$ is given. Then the β_i can be enumerated in stages so that at stage n only a finite number of elements of a finite number of the β_i have been enumerated. Let β_i^n be the set of elements of β_i enumerated by the nth step. Let $f_n(i)$ be the number of elements of β_i^n.

THEOREM 3.2. *A combinatorial series f is* R↑ *if and only if $f=\varphi^{\#}$ for at least one partial recursive combinatorial operator φ.*

Proof. Let $f=\sum c(i_1,\ldots,i_k)\binom{v_1}{i_1}\ldots\binom{v_k}{i_k}$. Suppose that c is R↑. Use (3.2) to produce a family $\{\beta_i\}_{i\in\times^k E}$ with β_i having $c(i)$ elements for all $i\in\times^k E$. Use notation as in (1.1), define $\varphi^e\colon \times^k\mathfrak{I}_{\text{fin}}(E)\to\mathfrak{I}(E)$ by

$$\varphi^e(\alpha)=[j(\alpha,n)\,|\; n\in\beta_{\langle\alpha\rangle}]\quad\text{for}\quad \alpha\in\times^k\mathfrak{I}_{\text{fin}}(E).$$

Then φ^e is precombinatorial and induces a combinatorial operator φ. Obviously $\varphi^e(\alpha)$ has the same number of elements as $\beta_{\langle\alpha\rangle}$, i.e., $c(i_1,\ldots,i_k)$, so $\varphi^{\#}=f$. φ is partial recursive since the assumption that $\{\beta_i\}_{i\in\times^k E}$ is a recursively enumerable family entails that all $(j(\alpha,n),\mathfrak{G}_k(\alpha))$ can be recursively enumerated for $n\in\beta_{\langle\alpha\rangle}$ and $\alpha\in\times^k\mathfrak{I}_{\text{fin}}(E)$; this is precisely the set of $(x,\mathfrak{G}_k(x_\varphi))$. Conversely, suppose that $\varphi\colon \times^k\mathfrak{I}(E)\to\mathfrak{I}(E)$ is partial recursive combinatorial. Choose a recursively enumerable family $\{\mathfrak{G}_k(\alpha(i))\}_{i\in\times^k E}$ such that for all i, $\alpha(i)\in\times^k\mathfrak{I}_{\text{fin}}(E)$ and $\langle\alpha(i)\rangle=i$. Then (3.1) for φ implies that $\{\varphi^e(\alpha(i))\}_{i\in\times^k E}$ is a recursively enumerable family of recursively enumerable sets. Of course by (1.1) $\varphi^e(\alpha(i))$ has $c(i)$ elements. Thus (3.2) implies that c is R↑.

COROLLARY. *The* R↑ *combinatorial series are closed under formal composition.*

The R↑ series of course include the recursive series, but the recursive series are not themselves closed under formal composition. For an example, first choose a recursive relation $R\subseteq E\times E$ such that (i) $R\cap(E\times\{0\})=\emptyset$; (ii) the set S of all x such that for some y, $(x,y)\in R$ is recursively enumerable and not recursive. Make the obvious notational conventions and substitute the constant series $\aleph_0$ for u_2 in the series $\sum_{(i,j)\in R}\binom{u_1}{i}\binom{u_2}{j}$ to get the series $\sum_{i\in S}\Big(\sum_{(i,j)\in R}\binom{\aleph_0}{j}\Big)\binom{u_1}{i}$. Since $\binom{\aleph_0}{j}=\aleph_0$ for $j>0$, and for each $i\in S$ there is a j with $(i,j)\in R$, we conclude that $\sum_{(i,j)\in R}\binom{\aleph_0}{j}=\aleph_0$ for $i\in S$. Thus the resulting series is $\sum_{i\in S}\aleph_0\binom{u_1}{i}$, which is not recursive even though the constant series $\aleph_0$ and the series $\sum_{(i,j)\in R}\binom{u_1}{i}\binom{u_2}{j}$ are both recursive.

THEOREM 3.3. *Let $f=\sum c(i_1,\ldots,i_k)\binom{v_1}{i_1}\ldots\binom{v_k}{i_k}$ be an $\mathrm{R}\uparrow$ combinatorial series. Then f induces a well-defined map $f_\Omega\colon \times^k\Omega\to\Omega$ such that for $x\in\times^k\Omega$, $f_\Omega(x)=\langle\varphi(\alpha)\rangle$, where $\alpha\in\times^k\mathfrak{I}(E)$ is such that $\langle\alpha\rangle=x$ and φ is a partial recursive combinatorial operator such that $\varphi^{\#}=f$.*

Proof. It must be shown that $f_\Omega(x)$ is independent of the choice of α and φ. Suppose $\langle\alpha\rangle=\langle\beta\rangle=x$. Suppose $\varphi^{\#}=\psi^{\#}=f$. Suppose that α_i is contained in domain p_i, p_i is 1-1 partial recursive, $p_i(\alpha_i)=\beta_i$. Let A consist of all $\gamma\in\times^k\mathfrak{I}_{\mathrm{fin}}(E)$ such that γ_i is a subset of the domain of p_i for $i=1,\ldots,n$. Let B consist of all $(p_1(\gamma_1),\ldots,p_k(\gamma_k))$ such that $\gamma\in A$. Let $p\colon A\to B$ be the 1-1 onto map such that $p\gamma=(p_1\gamma_1,\ldots,p_k\gamma_k)$. Since $p_1,\ldots,p_k$ are partial recursive, certainly $[\mathfrak{G}_k(\gamma)|\ \gamma\in A]$ is recursively enumerable. Thus by (3.1) $\varphi^e(\gamma)$ can be recursively enumerated without repetitions uniformly in $\mathfrak{G}_k(\gamma)$ for $\gamma\in A$. For the same reason $[\mathfrak{G}_k(\gamma')|\ \gamma'\in B]$ is recursively enumerable and $\psi^e(\gamma')$ can be recursively enumerated without repetitions uniformly in $\mathfrak{G}_k(\gamma')$ for $\gamma'\in B$. But since the p_i are 1-1, for $\gamma\in A$, we have $\langle\gamma\rangle=\langle p\gamma\rangle$. Thus by (2.2) and the fact that $\varphi^{\#}=\psi^{\#}$, certainly $\varphi^e(\gamma)$ and $\psi^e(p\gamma)$ have the same number of elements (finite or infinite). Map the nth element of $\varphi^e(\gamma)$ onto the nth element of $\psi^e(p\gamma)$ in the enumerations without repetition chosen above, for all $\gamma\in A$. By (2.1) and (2.2) for φ and ψ, we conclude that this yields a 1-1 partial recursive q with domain $\bigcup_{\gamma\in A}\varphi^e(\gamma)$, range $\bigcup_{\gamma'\in B}\psi^e(\gamma')$ such that for each $\gamma\in A$, $\varphi^e(\gamma)$ is mapped 1-1 onto $\psi^e(p\gamma)$.

But $\varphi(\alpha)$ is a union of $\varphi^e(\gamma)$'s and $\psi(\beta)$ is the union of the corresponding $\psi^e(p\gamma)$'s, so q maps $\varphi(\alpha)$ 1-1 onto $\psi(\beta)$. Thus $\langle\varphi(\alpha)\rangle=\langle\psi(\beta)\rangle$.

