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**1. Summary and notation.** The theory of recursive equivalence types is an effective analogue of the theory of cardinal numbers. It originated with Dekker [1] and has been developed at least by Dekker, Myhill, and the author [2], [3], [4], [6], [7], [8], [9], [10], [11], [12], [13], [14]. Let \( E = \{0, 1, 2, \ldots \} \) be the natural numbers. If \( \alpha, \beta \subseteq E \), call \( \alpha \) **recursively equivalent** to \( \beta \) if there exists a 1-1 partial recursive function \( p \) whose domain contains \( \alpha \) and which is such that \( p(\alpha) = \beta \). The equivalence class \( [\alpha] \) of \( \alpha \) under recursive equivalence is called a **recursive equivalence type** (RET). The set of RET's is denoted by \( \mathcal{D} \). With each \( n \) in \( E \) is associated the RET of all \( n \)-element subsets of \( E \). If \( n \) is identified with this associated RET, then \( E \) becomes a subset of \( \mathcal{D} \). For \( \alpha, \beta \subseteq E \), define \( \alpha \oplus \beta = [2^\alpha] \oplus [2^\beta] = [2^{\alpha+\beta}] \) and \( \alpha \ominus \beta = [2^\alpha] \ominus [2^\beta] = [\pi^\alpha \ominus \beta] \). Define addition and multiplication in \( \mathcal{D} \) by \( [\alpha] + [\beta] = [\alpha \oplus \beta], [\alpha] \times [\beta] = [\alpha \ominus \beta] \). The set \( A \) of isols consists of all those \( x \in \mathcal{D} \) such that \( x+1 \neq x \). Then \( E \subseteq A \subseteq \mathcal{D} \). The arithmetic of \( A \) is fairly well understood ([2], [3], [4], [6], [7], [8], [12], [13]). A fundamental tool for the analysis of \( A \) is the notion of a recursive combinatorial function \( f: \times^*E \to E \) and its induced normal function \( f_A: \times^*D \to D \). This was introduced by Myhill [8] as follows. Any function \( f: \times^*E \to E \) has a uniquely determined expansion function \( (f_0, \ldots, f_n) \)

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Fundamenta Mathematicae, T. LVIII
Combinatorial series

The sum \( \sum c(i_1, \ldots, i_a) \left[ \begin{array}{c} i_1 \\ \vdots \\ i_a \end{array} \right] \) is a polynomial, where \( E' = (0, \pm 1, \pm 2, \ldots) \) is the ring of integers and \( \left[ \begin{array}{c} i_1 \\ \vdots \\ i_a \end{array} \right] \) is the number of \( i \)-element subsets of a ring \( A \) of integers. An \( f \) is a \( \times E' \) function if for all \( c(i_1, \ldots, i_a) \), \( f(i_1, \ldots, i_a) = 0 \).

Let \( \delta(E) \) be the class of all subsets of \( E \). Let \( \delta_m(E) \) be the class of all finite subsets of \( E \). Let \( f : \times \delta_m(E) \rightarrow E \) be a function.

For any \( a \in A \), \( f(a) = f(\delta_a) \) is a function defined on \( \delta_a \). When \( f \) is a \( \times E' \) function, \( f(\delta_a) \) is called a \( \times E' \) function if it is recursive and only if it is:

\[ f(i_1, \ldots, i_a) = \sum c(i_1, \ldots, i_a) \left[ \begin{array}{c} i_1 \\ \vdots \\ i_a \end{array} \right] \]

recursive. In the natural sense, a \( \times E' \) function \( f \) is recursive if \( f \) is \( \times E' \) and \( f \) is recursive in the usual sense.

A counterexample in §3 shows that such \( \times E' \) functions are not closed under composition. This is bypassed by considering a wider class of \( E' \) than \( \times E' \) as defined above. Call \( c : \times E' \rightarrow \times E' \) a \( R \uparrow \) function if it is a \( \times E' \) function. Then \( c \)  is called a \( \times E' \) function if it is recursive in the usual sense.

A \( \times E' \) function is a \( \times E' \) function if it is recursive in the usual sense.

Proposition 1. The \( R \uparrow \) combinatorial series are closed under composition.

\[ f \circ g \circ h = (f \circ g \circ h)(x) \]

Consequently any formal identity between \( R \uparrow \) series yields an identity between corresponding normal functions.