COROLLARY. *Let $f(v_1,\ldots,v_n)$, $g^1(v_1,\ldots,v_k),\ldots,g^n(v_1,\ldots,v_k)$ be $\mathrm{R}\uparrow$ combinatorial series. Then*

$$f_\Omega\big(g^1_\Omega(x_1,\ldots,x_k),\ldots,g^n_\Omega(x_1,\ldots,x_k)\big)=\big(f\circ(g^1,\ldots,g^n)\big)_\Omega(x_1,\ldots,x_k).$$

Proof. Let $\varphi\colon\times^nE\mathfrak{I}(E)\to\mathfrak{I}(E)$, $\psi^1\colon\times^nE\mathfrak{I}(E)\to\mathfrak{I}(E),\ldots,\psi^n\colon\times^nE\mathfrak{I}(E)\to\mathfrak{I}(E)$ be partial recursive combinatorial operators with $\varphi^{\#}=f$, $\psi^{1\#}=g^1,\ldots,\psi^{n\#}=g^n$. By Lemma 3.1, $\varphi(\psi^1,\ldots,\psi^n)$ is a partial recursive combinatorial operator. Further, Theorem 2.2 assures that $\big(\varphi(\psi^1,\ldots,\psi^n)\big)^{\#}=\varphi^{\#}\circ\big((\psi^1)^{\#},\ldots,(\psi^n)^{\#}\big)$. Let $x_1=\langle\alpha_1\rangle,\ldots,x_k=\langle\alpha_k\rangle$. By Theorem 3.3 and the remarks just made

$$\begin{aligned}\big(f\circ(g^1,\ldots,g^n)\big)_\Omega(x_1,\ldots,x_k)&=\langle\varphi(\psi^1,\ldots,\psi^n)(\alpha_1,\ldots,\alpha_k)\rangle\\&=\big\langle\varphi\big(\psi^1(\alpha_1,\ldots,\alpha_k),\ldots,\psi^n(\alpha_1,\ldots,\alpha_k)\big)\big\rangle\\&=f_\Omega\big(\langle\psi^1(\alpha_1,\ldots,\alpha_k)\rangle,\ldots,\langle\psi^n(\alpha_1,\ldots,\alpha_k)\rangle\big)\\&=f_\Omega\big(g^1_\Omega(x_1,\ldots,x_k),\ldots,g^n_\Omega(x_1,\ldots,x_k)\big).\end{aligned}$$

Finally, Proposition I (§ 1) follows immediately from this Corollary and the fact that the extension of the projection $p_i\colon\times^kE\to E$, $p_i(x_1,\ldots,x_k)=x_i$, is the projection $p_{i_\Omega}\colon\times^k\Omega\to\Omega$, $p_{i_\Omega}(x_1,\ldots,x_k)=x_i$ (cf. [12], § 10).

4. Inequalities. Proposition II (§ 1) will be proved by application of the next six lemmas. The category method of [12], § 7 will be employed.

LEMMA 4.1. *Let $f(u)=\sum c(i)\binom{u}{i}$, $g(u)=\sum d(i)\binom{u}{i}$ be $\mathrm{R}\uparrow$ combinatorial series. Then* (i), (ii) *below are equivalent.*

(i) *There exists a $k\in E$ and there exists an $\mathrm{R}\uparrow$ series $h(u)$ such that $f(u+k)+h(u)=g(u+k)$ is a (formal) identity.*

(ii) *There exists a $k\in E$ and an $\mathrm{R}\uparrow$ function $e\colon E\to E$ such that for all i, $e(i)+c(i)\binom{k}{0}+\ldots+c(i+k)\binom{k}{k}=d(i)\binom{k}{0}+\ldots+d(i+k)\binom{k}{k}$.*

Proof. Assume that (i) holds. Write $h(u)=\sum e(i)\binom{u}{i}$. The combinatorial identity

$$\binom{u+k}{i}=\binom{u}{i}\binom{k}{0}+\ldots+\binom{u}{i-k}\binom{k}{k}$$

yields without difficulty that

$$f(u+k)=\sum_{i=0}^{\infty}\left(c_i\binom{k}{0}+\ldots+c_{i+k}\binom{k}{k}\right)\binom{u}{i},$$

$$g(u+k)=\sum_{i=0}^{\infty}\left(d_i\binom{k}{0}+\ldots+d_{i+k}\binom{k}{k}\right)\binom{u}{i}.$$

The identity $f(u+k)+h(u)=g(u+k)$ yields (ii) when corresponding coefficients of $\binom{u}{i}$ are compared. The converse is similar.

LEMMA 4.2. *Suppose that $f(u)$, $g(u)$ are $\mathrm{R}\uparrow$ combinatorial series and that there is a k and an $\mathrm{R}\uparrow$ combinatorial series $h(u)$ such that $f(u+k)+h(u)=g(u+k)$ is a formal identity. Then $f_\Omega(x)\leqslant g_\Omega(x)$ for almost all $x\in\Omega$.*

Proof. Due to Proposition I, $f_\Omega(x+k)+h_\Omega(x)=g_\Omega(x+k)$ for all $x\in\Omega$. Therefore $f_\Omega(y)\leqslant g_\Omega(y)$ for any $y\in\Omega-\{0,1,\ldots,k-1\}$, since any such y is of the form $x+k$.

Topologize $\mathfrak{I}(E)$ as follows. Suppose that δ_1,δ_2 are disjoint finite subsets of E. Let $U(\delta_1,\delta_2)$ consist of all subsets of E which include δ_1 and are disjoint from δ_2. Choose a base for open sets in $\mathfrak{I}(E)$ consisting of all $U(\delta_1,\delta_2)$ for δ_1,δ_2 disjoint finite subsets of E. Then $\mathfrak{I}(E)$ is a Cantor space (a homeomorph of the Cantor set).

LEMMA 4.3. *The set X_R of all $\alpha\in\mathfrak{I}(E)$ such that $\langle\alpha\rangle=\langle\alpha\rangle+1$ is of the first category.*

LEMMA 4.4. *Let* $f(u) = \sum c(i)\binom{u}{i}$, $g(u) = \sum d_i\binom{u}{i}$ *be* R↑ *combinatorial series. Suppose that* X *is the set of all* $\alpha \in \mathfrak{I}(E)$ *such that for some* $y \in \Omega$, $f_\Omega(\langle\alpha\rangle) + y = g_\Omega(\langle\alpha\rangle)$. *Then* X *is of the first category unless there exists a neighborhood* $U = U(\delta_1, \delta_2)$, *a* 1-1 *partial recursive function* p, *and partial recursive combinatorial operators* φ, ψ *with* $\varphi^{\#} = f$ *and* $\psi^{\#} = g$ *satisfying* (4.1) *below.*

(4.1) *For any finite* δ *in* U, *there exists a* $\beta \in \mathfrak{I}(E)$ *such that* $\varphi(\delta) \oplus \beta$ *is a subset of the domain of* p *and* $p(\varphi(\delta) \oplus \beta) = \psi(\delta)$.

Proof. Suppose that no such U, p, φ, ψ exist. It must be then shown that X is of the first category. There are only a countable number of p, φ, ψ. Therefore it suffices to show that for each choice of p, φ, ψ, the following set $X(p)$ is nowhere dense: $X(p)$ is the set of all $\alpha \in \mathfrak{I}(E)$ such that there exists a $\beta \in \mathfrak{I}(E)$ with $\varphi(\alpha) \oplus \beta$ a subset of the domain of p and $p(\varphi(\alpha) \oplus \beta) = \psi(\alpha)$.

Given a neighborhood $U = U(\delta_1, \delta_2)$, we produce a non-empty sub-neighborhood V of U disjoint from $X(p)$. According to the negation of (4.1) there must be a finite $\delta \in U$ such that for no $\beta \in \mathfrak{I}(E)$ do we have that $\varphi(\delta) \oplus \beta$ is a subset of the domain of p and $p(\varphi(\delta) \oplus \beta) = \psi(\delta)$.

If there is no $\delta^1 \supseteq \delta$, $\delta^1 \in U$, for which there exists a β^1 with $\varphi(\delta^1) \oplus \beta^1$ a subset of the domain of p and $p(\varphi(\delta^1) \oplus \beta^1) = \psi(\delta^1)$, then certainly $V = U(\delta, \delta_2)$ will do. Otherwise, there is a smallest $\delta^1 \supseteq \delta$ with this property. It is easily seen that this δ^1 is then the smallest $\nu \supseteq \delta$ such that: (1) If $x \in E$ and x_φ is defined and $x^\varphi \subseteq \nu$, then $(p(2x))_\varphi \leqslant \nu$; (2) If $x \in E$ and x_ψ is defined and $x_\psi \leqslant \nu$, and $p^{-1}(x)$, is even, then $(1/2p^{-1}(x))_\varphi \leqslant \nu$.

By choice of δ, certainly $\delta^1 \underset{+}{\supset} \delta$. Therefore there is an $x \in \delta^1 - \delta$. The minimality of δ^1 assures that $V = U(\delta, \delta_2 \cup \{x\})$ will do.

LEMMA 4.5. *Let* $f(u) = \sum c(i)\binom{u}{i}$, $g(u) = \sum d(i)\binom{u}{i}$ *be* R↑ *combinatorial series. Suppose that* $f_\Omega(x) \leqslant g_\Omega(x)$ *for almost all* $x \in \Omega$. *Then there exists a* k *and an* R↑ *combinatorial series* $h(u)$ *such that* $f(u+k) + h(u) = g(u+k)$ *is a formal identity.*

Proof. The proof proceeds through Lemma 4.4. Suppose that no choice of U, φ, ψ, p satisfies (4.1). Then Lemmas 4.4 and 4.3 imply that $X \cup X_R$ is of the first category. The Baire category theorem assures that there is an infinite $\alpha \in P(E) - (X \cup X_R)$.

Thus $x_0 = \langle\alpha\rangle \in \Lambda - E$. The definition of X assures that $f_\Omega(x_0) \not\leqslant g_\Omega(x_0)$. Thus $f_\Omega(x) \leqslant g_\Omega(x)$ does not hold for almost all $x \in \Omega$, contrary to hypothesis.

Therefore there is a choice of $U = U(\delta_1, \delta_2)$, φ, ψ, p satisfying (4.1). Let $q\colon E \to E$ be a 1-1 recursive function with range a subset of $E - (\delta_1 \cup \delta_2)$.

Then $\zeta(\alpha) = \delta_1 \cup q(\alpha)$ is a recursive combinatorial operator $\zeta\colon \mathfrak{I}(E) \to \mathfrak{I}(E)$. Also $\zeta^{\#} = u + k$, where k is the cardinality of δ_1. Define $\varphi_1, \psi_1\colon \mathfrak{I}(E) \to \mathfrak{I}(E)$ by $\varphi_1(\alpha) = \varphi(\zeta(\alpha))$, $\psi_1(\alpha) = \psi(\zeta(\alpha))$. Then φ_1, ψ_1 are partial recursive combinatorial operators such that $\varphi_1^{\#} = f(x+k)$, $\psi_1^{\#} = g(x+k)$. Applying (4.1) it follows that for all finite α, there exists a β with $\varphi_1(\alpha) \oplus \beta$ a subset of the domain of p and $p(\varphi_1(\alpha) \oplus \beta) = \psi_1(\alpha)$. Without difficulty, β is defined uniquely by α.

Define $\tau^e\colon \mathfrak{I}_{\text{fin}}(E) \to \mathfrak{I}(E)$ by requiring that $\tau^e(\alpha)$ consist of all $x \in E$ for which: (1) $p(2x+1)$ is defined; (2) $(p(2x+1))_{\psi_1}$ is defined; (3) $(p(2x+1))_{\psi_1} = \alpha$. Then τ^e has one of the two properties of a precombinatorial operators, namely for $\alpha \neq \alpha'$ certainly $\tau^e(\alpha)$ is disjoint from $\tau^e(\alpha')$. (However, τ^e does not necessarily have the other property of precombinatorial operators; it may well be that $\langle\alpha\rangle = \langle\alpha'\rangle$ but $\tau^e(\alpha)$ and $\tau^e(\alpha')$ have different numbers of elements.) For finite α, let $\tau(\alpha) = \bigcup_{\beta \text{ finite}, \beta \leqslant \alpha} \tau^e(\beta)$.

The point of introducing $\tau(\alpha)$ is that $\tau(\alpha)$ is the unique β referred to before. That is, it is easily seen that for finite $\alpha \in \mathfrak{I}(E)$, $\varphi_1(\alpha) \oplus \tau(\alpha)$ is a subset of the domain of p and $p(\varphi_1(\alpha) \oplus \tau(\alpha)) = \psi_1(\alpha)$.

The fact that p is partial recursive and (3.1) for ψ_1 can be applied to show that $\tau^e(\alpha)$ can be recursively enumerated without repetitions uniformly in $\mathfrak{G}_1(\alpha)$ for $\alpha \in \mathfrak{I}_{\text{fin}}(E)$.

If $e(n)$ is the number of elements in $\tau^e([x \in E|\ x < n])$, then it follows that $e(n)$ is R↑ by (3.2). Further, a simple induction shows that for $\alpha \in \mathfrak{I}_{\text{fin}}(E)$, $p(\varphi_1^e(\alpha) \oplus \tau^e(\alpha)) = \psi_1^e(\alpha)$. Therefore if $f(u+k) = \sum c'(i)\binom{u}{i}$ and $g(u+k) = \sum d'(i)\binom{u}{i}$, we have $c'(i) + e(i) = d'(i)$. If $h(u) = \sum e(i)\binom{u}{i}$, this means that $f(u+k) + h(u) = g(u+k)$.

LEMMA 4.6. *There exist* c *isols* x *such that* $f_\Omega(x) \not\leqslant g_\Omega(x)$ *whenever* f, g *are* R↑ *combinatorial series for which every choice of* k *and an* R↑ *combinatorial series* $h(u)$ *yields* $f(u+k) + h(u) \neq g(u+k)$.

Proof. Call the set X produced in Lemma 4.4 $X(f, g)$ to indicate dependence on f, g. Let Y be the union of all $X(f, g)$ for (f, g) ranging over the countably many pairs mentioned in Lemma 4.6. Then any α in $\mathfrak{I}(E) - (Y \cup X_R)$ yields a suitable $x = \langle\alpha\rangle$. Since $Y \cup X_R$ is of the first category, there are c α's. Since each x contains only a countable number of α's, there are cx's.

THEOREM 4.2. *Let* $f(u) = \sum c_i\binom{u}{i}$, $g(u) = \sum d_i\binom{u}{i}$ *be* R↑ *combinatorial series. Then the following conditions are equivalent.*

(i) $f_\Omega(x) = g_\Omega(x)$ *for almost all* $x \in \Omega$.

(ii) *There exists a* k *such that* $f(u+k) = g(u+k)$.

(iii) *There exists a k such that for all i,*

$$c_i\binom{k}{0}+c_{i+1}\binom{k}{1}+\ldots+c_{i+1}\binom{k}{k}=d_i\binom{k}{0}+d_{i+1}\binom{k}{1}+\ldots+d_{i+k}\binom{k}{k}.$$

Moreover, there are c isols x such that $f_\Omega(x)\neq g_\Omega(x)$ whenever f, g are R↑ *series which fail to satisfy* (iii).

Proof. Assume that (i) holds. Apply Proposition II. For some $k_1, k_2 \in E$ and some R↑ combinatorial series $h_1(u), h_2(u)$, we have $f(u+k_1)+h_1(u)=g(u+k_1)$ and $g(u+k_2)+h_2(u)=f(u+k_2)$. Let $k=\max(k_1,k_2)$. Then $f(u+k)+h_1(u+(k-k_1))=g(u+k)$, $g(u+k)+h_2(u+(k-k_2))=f(u+k)$. It follows immediately that $f(u+k)=g(u+k)$ upon examination of coefficients. Next, assume that (ii) holds. Apply Proposition I to get $f_\Omega(x)=g_\Omega(x)$ for any $x\in\Omega-\{0,1,\ldots,k-1\}$. As for (iii), the same argument applies as applied in the proof of Lemma 4.1 to show that (iii) is equivalent to (ii). The last part follows from the last part of Proposition II similarly.