The second principal proposition concerns inequalities. It is used to prove the fact that \( R \uparrow \) series are unique, not as values for the coefficients, but as values for the \( i \)-th element of the \( i \)-th subset of \( E \).

Corresponding to each such \( f \) is a function \( f : \times \delta_m(E) \rightarrow E \) given by (1.1), (1.2), (1.3). The only change is to allow \( c(\langle \alpha \rangle) \) the value \( a \).

There are two principal propositions, of which all other results are corollaries. The first concerns identities. A basic fact about recursive combinatorial functions is that they are closed under composition and any identity between them true in \( E \) yields an identity in \( \Omega \) between corresponding normal functions. There are two obstacles to extending this result to combinatorial series. One obstacle is that combinatorial
The inequality theorem is as follows. Say that almost all \( x \in \Omega \) have property \( P \) if there exists an \( n \in \mathcal{E} \) such that \( P \) holds for all \( x \in \Omega \) with \( x = x' \).

**Proposition II** (§4). Let \( f(u) = \sum c(i)u^i \) and \( g(u) = \sum d(i)u^i \) be \( R \to \) combinatorial series. Then the following three conditions are equivalent.

1. \( f(a) \leq g(a) \) for almost all \( x \in \Omega \).
2. For some \( k \in \mathbb{E} \), there exists a recursive \( R \to \) series \( b(u) \) such that \( f(a + k) + b(u) = g(a + k) \).
3. For some \( h \in \mathbb{E} \), there exists a recursive \( R \to \) function \( c: E \to \mathbb{E} \) such that for all \( i \in \mathbb{E} \)

\[
equiv(i) = \equiv(i+1) + \ldots + \equiv(i+k) + \equiv(i)
= d(i) + d(i+1) + \ldots + d(i+k).
\]

Moreover, there are \( c \) isols \( x \) such that for any pair of \( R \to \) series \( f, g \) violating (iii), \( f(a) \neq g(a) \).

This proposition and (11.3) of [12] coincide when applied to recursive combinatorial functions of one variable. Proposition II can be generalized to functions of several variables. The main content is that algebraic properties of values of \( f \) are correlated with arithmetical properties of the coefficients \( c(i) \).

Proposition II can be applied (§4) to show that Proposition I fails with the weaker hypothesis that the series have coefficient functions \( c(i_1, \ldots, i_n) \) which are merely limits of recursive sequences of recursive functions rather than monotone limits.

Finally, Proposition I allows the resolution (§6) of some problems about the isole integers raised in [13]. Adopt the terminology of [13]. Suppose that \( \varphi \) is a quantified conjunction of atomic formulas. Suppose that \( \varphi \) is true in \( E^* \) and \( E^* \varphi \) has Skolem functions in both two number quantifier forms in the Kleene-Mostowski hierarchy. The main result is that then \( \varphi \) is true in \( A^* \).

2. Combinatorial series and operators. A generalization of combinatorial operators [8], [12] suitable for combinatorial series is needed. Construct \( \times^* \mathbb{E}(E) \) as a boolean algebra by defining for \( a, b \in \times^* \mathbb{E}(E) \), \( i = 1, \ldots, k \), \( (a \land b)_i = a_i \land b_i \), \( (a \lor b)_i = a_i \lor b_i \), \( (\neg a)_i = \neg a_i \).

A precombinatorial operator is a \( \varphi: \times^* \mathbb{E}(E) \to \mathcal{S}(E) \) such that (2.1) and (2.2) hold for \( a, b \in \times^* \mathbb{E}(E) \).

(2.1) \( \varphi \neq \beta \) implies \( \varphi(a) \land \varphi(\beta) = \emptyset \)

(where \( \emptyset \) denotes the null set).

(2.2) \( \langle a \rangle = \langle b \rangle \) implies that \( \varphi(a) \) and \( \varphi(\beta) \) have the same number of elements. (Note \( \varphi(a) \) may be infinite.)

A combinatorial operator is a \( \varphi: \times^* \mathbb{E}(E) \to \mathcal{S}(E) \) such that there exists a precombinatorial operator \( \psi: \times^* \mathbb{E}(E) \to \mathcal{S}(E) \) satisfying (2.3) below for all \( a \in \times^* \mathbb{E}(E) \).

(2.3) \( \varphi(a) \cup \varphi(n) = \mathcal{S}(E) \).