An example will show that Proposition I fails when applied to a slightly wider class of series than R↑ series. An identity between combinatorial functions will be exhibited (with the combinatorial functions expressible in both two number quantifier forms) which fails in Ω.

Let S be a recursively enumerable but not recursive set. Let M be the set of all $x\in E$ such that x is not a perfect square. Let $T=[2^x|\,x\in S]\cup M$. Then T is recursively enumerable but not recursive. Further, the fact that the difference of successive squares $(x+1)^2-x^2=2x+1$ is monotone increasing and unbounded has the consequence that for any k, there is an i_0 such that for $i\geqslant i_0$, if $i\in E-M$, then $i+1\in M,\ldots,i+k\in M$.

Let $c\colon E\to E$ be the characteristic function of T, let d be the characteristic function of $E-T$. Let $f(u)=\sum c(i)\binom{u}{i}$, $g(u)=\sum d(i)\binom{u}{i}$, let $f(x), g(x)$ denote the corresponding combinatorial functions. The identity $2^x=f(x)+g(x)$ holds in E (or equivalently, the formal identity $2^u=f(u)+g(u)$ holds in combinatorial series). Suppose that this identity yielded an identity in Ω. *A fortiori*, $2^x\geqslant f_\Omega(x)$ for all $x\in\Omega$. Hence Proposition II implies that for some k, $2^{u+k}=f(u+k)+h(u)$ for an R↑ series $h(u)$. Obviously $h(u)$ must have finite coefficients, so this yields an identity $2^{x+k}=f(x+k)+h(x)$ in combinatorial functions in E. But also $2^{x+k}=f(x+k)+g(x+k)$, so $g(x+k)=h(x)$. Let $h(x)=\sum e(i)\binom{x}{i}$ with e an R↑ function. As has been remarked, an i_0 can be chosen so that for $i\geqslant i_0$ and $i\in E-M$, we have $i+1\in M,\ldots,i+k\in M$. Thus for $i\in E-M$ and $i\geqslant i_0$

$$e(i)=d(i)+d(i+1)\binom{k}{1}+\ldots+d(i+k)\binom{k}{k}=d(i).$$

By choice e is R↑, so there exists a monotone increasing recursive sequence of recursive functions e_n such that $\lim_n e_n=e$. Define a monotone increasing recursive sequence of recursive functions g_n as follows.

(1) $g_n(i)=d(i)$ for $i<i_0$.

(2) $g_n(i)=e_n(i)$ for $i\geqslant i_0$ and $i\in E-M$.

(3) $g_n(i)=0$ for $i\geqslant i_0$ and $i\in M$.

Then for $i<i_0$, obviously $\lim_n g_n(i)=d(i)$. For $i\geqslant i_0$, if $i\in E-M$ certainly $\lim g_n(i)=\lim e_n(i)=e(i)=d(i)$. For $i\geqslant i_0$, if $i\in M$, then $\lim g_n(i)=0=d(i)$. Thus d is R↑. This is impossible since d is the characteristic function of the complement of a recursively enumerable but not recursive set.

5. Applications to $\Omega-\Lambda$. Throughout this section $f(u)=\sum c(i)\binom{u}{i}$ will be an R↑ combinatorial series.

THEOREM 5.1. $f_\Omega(x)=f_\Omega(x)+1$ *for almost all $x\in\Omega$ if and only if some c_i is $\aleph_0$.*

Proof. By Theorem 4.2 and Proposition I, the hypothesis $f_\Omega(x)=f_\Omega(x)+1$ for almost all x implies that for some k,

$$c(0)\binom{k}{0}+c(1)\binom{k}{1}+\ldots+c(k)\binom{k}{k}=\big(1+c(0)\big)\binom{k}{0}+c(1)\binom{k}{1}+\ldots+c(k)\binom{k}{k}.$$

Thus each side is $\aleph_0$ and one of $c(0),\ldots,c(k)$ is $\aleph_0$. The converse is similar.

This shows that any combinatorial series with an $\aleph_0$ coefficient (i.e., which does not correspond to a combinatorial function) has almost all values in $\Omega-\Lambda$. This is what is meant by saying that combinatorial series are slanted toward $\Omega-\Lambda$.

THEOREM 5.2. *The following conditions are equivalent.*

(i) $f_\Omega(x)=2f_\Omega(x)$ *for almost all $x\in\Omega$.*

(ii) *There exists a k such that if $f(u+k)=\sum d(i)\binom{u}{i}$, then all d_i have values* 0 *or $\aleph_0$.*

(iii) *There exists a k such that for all i, if one of $c_i,\ldots,c_{i+k}$ is non-zero, then one of $c(i),\ldots,c(i+k)$ is $\aleph_0$.*

Proof. Apply Proposition I and Theorem 4.2 to see that (ii) is equivalent to the assertion that there exists a k such that for all i, $c(i)\binom{k}{0}+\ldots+c(i+k)\binom{k}{k}$ has value 0 or $\aleph_0$, i.e., to the assertion that if one of $c(i),\ldots,c(i+k)$ is non-zero, then one of this list is $\aleph_0$. Thus (ii) and (iii) are equivalent. Now assume that (i) holds and apply Proposition I and Theorem 4.2 to obtain a k such that $f(u+k) = 2f(u+k)$. If $f(u+k) = \sum d(i)\binom{u}{i}$, this means that $d(i) = 2d(i)$ for all i, i.e., all e_i are 0 or $\aleph_0$. Thus (ii) holds. Conversely, suppose that (ii) holds. Then $2f(u+k) = \sum 2d(i)\binom{u}{i} = f(u+k)$, so Proposition I assures that $2f_\Omega(x+k) = f_\Omega(x+k)$ for all $x \in \Omega$. This shows that (i) holds.

Call f non-constant if some $c(i) > 0$ with $i > 0$.

THEOREM 5.3. *Suppose that f is non-constant. Then the following are equivalent.* (i) $f_\Omega(x) = \big(f_\Omega(x)\big)^2$ *for almost all* $x \in \Omega$. (ii) *There is a k such that* $f(u+k) = \sum_{i=0}^{\infty} \aleph_0 \binom{u}{i}$. (iii) *There is a k such that for all i, one of $c(i), \ldots, c(i+k)$ is $\aleph_0$.*

Proof. Suppose that (ii) holds. By Theorem 4.2, this means that for all i, $c(i)\binom{k}{0} + \ldots + c(i+k)\binom{k}{k} = \aleph_0$—i.e., one of $c(i), \ldots, c(i+k)$ is $\aleph_0$. Conversely if (iii) holds, Theorem 4.2 implies $f(u+k) = \sum_{i=1}^{\infty} \aleph_0 \binom{u}{i}$; so (ii) holds. Now assume that (i) holds. Then Proposition I and Theorem 4.2 imply that there is a k for which $f(u+k) = \big(f(u+k)\big)^2$. Since f is non-constant, there is an $i_1 > 0$ with $c(i_1) > 0$. Let $g(u) = f(u+k+i_1+1)$ $= \sum e(i)\binom{u}{i}$. By computing coefficients it is seen that $c(i_1) \neq 0$ implies $e(i_1) \neq 0$, $e(i_1-1) \neq 0, \ldots, e(0) \neq 0$. Further, $e(0)$ is a sum of terms, one of which is $\binom{k+i_1+1}{i_1} c(i_1)$. But $i_1 > 0$ implies $\binom{k+i_1+1}{i_1} \geqslant \binom{i_1+1}{i_1} \geqslant 2$, so $e(0) \geqslant 2$. Thus there is a combinatorial series $h(u)$ such that $g(u) = 2 + h(u)$. Since $\big(g(u)\big)^2 = g(u)$, certainly $g(u)\big(2+h(u)\big) = g(u)$. It follows immediately from this equation that $2g(u) = g(u)$. Thus each $e(i)$ is 0 or $\aleph_0$. Since $e(0), \ldots, e(i)$ are non-zero, $e(0) = \ldots = e(i_1) = \aleph_0$. We claim that $e(i) = \aleph_0$ for all i. If not, there is a least i_2 with $e(i_2) = 0$. Obviously $i_2 > i_1 > 0$, so $i_2 - 1 \geqslant i_1 > 0$. By assumption, $e(i_2-1) = \aleph_0$. Now $g(u)$ $= \big(g(u)\big)^2 = \sum_{i,j} e(i)e(j)\binom{u}{i}\binom{u}{j}$. When expanded as a combinatorial series, there is a summand $\big(e(i)\big)^2\binom{u}{i}^2$. But $\binom{u}{i}^2 = \sum_{t=i}^{2i} v(t)\binom{u}{t}$ with $v(i) \neq 0, \ldots,$ $v(2i) \neq 0$. Thus for $i = i_2 - 1$ and $t = i_2$ we get a term $\big(e(i_2-1)\big)^2\binom{u}{i_2}v(i_2)$ $= \aleph_0\binom{u}{i_2}$. Therefore $e(i_2) = \aleph_0$, a contradiction. Thus (ii) holds.

Conversely, suppose that (ii) holds. Then $f_\Omega(u+k) = \sum \aleph_0 \binom{u}{i}$. But $\left(\sum \aleph_0 \binom{u}{i}\right)^2 = \sum \aleph_0 \binom{u}{i}$, so $\big(f(u+k)\big)^2 = f(u+k)$. Apply Proposition I, get $f_\Omega(x+k) = \big(f_\Omega(x+k)\big)^2$ for all $x \in \Omega$. Thus (i) holds.

THEOREM 5.4. *The following conditions are equivalent.*

(i) $\big(f_\Omega(x)\big)^n = 2\big(f_\Omega(x)\big)^n$ *for almost all* $x \in \Omega$.

(ii) *There exists a k such that* $\big(f(u+k)\big)^n = \sum e(i)\binom{u}{i}$ *with each $e(i)$ having value 0 or $\aleph_0$.*

(iii) *There exists a k such that for all i* (iii)(a) *below implies* (iii)(b) *below.*

(iii) (a) *There exists a $c(j) \neq 0$ such that $j - k \leqslant i \leqslant nj$.*

(iii) (b) *There exist $c(j_1) \neq 0, \ldots, c(j_n) \neq 0$, one of $c(j_1), \ldots, c(j_n)$ being $\aleph_0$, and an i' with $i \leqslant i' \leqslant i+k$, such that* $\max(j_1, \ldots, j_n) \leqslant i'$ $\leqslant j_1 + \ldots + j_n$.

Proof. Propositions I, II, and Theorem 5.2 show that (i) and (ii) are equivalent. The equivalence of (ii) and (iii) will follow from a close examination of $\left(\sum c(i)\binom{u}{i}\right)^n = \sum_{i_1 \ldots i_n} c(i_1) \ldots c(i_n)\binom{u}{i_1} \ldots \binom{u}{i_n}$. Observe that $\binom{u}{i_1} \ldots \binom{u}{i_n} = \sum c(i, i_1, \ldots, i_n)\binom{u}{i}$, where the latter summation extends over all i satisfying $\max(i_1, \ldots, i_n) \leqslant i \leqslant i_1 + \ldots + i_n$ and for each such i, $c(i, i_1, \ldots, i_n) > 0$. Thus

$$\left(\sum_i c(i)\binom{u}{i}\right)^n = \sum_i \left(\sum\nolimits^* c(i, i_1, \ldots, i_n) c(i_1) \ldots c(i_n)\binom{u}{i}\right)$$

where the asterisk * indicates summation over all $(i_1, \ldots, i_n)$ such that $\max(i_1, \ldots, i_n) \leqslant i \leqslant i_1 + \ldots + i_n$. From this we derive

(5.1) $\sum^* c(i, i_1, \ldots, i_n) c(i_1) \ldots c(i_n) > 0$ *if and only if there exists a $c(j) \neq 0$ such that $j \leqslant i \leqslant jn$.*

Proof of (5.1). Suppose that such a j exists. Then $\max(j, \ldots, j)$ $\leqslant i \leqslant jn = j + \ldots + j$ and $c_j \neq 0$, so

$$\sum\nolimits^* c(i, i_1, \ldots, i_n) c(i_1) \ldots c(i_n) \geqslant c(i, j, \ldots, j)\big(c(j)\big)^n > 0\,.$$

Conversely, if $\sum^* c(i, i_1, \ldots, i_n)\, c(i_1) \ldots c(i_n) > 0$, then for some $i_1, \ldots, i_n$ with $\max(i_1, \ldots, i_n) \leqslant i \leqslant i_1 + \ldots + i_n$, we have $c(i_1) > 0, \ldots, c(i_n) > 0$. If $j = \max(i_1, \ldots, i_n)$, then $j \leqslant i \leqslant nj$ and $c(j) > 0$.

Thus (5.1) and the expansion of $\binom{u+k}{i}$ yield

(5.2) *Let $\big(f(u+k)\big)^n = \sum e(i)\binom{u}{i}$. Then $e(i) > 0$ if and only if there exists a j with $c(j) > 0$ and $j - k \leqslant i \leqslant jn$.*

A similar argument shows that

(5.3) $\sum^* c(i, i_1, \ldots, i_n) c(i_1) \ldots c(i_n) = \aleph_0$ *if and only if there exist* $c(j_1) \neq 0, \ldots, c(j_n) \neq 0$ *such that some one of* $c(j_1), \ldots, c(j_n)$ *is* $\aleph_0$ *and* $\max(j_1, \ldots, j_n) \leqslant i \leqslant j_1 + \ldots + j_n$.

Combine this with the expansion of $\binom{u+k}{i}$ to get

(5.4) *If* $f(u+k) = \sum e(i)\binom{u}{i}$, *then* $e(i) = \aleph_0$ *if and only if there exist* $c(j_1) \neq 0, \ldots, c(j_n) \neq 0$ *with some one of* $c(j_1), \ldots, c(j_n)$ *being* $\aleph_0$, *such that* $\max(j_1, \ldots, j_n) \leqslant i' \leqslant j_1 + \ldots + j_n$ *for some* i' *such that* $i \leqslant i' \leqslant i+k$.

Then (5.2) and (5.4) show that (ii) and (iii) are equivalent.

THEOREM 5.5. *The following conditions are equivalent for non-constant* f.

(i) $\big(f_\Omega(x)\big)^n = \big(f_\Omega(x)\big)^{n+1}$ *for almost all* $x \in \Omega$.

(ii) *There exists a* k *such that* $\big(f(u+k)\big)^n \sum_{i=0}^{\infty} = \aleph_0 \binom{u}{i}$.

(iii) *There exists a* k *such that for all* i *there exist* $c(j_1) \neq 0, \ldots, c(j_n) \neq 0$ (*with one of* $c(j_1), \ldots, c(j_n) = \aleph_0$) *such that* $\max(j_1, \ldots, j_n) \leqslant i' \leqslant j_1 + \ldots + j_n$ *for some* i' *with* $i \leqslant i' \leqslant i+k$.