Then \( \varphi \) is uniquely determined by \( \psi \) since by (2.3) and (2.1) \( \varphi(\beta) = \psi(\beta) \cup \psi(\beta) \) for all \( \beta \in \times^* \mathbb{E}(E) \). It is thus legitimate to write \( \varphi \) as \( \varphi(\alpha) \) and we have

(2.4) \( \varphi(a) = \varphi(a) \cup \mathcal{S}(E) \).

For \( a \in \varphi(\times^* \mathbb{E}) \), define \( s_\varphi \) as the \( a \in \times^* \mathbb{E}(E) \) such that \( \varphi(a) \). By (2.2) and (2.1) each combinatorial operator \( \varphi: \times^* \mathbb{E}(E) \to \mathcal{S}(E) \) yields a function \( E \times \mathbb{E}(E) \land \{s_\varphi\} \) given for \( \varphi(a) \) by \( c \equiv \) cardinality of \( \varphi(a) \) whenever \( \langle a \rangle = \beta \). We call \( \sum_{c(\beta)} \sum_{c(\beta)} \sum_{c(\beta)} \sum_{c(\beta)} \) the combinatorial series \( s_\varphi \) induced by \( \varphi \). Note that by (1.1), (1.2), (1.3), every combinatorial series is induced by at least one combinatorial operator.

**Theorem 2.1.** Combinatorial operators are closed under composition. That is, if \( \varphi: \times^* \mathbb{E}(E) \to \mathcal{S}(E) \), \( \psi_1: \times^* \mathbb{E}(E) \to \mathcal{S}(E) \), \( \ldots, \psi_n: \times^* \mathbb{E}(E) \to \mathcal{S}(E) \) are combinatorial operators, then \( \psi = (\psi_1, \ldots, \psi_n): \times^* \mathbb{E}(E) \to \mathcal{S}(E) \) is also a combinatorial operator. (Here we put \( \psi = (\psi_1, \ldots, \psi_n) \).)

Further, suppose \( \psi: \times^* \mathbb{E}(E) \to \mathcal{S}(E) \), \( \psi_1: \times^* \mathbb{E}(E) \to \mathcal{S}(E) \), \( \ldots, \psi_n: \times^* \mathbb{E}(E) \to \mathcal{S}(E) \) are combinatorial operators such that \( \varphi \equiv \psi_1, \ldots, \psi_1 \equiv \psi_2, \ldots, \psi_2 \equiv \psi_3, \ldots, \psi_n \equiv \psi \). Then \( \varphi = (\psi_1, \ldots, \psi_n) \).

**Proof.** Define \( \varphi: \times^* \mathbb{E}(E) \to \mathcal{S}(E) \) by (2.5) \( \varphi(a) \) if and only if: (i) \( a \) exists; (ii) \( \varphi(a) \) exists for all \( \varphi(a) \); (iii) \( \alpha = \bigvee_{i=1}^{\varphi(a)} \varphi(a) \).

It will eventually be shown that \( \varphi \) is precombinatorial (Lemma 2.3), that \( \varphi \) yields a combinatorial operator which is \( \psi = (\varphi(a), \ldots, \varphi(n)) \) (Lemma 2.4), and that the coefficient function for \( \psi = (\varphi(a), \ldots, \varphi(n)) \) is determined by the coefficient functions for \( \varphi(a), \ldots, \varphi(n) \) (Lemma 2.4). This will prove the theorem.

For \( a \in \times^* \mathbb{E}(E) \), let \( I(a) \) consist of all \( \beta \in \times^* \mathbb{E}(E) \) with \( \beta \leq a \). Call a 2m-tuple \( (E_1, \ldots, E_n, t_1, \ldots, t_n) \) admissible for \( a \) if

(2.6) \( (i) F_1, \ldots, F_n \subseteq T(a); \quad (ii) f_1 \) is a function with domain \( F_1; \quad (iii) \beta \in F_1, \emptyset \neq f_1(\beta) \subseteq \mathcal{S}(E) \); (iv) \( a = \bigvee_{i=1, \ldots, n} f_1(\beta) \).
Lemma 2.2. \( x \in \theta(a) \) if and only if there exists an \((F_1, ..., F_n, f_1, ..., f_n)\) admissible for \( a \) such that
\[ x \in \nu(\bigcup_{\beta \in \theta(a)} f_1(\beta), ..., \bigcup_{\beta \in \theta(a)} f_n(\beta)). \]

Proof. Suppose \( x \in \theta(a) \). Define \( F_i \) to be the set of \( u \in \nu(a) \) such that \( u \in \nu(a_i), i = 1, ..., n \). For \( \beta \in \theta(a) \), define \( f_\beta(\beta) = \nu(a_\beta) \cap \nu(\beta) \). We prove that \((F_1, ..., F_n, f_1, ..., f_n)\) is admissible for \( a \). Note that (2.6) (ii) is clear, while (2.5) (iii) implies (2.6) (i) (iv). As for (2.6) (iii), suppose \( \beta \in F_i \). By definition \( \beta = u \) for some \( u \in \nu(a_i) \), so \( x \in \nu(\beta) \). Thus \( x \in \nu(\bigcup_{\beta \in \theta(a)} f_1(\beta), ..., \bigcup_{\beta \in \theta(a)} f_n(\beta)) \). As above, for \( u \in \nu(a) \) we have \( u \in f_\beta(\beta) \)
where \( \beta = u \). Thus \( \nu(a) \subset \bigcup_{\beta \in \theta(a)} f_\beta(\beta) \). The converse inclusion is evident, so \( x \in \nu(\bigcup_{\beta \in \theta(a)} f_1(\beta), ..., \bigcup_{\beta \in \theta(a)} f_n(\beta)) \), and the desired conclusion follows.

Conversely, let \((F_1, ..., F_n, f_1, ..., f_n)\) be admissible for \( a \) with \( x \in \nu(\bigcup_{\beta \in \theta(a)} f_1(\beta), ..., \bigcup_{\beta \in \theta(a)} f_n(\beta)) \). Then \( u \in \nu(a) \) exists by definition, verifying (2.5) (i).

Certainly also \( u \in \bigcup_{\beta \in \theta(a)} f_\beta(\beta) \). Thus \( \nu(a) \subset \bigcup_{\beta \in \theta(a)} f_\beta(\beta) \). If \( u \in \nu(a) \), this implies that \( u \in f_\beta(\beta) \) for a \( \beta \in F_i \). By (2.6) (ii), \( f_\beta(\beta) \subset \nu(\beta) \), so \( u \in \nu(\beta) \). Thus (2.5) (ii) holds. Further, by (2.6) (iv), \( \beta \leq a \), so \( u \in \nu(\beta) \). By (2.6) (iii), \( f_\beta(\beta) \) is non-empty. Let \( u \in f_\beta(\beta) \). By (2.6) (iii), \( u \in \nu(\beta) \), so \( x \in \nu(a) \).

Lemma 2.3. \( \theta \) is a precombinatorial operator.

Proof. To verify (2.1), simply note that \( x \in \theta(a) \) implies that \( a \) is determined by \( x \), since \( a = \bigcup_{\beta \in \theta(a)} u \). Thus \( x \in \theta(a) \) implies \( a = \beta \). To verify (2.2), suppose that \( a, a' \in \nu(a) \), and \( (x, y) = \langle a', a \rangle \). Then there exists \( 1 \leq i \leq n \) such that \( a_i = a_i \) for \( i = 1, ..., n \). Let \( y, f_{\beta}(\beta) \) be the 1-1 map such that for \( \beta \in I(a), y(\beta) = (y(\beta), y(\beta)) \). For \( j = 1, ..., n \), define a 1-1 map \( y_i : f_{\beta}(\beta) \to \bigcup_{\beta \in \theta(a)} f_{\beta}(\beta) \) such that \( \beta' = \beta \). Then \( \nu(a) \subseteq \nu(a) \). (Such maps \( y_i \) exist since \( \beta' = \beta \) and \( \nu(a) \) satisfies (2.1) and 2.2.)