Proof. Suppose that (ii) holds. Then f non-constant implies $f(u+k)$ non constant. Thus

$$f(u+k)\Big(\sum_{i=0}^{\infty} \aleph_0 \binom{u}{i}\Big) = \sum_{i=0}^{\infty} \aleph_0 \binom{u}{i} = \big(f(u+k)\big)^{n+1}.$$

By Proposition I, $\big(f_\Omega(x+k)\big)^n = \big(f_\Omega(x+k)\big)^{n+1}$ for all $x \in \Omega$, i.e., (i) follows. Suppose that (i) holds. Then

$$\big(f_\Omega(x)\big)^n = \big(f_\Omega(x)\big)^{2n}$$

for almost all $x \in \Omega$. Apply Proposition I and Theorem 5.3 to obtain a k such that $\big(f(u+k)\big)^n = \sum_{i=0}^{\infty} \aleph_0 \binom{u}{i}$. This is (ii). To see that (ii) is equivalent to (iii) apply (ii) and (iii) of Theorem 5.4.

COROLLARY. *Suppose that all* $c(i)$ *are zero or* $\aleph_0$. *Then the following are equivalent.*

(i) $\big(f_\Omega(x)\big)^n = \big(f_\Omega(x)\big)^{n+1}$ *for almost all* $x \in \Omega$.

(ii) *There exists a* k *such that for all* i *there exists a* j, $j-k \leqslant i \leqslant nj$, *such that* $c(j) = \aleph_0$.

EXAMPLE. Idemmultiplicity ∞, idempotence ∞. Let $f(u) = u + \aleph_0$. Certainly for no $n \geqslant 1$ and $k \geqslant 0$ is $\big(f(u+k)\big)^n = \sum_{i=0}^{\infty} \aleph_0 \binom{u}{i}$; also for no $m \geqslant 1$ and $k \geqslant 0$ is $\big(f(u+k)\big)^m = \sum e(i)\binom{u}{i}$ with all $e(i)$ equal to $\aleph_0$ or 0. Therefore Theorems 4.2, 5.4, 5.5 show that there are c isols x for which $\big(f_\Omega(x)\big)^n \neq \big(f_\Omega(x)\big)^{n+1}$ for all $n \geqslant 1$; and simultaneously $f_\Omega(x)^m \neq 2\big(f_\Omega(x)\big)^m$ for all $m \geqslant 1$.

EXAMPLE. Idemmultiplicity 1, idempotence $1 < n < \infty$. Let $f(u) = \sum_{i=0}^{\infty} \aleph_0 \binom{u}{n^i}$. It is easily seen (*via* Theorem (5.5) (ii), (iii) or directly) that $\big(f(u)\big)^n = \sum_{i=0}^{\infty} \aleph_0 \binom{u}{i}$; and hence that $\big(f_\Omega(x)\big)^n = \big(f_\Omega(x)\big)^{n+1}$ for all $x \in \Omega$. Obviously since f has all coefficients $\aleph_0$ or 0, $f_\Omega(x) = 2f_\Omega(x)$ for all $x \in \Omega$. Note that $\big(f(u)\big)^{n-1} = \sum v(i)\binom{u}{i} = \sum_i \Big(\sum_{n^i \leqslant j \leqslant (n-1)n^i} \aleph_0 \binom{u}{j}\Big)$. Therefore $v\big((n-1)n^i\big) = \aleph_0$ and $v(j) = 0$ for $(n-1)n^i < j < n^{i+1}$. Since $n^{i+1} - (n-1)n^i = n^i$ and $n^i \to \infty$ as $i \to \infty$, condition (ii) in the corollary to Theorem 5.5 is not satisfied. Therefore Theorem 4.2 assures that there are c isols x for which $f_\Omega^{n-1}(x) \neq f_\Omega(x)$.

EXAMPLE. Idemmultiplicity m, idempotence n, with $1 \leqslant m \leqslant n < \infty$. The argument above shows that it suffices to produce a recursive combinatorial series $f(u)$ such that: (i) $\big(f(u+m)\big)^n = \sum_{i=0}^{\infty} \binom{u}{i} \aleph_0$. (ii) $\big(f(u+m)\big)^m = \sum e(i)\binom{u}{i}$ with each $e(i)$ either $\aleph_0$ or 0. (iii) If $m > 1$ and $\big(f(u)\big)^{m-1} = \sum p(i)\binom{u}{i}$, then for all k there exists an i with $0 < p(i) < \aleph_0$ and $p(i+1) < \aleph_0, \ldots, p(i+k) < \aleph_0$. (iv) If $n > 1$ and $\big(f(u)\big)^{n-1} = \sum q(i)\binom{u}{i}$, then for all k there exists an i with $q(i) \neq \aleph_0 \ldots = q(i+k) \neq \aleph_0$. Such an $f(u)$ is

$$\sum_{i=0}^{\infty} \Big[\aleph_0 \binom{u}{a_{3i}} + \binom{u}{a_{3i+1}} + \aleph_0 \binom{u}{a_{3i+2}}\Big],$$

where the a_i are defined recursively as $a_0 = 0$, $a_{3i+1} = a_{3i} + 1$, $a_{3i+2} = m a_{3i+1}$, $a_{3i+3} = n a_{3i+2}$. To see this, first observe that the monotonicity of the a_n and the location of $\aleph_0$'s in f insure that for $z \geqslant 1$,

$$\big(f(u)\big)^z = \sum_{i=0}^{\infty} \Big(\aleph_0 \binom{u}{a_{3i}} + \binom{u}{a_{3i+1}} + \aleph_0 \binom{u}{a_{3i+2}}\Big)^z.$$

Observe that $\big(f(u+m)\big)^n = \sum_{i=0}^{\infty} \Big(\sum_{a_{3i} \leqslant j \leqslant n a_{3i+2}} \aleph_0 \binom{u}{j}\Big)$ and $n a_{3i+2} = a_{3i+3}$; therefore (i) holds. Observe that $\big(f(u+m)\big)^m = \sum_{i=0}^{\infty} \Big(\sum_{a_{3i} - m \leqslant j \leqslant m a_{3i+2}} \aleph_0 \binom{u}{j}\Big)$; therefore (ii) holds.

Observe that

$$(f(u))^{m-1} = \sum_{i=0}^{\infty}\Big(\sum_{a_{3i}\leqslant j\leqslant (m-1)a_{3i+1}-1}\aleph_0\binom{u}{j} + \sum_{(m-1)a_{3i}<j\leqslant(m-1)a_{3i+1}} p(j)\binom{u}{j} + \\ + \sum_{a_{3i+2}\leqslant j\leqslant (m-1)a_{3i+2}}\aleph_0\binom{u}{j}\Big).$$

Then $\aleph_0 > p\big((m-1)a_{3i+1}\big) > 0$, while for $(m-1)a_{3i+1} < j < a_{3i+2}$, $p(j) = 0$. But $a_{3i+2}-(m-1)a_{3i+1} = a_{3i+1}$, and $a_{3i+1}\to\infty$ as $i\to\infty$. Thus (iii) holds. Finally, observe that $f^{n-1}(u)$ is

$$\sum_{i=0}^{\infty}\Big[\Big(\sum_{a_{3i}\leqslant j\leqslant (n-1)a_{3i+2}-1} q(j)\binom{u}{j}\Big)+\aleph_0\binom{u}{(n-1)a_{3i+2}}\Big].$$

Then $q\big((n-1)(a_{3i+2})\big) = \aleph_0$, while for $(n-1)a_{3i+2} < j < a_{3i+3}$, $q(j) = 0$. But $a_{3i+3}-(n-1)a_{3i+2} = a_{3i+2}$, and $a_{3i+2}\to\infty$ as $i\to\infty$. Thus (iv) holds.

EXAMPLE. Idemmultiplicity $m < \infty$, idempotence ∞. Examine

$$f(u) = \sum_{i=0}^{\infty}\Big(\aleph_0\binom{u}{2^{2^i}}+\binom{u}{2^{2^i}+1}+\aleph_0\binom{u}{m(2^{2^i})+1}\Big)$$

by the above method.