There is a 1-1 correspondence between admissible 2n-tuples for \( a \) and admissible 2n-tuples for \( a' \) given by

\[ (F_1, ..., F_n, f_1, ..., f_n) \leftrightarrow (F_1, ..., F_n, a', f_1, ..., f_n), \]

where \( F_i = \{ p \in F_i | \beta \in \theta(a) \} \) and \( f_{\beta}(\beta) = g_f(\beta) \) for \( \beta \in F_i \) and \( \beta' = p \). Then since \( g_\beta \) is 1-1 onto \( f_\beta(\beta) \), \( f_{\beta}(\beta) \) has the same number of elements as \( f_{\beta} \beta \). Due to (2.2), for fixed \( i \) and distinct \( f_{\beta} \beta, f_{\beta} \beta \) we have that \( f_{\beta} \beta \) is disjoint from \( f_{\beta} \beta \). Thus \( f_{\beta}(\beta) \) has the same number of elements as \( \bigcup_{\beta \in \theta(a)} f_{\beta}(\beta) \). Therefore (2.2) for \( \nu(a) \) yields that \( \nu(a) \bigcup_{\beta \in \theta(a)} f_{\beta}(\beta) \) and \( \bigcup_{\beta \in \theta(a)} f_{\beta}(\beta) \) have the same number of elements. Further, if \((F_1, ..., F_n, f_1, ..., f_n) \) and \((G_1, ..., G_n, g_1, ..., g_n) \) are admissible for \( a \), we claim that \( \nu(a) \bigcup_{\beta \in \theta(a)} f_{\beta}(\beta) \) and \( \nu(a) \bigcup_{\beta \in \theta(a)} g_{\beta}(\beta) \) are disjoint. Otherwise, there is an \( x \) in both these sets. Then \( \nu(a) \bigcup_{\beta \in \theta(a)} f_{\beta}(\beta) \). Since \( f_{\beta}(\beta) \leq \nu(\beta) \) and (2.3) holds for \( \nu(a) \), it follows that \( f_{\beta}(\beta) \). Then \( F_i \bigcup_{\beta \in \theta(a)} f_{\beta}(\beta) \). Since the same argument applies with \( G_i \) replacing \( F_i \) and \( G_i \) replacing \( F_i \), we conclude that

\[ (F_1, ..., F_n, f_1, ..., f_n) = (G_1, ..., G_n, g_1, ..., g_n). \]

Taking a union over all 2n-tuples admissible for \( a \) in one case and for \( a' \) in the other shows that

\[ \theta(a) = \bigcup_{\beta \in \theta(a)} f_{\beta}(\beta), ..., \bigcup_{\beta \in \theta(a)} f_n(\beta) \]

and

\[ \theta(a') = \bigcup_{\beta \in \theta(a')} f_{\beta}(\beta), ..., \bigcup_{\beta \in \theta(a')} f_n(\beta) \]

have the same number of elements.

Lemma 2.4. The combinatorial operator induced by the precombinatorial operator \( \theta \) is \( \varphi = \varphi(a) \). Moreover, the coefficient function for \( \varphi = \varphi(a) \) is determined by the coefficient functions for \( \varphi \), \( \varphi \), \( \varphi \).

Proof. \( x \in \varphi(a) \) if and only if \( x \in \theta(a) \). Thus \( \theta(a) \) is admissible for \( a \) and \( \nu(\beta) \) is equal to \( \nu(\beta) \). Hence \( \theta(a) = \nu(\beta) \). Finally, note that \( \nu(a) \). In turn this is equivalent to the assertion that \( a \), the cardinalities of various \( \nu(\beta) \), \( \bigcup_{\beta \in \theta(a)} f_{\beta}(\beta) \), the latter depending only on \( \nu(\beta) \), \( \nu(\beta) \). Thus \( \nu(a) \) depends only on \( \nu(\beta) \).

From Theorem 2.1 it follows that there is an operation on combinatorial series induced by composition of combinatorial operators. It is
appropriate to characterize the induced operation on series directly. Define the sum and scalar multiple of combinatorial series as follows. Let
\[ f = \sum c(i_1, \ldots, i_k) \left( \binom{i_1}{n_1} \ldots \binom{i_k}{n_k} \right), \]
\[ g = \sum d(i_1, \ldots, i_k) \left( \binom{i_1}{n_1} \ldots \binom{i_k}{n_k} \right), \]
let \( s \in E \cup \{a_0\} \). Then
\[ f + g = \sum \left( c(i_1, \ldots, i_k) + d(i_1, \ldots, i_k) \right) \left( \binom{i_1}{n_1} \ldots \binom{i_k}{n_k} \right), \]
\[ sf = \sum sc(i_1, \ldots, i_k) \left( \binom{i_1}{n_1} \ldots \binom{i_k}{n_k} \right). \]

(All operations on coefficients are in the sense of cardinal arithmetic in \( E \cup \{a_0\} \) throughout.) Let 0 be the series which is the identity with respect to addition—i.e., which has an identically zero coefficient function.

Since combinatorial series \( f \) with finite coefficients are in a 1-1 correspondence with combinatorial functions \( f \) of natural numbers, and combinatorial functions of natural numbers are closed under composition, there is an operation on combinatorial series with finite coefficients which corresponds to composition of functions. As temporary notation, if \( f(i_1, \ldots, i_n) = g(i_1, \ldots, i_n), h(i_1, \ldots, i_n) \) are combinatorial series with finite coefficients, let \( f * (g_1, \ldots, g_m) \) denote the combinatorial series \( h \) (with finite coefficients) such that for \( (a_1, \ldots, a_n) \in \times \times E, \)
\[ h(a_1, \ldots, a_n) = f \left( g_1(a_1, \ldots, a_n), \ldots, g_m(a_1, \ldots, a_n) \right). \]

If
\[ F = \sum c(i_1, \ldots, i_k) \left( \binom{i_1}{n_1} \ldots \binom{i_k}{n_k} \right) \]
is a combinatorial series, define combinatorial series
\[ F_1 = \sum d(i_1, \ldots, i_k) \left( \binom{i_1}{n_1} \ldots \binom{i_k}{n_k} \right), \]
\[ F_2 = \sum e(i_1, \ldots, i_k) \left( \binom{i_1}{n_1} \ldots \binom{i_k}{n_k} \right) \]
as follows. For \( c(i_1, \ldots, i_k) = 0 \), let \( d(i_1, \ldots, i_k) = 0 \) and \( e(i_1, \ldots, i_k) = 1 \). For \( c(i_1, \ldots, i_k) < 0 \), let \( d(i_1, \ldots, i_k) = c(i_1, \ldots, i_k) \) and \( e(i_1, \ldots, i_k) = 0 \).

It follows immediately that \( F = F_1 + F_2 \), and obviously \( F_1, F_2 \) have finite coefficients. Next suppose given a combinatorial series \( F = \sum c(i_1, \ldots, i_k) \left( \binom{i_1}{n_1} \ldots \binom{i_k}{n_k} \right) \) with finite coefficients. Define a combinatorial series with finite coefficients
\[ F_{a_0}(i_1, \ldots, i_n, a_{n+1}, \ldots, a_k) = \sum d(i_1, \ldots, i_k) \left( \binom{i_1}{n_1} \ldots \binom{i_k}{n_k} \right) \]
as follows. Let \( d(i_1, \ldots, i_k) \) have as values only 0, 1, and let \( d(i_1, \ldots, i_k) = 1 \) if and only if there exists a \( c(i_1, \ldots, i_k) > 0 \) such that: (i) \( i_1 < j_1, \ldots, i_k < j_k \), with some inequality strict; (ii) \( i_1 + a_{n+1} i_2 + \cdots + a_k i_k = j_1 + a_{n+1} j_2 + \cdots + a_k j_k \) whenever \( i_1 + a_{n+1} i_2 + \cdots + a_k i_k = 0 \), we have \( i_1 = j_1, \ldots, i_k = j_k \).

The formal composition of combinatorial series can be defined as follows using these definitions.

**Definition.** Suppose that \( F = \sum_{i} a_i \binom{i}{n} \) and \( G = \sum_{i} b_i \binom{i}{n} \) are combinatorial series. Then \( F \circ (G_1, \ldots, G_m) \) is the combinatorial series
\[ F \circ (G_1, \ldots, G_m) = \sum_{i} a_i \binom{i}{n} \left( \sum_{j=1}^{m} \binom{j}{n} b_j(i, j) \right), \]
where \( \binom{j}{n} b_j(i, j) = 0 \) whenever \( b_j(i, j) = 0 \).

(2.7) Suppose that \( F, G_1, \ldots, G_m \) have finite coefficients. Then \( F \circ (G_1, \ldots, G_m) \) also.

**Proof.** Let \( F = \sum a_i \binom{i}{n} \) and \( G_j = \sum b_j(i, j) \binom{i}{n} \). Then \( F \circ (G_1, \ldots, G_m) = \sum a_i \left( \sum_{j=1}^{m} b_j(a_i, j) \right) \binom{i}{n} \), so the \( F \circ (G_1, \ldots, G_m) \) has finite coefficients.