6. Applications to isolic integers. The only result of preceding sections that is relevant is Proposition I for series with finite coefficients. This special case can be proved very simply by the method used for recursive combinatorial functions in [12], thus avoiding Lemma 2.2. The gain over [12], [13] arises solely from using $R\uparrow$ coefficient functions rather than just recursive coefficient functions. Notation and terminology are from [12], [13]. Several elementary lemmas are required. The first two are well known.

LEMMA 6.1. *If $f\colon E\to E$ is $\exists\forall$ (relative to recursive predicates), then f is $\forall\exists$.*

Proof. If f is $\exists\forall$, then there exists a recursive predicate $R(u, v, x, y)$ such that $f(x) = y$ if and only if $\exists u\forall vR(u, v, x, y)$. Let $j\colon E\times E\to E$ be a 1-1 onto general recursive function, let $k\colon E\to E$, $l\colon E\to E$ be such that $j\big(k(a), l(a)\big) = a$ for all $a\in E$. Define $S(u, v, x, y)$ by $S(u, v, x, y)\leftrightarrow R\big(k(u), v, x, l(u)\big)\wedge l(u)\neq y$. Then $f(x) = y\leftrightarrow\forall u\exists v\neg S(u, v, x, y)$.

LEMMA 6.2. *$f\colon E\to E$ is $\exists\forall$ if and only if there exists a recursive sequence of recursive functions $f_n\colon E\to E$ such that $f = \lim_n f_n$.*

Proof. Suppose that f is such a limit. Then define $R(u, v, x, y)$ by $R(u, v, x, y)\leftrightarrow\big(v\geqslant u\to f(v, x) = y\big)$. Then $f(x) = y\leftrightarrow\exists u\forall vR(u, v, x, y)$.

Conversely, suppose that f is $\exists\forall$. Choose R, S as in the proof of Lemma 6.1. Define $u_n(x, y)$ as the least u such that $(\forall v\leqslant n)R(x, y, u, v)\vee\vee(\forall v\leqslant n)S(x, y, u, v)$. Define $f_n(x)$ by requiring that $f_n(x)$ is the least $y\leqslant n$ such that $(\forall v\leqslant n)R\big(x, y, u_n(x, y), v\big)$, or is 0 if no such y exists. We sketch a proof that $\lim f_n = f$. Define $u(x, y)$ as the least u satisfying $\forall vR(x, y, u, v)\vee\forall vS(x, y, u, v)$. Suppose $f(x_0) = y_0$. For $0\leqslant y < y_0$ define $v(x_0, y)$ as the least v such that $\neg R\big(x_0, y, u(x_0, y), v\big)$. For $0\leqslant u < u(x, y)$ define $w(x, y, u)$ as the least w such that $\neg R\big(x, y, u, (w)_0\big)\vee\vee\neg S\big(x, y, u, (w)_1\big)$. Suppose that n is larger than y_0, and is also larger than each $v(x_0, y)$ for $y < y_0$; and is also larger than each $w(x_0, y, u)$ for $y\leqslant y_0$ and $u < u(x_0, y)$. Then an easy argument shows $f_n(x_0) = y_0$.

LEMMA 6.3. *Let $F\colon \times^k E^*\to E^*$ be the limit of a recursive sequence of recursive functions. Then there exist combinatorial functions*

$$f(x_1, \ldots, x_{2k}) = \sum c(i_1, \ldots, i_{2k})\binom{x_1}{i_1}\cdots\binom{x_{2k}}{i_{2k}},$$

$$g(x_1, \ldots, x_{2k}) = \sum d(i_1, \ldots, i_{2k})\binom{x_1}{i_1}\cdots\binom{x_{2k}}{i_{2k}}$$

such that c, d are $R\uparrow$ and $F(x_1-x_2, \ldots, x_{2k-1}-x_{2k}) = f(x_1, \ldots, x_{2k})-g(x_1, \ldots, x_{2k})$ identically.

Proof. Consider $k = 1$ for convenience. Let $F_n(x)$ be a recursive sequence of recursive functions with limit F. We construct two sequences of recursive combinatorial functions

$$f_n = \sum c_n(i, j)\binom{x_1}{i}\binom{x_2}{j},\qquad g_n = \sum d_n(i, j)\binom{x_1}{i}\binom{x_2}{j}$$

such that

(1) $F_n(x_1-x_2) = f_n(x_1, x_2)-g_n(x_1, x_2)$ identically,

(2) $c_n(i, j)\leqslant c_{n+1}(i, j)$, $d_n(i, j)\leqslant d_{n+1}(i, j)$ for all i, j, n.

(3) $c_n(i, j)$ converges to a $c(i, j)\in E$, $d_n(i, j)$ converges to a $d(i, j)\in E$ for all i, j.

(4) $F(x_1-x_2) = f(x_1, x_2)-g(x_1, x_2)$ identically, where $f = \sum c(i, j)\binom{x_1}{i}\binom{x_2}{j}$ and $g = \sum d(i, j)\binom{x_1}{i}\binom{x_2}{j}$. This will conclude the proof provided that $c_n(i, j)$ and $d_n(i, j)$ are recursive functions of i, j, n. For $x = 0, \pm 1, \pm 2, \ldots$ enumerate $S(x) = [(y_1, y_2)\in E\times E \mid x = y_1-y_2]$ effectively without repetitions uniformly in x. Partially order $E\times E$ by defining $(x_1, x_2)\leqslant(x_2', x_2')$ if $x_1\leqslant x_1'$ and $x_2\leqslant x_2'$. The construction is as follows.

Suppose that $f_0(i, j)$ and $g_0(i, j)$ have been defined for $(i, j) < (x_1, x_2)$. Then $c_0(i, j)$, $d_0(i, j)$ have been defined for $(i, j) < (x_1, x_2)$.

Let (y_1, y_2) be the first element in the enumeration of $S(F_0(x_1-x_2))$ such that

$$y_1 \geqslant \sum_{(i,j)<(x_1,x_2)} c_0(i,j)\binom{x_1}{i}\binom{x_2}{j}$$

and also

$$y_2 \geqslant \sum_{(i,j)<(x_1,x_2)} d_0(i,j)\binom{x_1}{i}\binom{x_2}{j}.$$

Then define $f_0(x_1, x_2) = y_1$ and $g_0(x_1, x_2) = y_2$.

Now suppose that $n > 0$ and that $f_0, \ldots, f_{n-1}, g_0, \ldots, g_{n-1}$ have been defined. Suppose further that $f_n(i,j)$ and $g_n(i,j)$ have been defined for $(i,j) < (x_1, x_2)$. Then $c_n(i,j)$, $d_n(i,j)$ have been defined for $(i,j) < (x_1, x_2)$. Also certainly $c_{n-1}(x_1, x_2)$ and $d_{n-1}(x_1, x_2)$ also have been defined. Let (y_1, y_2) be the first member of $S(F_n(x_1-x_2))$ in the order of enumeration such that

$$y_1 \geqslant \sum_{(i,j)<(x_1,x_2)} c_n(i,j)\binom{x_1}{i}\binom{x_2}{j} \quad \text{and} \quad y_2 \geqslant \sum_{(i,j)<(x_1,x_2)} d_n(i,j)\binom{x_1}{i}\binom{x_2}{j}$$

and also

$$c_n(x_1, x_2) \geqslant c_{n-1}(x_1, x_2), \qquad d_n(x_1, x_2) \geqslant d_{n-1}(x_1, x_2).$$

Define

$$f_n(x_1, x_2) = y_1 \quad \text{and} \quad g_n(x_1, x_2) = y_2.$$

Requirements (1) and (2) have been established by construction. Requirements (3) and (4) remain to be established by induction on (x_1, x_2) in the partially ordered set $E\times E$.

Suppose $(x_1, x_2) = (0,0)$. Since $F_n(0) \to F(0)$, there is an n_0 such that $F_n(0) = F(0)$ for $n \geqslant n_0$. A look at definitions shows that certainly for $n \geqslant n_0$, $f_n(0,0) = f_{n_0}(0,0)$ and $g_n(0,0) = g_{n_0}(0,0)$. Then $c(0,0) = f(0,0) = f_{n_0}(0,0)$ and $d(0,0) = d_{n_0}(0,0) = g_{n_0}(0,0)$ have the desired property since $F(0) = F_{n_0}(0) = f_{n_0}(0,0) - g_{n_0}(0,0) = f(0,0) - g(0,0)$. Now suppose $(x_1, x_2) > (0,0)$. Take as inductive hypothesis that for $(i,j) < (x_1, x_2)$ it is known that $c_n(i,j) \to c(i,j)$, $d_n(i,j) \to d(i,j)$, and $F(i-j) = f(i,j) - g(i,j)$. Choose n_0 so large that for $n \geqslant n_0$ we have $c_n(i,j) = c(i,j)$ and $d_n(i,j) = d(i,j)$ for all $(i,j) < (x_1, x_2)$; and have also $F_n(i-j) = F(i-j)$ for all $(i,j) \leqslant (x_1, x_2)$. Then for $n \geqslant n_0$, $(i,j) < (x_1, x_2)$, we have $F_n(i-j) = F_{n_0}(i-j)$ and $c_n(i,j) = c_{n_0}(i,j)$, $d_n(i,j) = d_{n_0}(i,j)$. A look at the definition of $f_n(x_1, x_2)$ and $g_n(x_1, x_2)$ shows that it follows that $f_n(x_1, x_2) = f_{n_0}(x_1, x_2)$ and $g_n(x_1, x_2) = g_{n_0}(x_1, x_2)$ for $n \geqslant n_0$. Thus if $f(x_1, x_2) = f_{n_0}(x_1, x_2)$ and $g(x_1, x_2) = g_{n_0}(x_1, x_2)$, the requirements are satisfied since $F(x_1-x_2) = F_{n_0}(x_1-x_2) = f_{n_0}(x_1, x_2) - g_{n_0}(x_1, x_2) = f(x_1, x_2) - g(x_1, x_2)$.

Let φ be a statement which is a quantified conjunction of atomic formulas involving relation symbols, function symbols, and the identity

symbol. Suppose that with E^* as a domain of individuals each relation symbol denotes a recursive relation and each function symbol denotes a recursive function. Then with Λ^* as domain of individuals let relation and function symbols denote respective extension to Λ^* in the sense of [13].

THEOREM 6.4. *Let φ satisfy the restrictions above. Then φ is true in Λ^* if and only if φ is true in E^* and has in E^* Skolem functions in both two-number quantifier forms in the Kleene-Mostowski hierarchy.*

Proof. One direction is Lemma 6.1 combined with the fundamental lemma of [13]. For the other, suppose that φ is true in E^* with such Skolem functions. An application of results in [13] shows that consideration may be limited to a φ which is a quantification of an atomic formula $F_1(v_1, \ldots, v_n) = F_2(v_1, \ldots, v_n)$, where F_1, F_2 denote respectively recursive functions $f_1\colon X^nE^* \to X^nE^*$, $f_2\colon E^* \to E^*$. For notational convenience, limit consideration to a typical special case. Let φ be

$$\forall v_1 \exists v_2 \forall v_3 \exists v_4 [F_1(v_1, v_2, v_3, v_4) = F_2(v_1, v_2, v_3, v_4)].$$

Let $h(v_1)$, $k(v_1, v_3)$ be the (both two number quantifier form) Skolem functions in E^*. Then for $x_1, x_3 \in E^*$,

$$f_1(x_1, h(x_1), x_3, k(x_1, x_3)) = f_2(x_1, h(x_1), x_3, h(x_1, x_3)). \tag{6.1}$$

Choose recursive combinatorial functions $p_i, q_i\colon \times^8 E \to E$ such that for $x_1, \ldots, x_8 \in E$,

$$f_i(x_1-x_2, x_3-x_4, x_5-x_6, x_7-x_8) = p_i(x_1, \ldots, x_8) - q_i(x_1, \ldots, x_8), \qquad i = 1, 2.$$

Under a 1-1 effective correspondence between E and E^*, Lemma 6.2 yields a corresponding version for E^*. Use Lemma 6.2 and Lemma 6.3 to choose combinatorial functions $r_1, s_1\colon \times^2 E \to E$ and $r_2, s_2\colon \times^4 E \to E$ which have R↑ combinatorial series, and are such that for $x_1, x_2, x_3, x_4 \in E$,

$$h(x_1-x_2) = r_1(x_1, x_2) - s_1(x_1, x_2),$$
$$k(x_1-x_2, x_3-x_4) = r_2(x_1, x_2, x_3, x_4) - s_2(x_1, x_2, x_3, x_4).$$

Let (X) abbreviate the following expression:

$$(x_1, x_2, r_1(x_1, x_2), s_1(x_1, x_2), x_5, x_6, r_2(x_1, x_2, x_5, x_6), s_2(x_1, x_2, x_5, x_6)).$$

Then (6.1) yields that for all $x_1, x_2, x_5, x_6 \in E$,

$$p_1(X) + q_2(X) = p_2(X) + q_1(X).$$

Since all functions involved have R↑ combinatorial series, Proposition I applies to show that the corresponding identity is true in Λ. But it follows easily from [13] that the corresponding identity in Λ can be rephrased as asserting that for x_1, x_2, x_5, x_6 in Λ, $f_{1\Lambda^*}(x_1-x_2, r_{1\Omega}(x_1, x_2)- -s_{1\Omega}(x_1, x_2), x_5-x_6, r_{2\Omega}(x_1, x_2, x_5, x_6)-s_{2\Omega}(x_1, x_2, x_5, x_6)) = f_{2\Lambda^*}$ (same).

But this shows that φ is true in Λ^* since x_1-x_2 and x_5-x_6 range over Λ^*. (However, it has not been shown that the element $r_{1\Omega}(x_1, x_2)- -s_{2\Omega}(x_1, x_2)$ of Λ^* depends only on x_1-x_2.)

If the quantifier prefix in φ is of one of the forms $\forall$, $\exists$, $\forall\exists$, $\exists\forall$, $\forall\exists\forall$, or $\exists\forall\exists\forall$, there are always Skolem functions in both number quantifier forms. In paper [13] it was observed that for the $\forall\exists\forall\exists\forall$ prefix (and hence for all higher prefixes) there need not be Skolem functions in both two-number quantifier forms. This leaves out the case $\forall\exists\forall\exists$ which can be disposed of easily as follows. It will follow that there is an $\forall\exists\forall\exists$ prefix statement φ without Skolem functions in $\exists\forall$ form if there exists an $\forall\exists$ function $f\colon E^*\to E^*$ which is not $\exists\forall$. The reason is simply that if R is a recursive relation such that $f(x) = y \leftrightarrow \forall u \exists v R(u, v, x, y)$, then the statement $\forall x \exists y \forall u \exists v R(u, v, x, y)$ is true in E^* and has f as its Skolem function for the outermost existential quantifier. Without difficulty, it suffices to produce an $f\colon E\to E$ which is $\forall\exists$ but not $\exists\forall$. Myhill has made the comment that if β is a retraceable set retraced by a general recursive retracing function and $E-\beta$ is recursively enumerable and not recursive, then the function g enumerating β in order of magnitude is $\forall$ but not $\exists$, i.e., $E\times E-f$ is recursively enumerable but not recursive. Such β are exhibited in [5].

Modify this remark as follows. Apply the construction of [5] to obtain a set α which is retraced by a fully defined $\exists\forall$ function such that $E-\alpha$ is an $\exists\forall$ set which is not an $\forall\exists$ set. Then the function f enumerating α in order of magnitude will do.

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