(2.8) Suppose that \( F \) has finite coefficients. Then \( F \circ (G_1, \ldots, G_m) \) also.

**Proof.** Let \( F = \sum a_i \binom{i}{n} \) and \( G_j = \sum b_j(i, j) \binom{i}{n} \). Then \( F \circ (G_1, \ldots, G_m) = \sum a_i \left( \sum_{j=1}^{m} b_j(a_i, j) \right) \binom{i}{n} \), so the \( F \circ (G_1, \ldots, G_m) \) has finite coefficients.

(2.9) Let \( F, G : \times \times E \to \times \times E \) be combinatorial operators. Define \( \times \times F \) as \( \times \times G \) by \( \times \times F = \times \times G \). Then \( \times \times F \) is a combinatorial operator and \( \times \times G \) is a combinatorial operator.

(2.10) Let \( F, G : \times \times E \to \times \times E \) be combinatorial operators. Define \( \times \times F \circ (G_1, \ldots, G_m) \) by \( \times \times F \circ (G_1, \ldots, G_m) = \times \times F \circ (G_1, \ldots, G_m) \).

(2.11) Let \( F, G : \times \times E \to \times \times E \) be combinatorial operators. Define \( \times \times F \circ (G_1, \ldots, G_m) \) by \( \times \times F \circ (G_1, \ldots, G_m) = \times \times F \circ (G_1, \ldots, G_m) \).

**Lemma 2.5.** Let \( F, G : \times \times E \to \times \times E \) be combinatorial operators with \( \times \times F = \sum a_i \binom{i}{n} \). Let \( \times \times G = \sum b_i \binom{i}{n} \). Then \( \times \times F \circ (G_1, \ldots, G_m) = \times \times (a_i \circ (b_1, \ldots, b_m)) \).

Then \( \times \times F \circ (G_1, \ldots, G_m) = \times \times (a_i \circ (b_1, \ldots, b_m)) \).

(By convention \( \times \times F \circ (G_1, \ldots, G_m) \) is here regarded as a series in \( v_1, \ldots, v_m \).)
Proof. If \( \gamma \subseteq E \), let \( p_\gamma(y) \) consist of all \( y \in E \) such that there exists an \( s \in E \) with \( [s] \cap [y] \neq \emptyset \). Use Lemma 2.2 and 2.4 or a easy direct argument to show that for finite \( a_1, \ldots, a_m \), \( \phi(A_1, \ldots, A_m) \) is a disjoint union of all \( \phi(a_0 \odot A_1, \ldots, a_0 \odot A_m) \) such that \( A_1 \ldots A_m \) are finite, and such that \( \gamma_1 \subseteq E \odot a_0 \) and \( p_\gamma(y) = a_0 \) for \( t = 1, \ldots, m \). Of course if \( a_0 = t \) is non-empty and has \( s \neq t \) elements and \( \gamma \neq 0 \), it is an elementary exercise that there are infinitely many \( \gamma \subseteq E \odot a_0 \) such that \( \gamma \) has \( j \) elements and \( p_\gamma(y) = a_0 \). This observation will be applied to evaluate \( \psi = \sum d(t_1, \ldots, t_m) (\psi_1 \ldots \psi_m) \). Suppose that \( t_1, \ldots, t_m \) are given. Choose \( a_0 \subseteq E \) of cardinality \( t_1, \ldots, t_m \). 

Case 1. There is a \( (t_1, \ldots, t_m) > 0 \) such that (i) \( t_1 < t_2, \ldots, t_n < t_m \), with some inequality strict; (ii) \( t_1 + t_2 + \ldots + t_m < t_1 \), whenever \( t_1 = 0 \), then \( t_1 = t_2 \). Then for \( t = 1, \ldots, n \). Then certainly for some \( t \) among \( 1, \ldots, n \), we have \( t_1 = 0 \). The remark above assures that there exist infinitely many distinct \( \gamma_1 \ldots \gamma_t \) with \( \gamma_t \) of cardinality \( t_1 \ldots t_m - 1 \) and

\[ \phi(a_0 \odot \gamma_1, \ldots, a_0 \odot \gamma_t) \subseteq \psi(a_0 \odot t_1, \ldots, a_0 \odot t_m) \]

Since each \( \phi(a_0 \odot \gamma_1, \ldots, a_0 \odot \gamma_t) \) has \( t_1, \ldots, t_m \) elements and for distinct \( \gamma_1 \ldots \gamma_t \) we have \( \phi(a_0 \odot \gamma_1, \ldots, a_0 \odot \gamma_t) \) and \( \phi(a_0 \odot \gamma_1, \ldots, a_0 \odot \gamma_t) \) are disjoint, obviously \( \psi(a_0 \odot t_1, \ldots, a_0 \odot t_m) \) is infinite. Therefore in this case \( d(t_1, \ldots, t_m) = \infty \).

Case II. Suppose that \( (t_1, \ldots, t_m) > 0 \) such that (i) \( t_1 + \ldots t_m < t_1, \ldots, t_n < t_m \); (ii) \( t_1 + t_2 + \ldots + t_m = t_1 \), then \( t_1 = t_2 \). Then \( t_1 = t_1, \ldots, t_n = t_m \).

In this case the only choice of \( \gamma_1 \ldots \gamma_t \) is \( \gamma_1 = \ldots = \gamma_t = 0 \), so \( \phi(a_0 \odot \gamma_1, \ldots, a_0 \odot \gamma_t) \) has the same number of elements as \( \phi(a_0 \odot t_1, \ldots, a_0 \odot t_m) \). Therefore \( d(t_1, \ldots, t_m) = |c(t_1, \ldots, t_m)\). Apply this calculation and the definition of \( (\psi_0)_{a_0} \) to get the conclusion.

Lemma 2.6. Let \( \psi: \times^* E \to \times E \), \( \times^* E \to \times E \), \( \psi: \times^* E \to \times E \) be combinatorial operators. Let \( \phi_0 = F, \phi_0 = G \). Suppose that \( \psi = \psi_0 \) has finite coefficients. Then

\[ (\psi_0, \ldots, \psi_m) \models F(G_1, \ldots, G_n) = \sum (\psi_0, \ldots, \psi_m) = \psi_0 \otimes (\psi_0, \ldots, \psi_m) \]

Proof. Let \( \psi \) be chosen as for Lemma 2.5. Let \( \zeta: \times^* E \to \times E \) be a combinatorial operator with \( \zeta = Fm \). Apply (2.9), (2.10), Lemma 2.5 to get that

\[ \psi = \psi_0 + \sum (\psi_0, \ldots, \psi_m) \models (\psi_0, \ldots, \psi_m) \]

Then the last assertion of Theorem 2.1 yields

\[ (\psi_0, \ldots, \psi_m) = \psi_0 \otimes (\psi_0, \ldots, \psi_m) \]

Now apply (2.9), (2.10), (2.11) to get that \( \psi = G = \psi_0 \otimes (\psi_1, \ldots, \psi_m) \), \( \psi = (\psi_0 \otimes (\psi_0, \ldots, \psi_m)) \). Then an application of the last assertion of Theorem 2.1 yields

\[ (\psi_0, \ldots, \psi_m) = \psi_0 \otimes (\psi_0, \ldots, \psi_m) \]

A look at the definition of \( \psi \) in the proof of Lemma 2.5 reveals \( (\psi, \ldots, \psi_m) = (\psi_0 \otimes (\psi_0, \ldots, \psi_m), \ldots, (\psi_0 \otimes (\psi_0, \ldots, \psi_m))) \). Combine (2.12) and (2.13) and this observation to get

\[ (\psi_0, \ldots, \psi_m) = \psi_0 \otimes (\psi_0, \ldots, \psi_m) \]

The right side can be rewritten using (2.9), (2.10) as

\[ \psi_0 \models (\psi_0, \ldots, \psi_m) = \psi_0 \otimes (\psi_0, \ldots, \psi_m) \]

But, \( \psi_0 = F, \psi_0 = G_1, \ldots, \psi_m = G_1, \ldots, \psi_m = G_n \), then all series with finite coefficients, i.e., represent combinatorial functions on \( E \). Composition for these series corresponds to composition of functions. Applying a known result for combinatorial functions [12] we get that

\[ (\psi_0, \ldots, \psi_m) = \psi_0 \otimes (\psi_0, \ldots, \psi_m) \]

and

\[ (\psi_0, \ldots, \psi_m) = \psi_0 \otimes (\psi_0, \ldots, \psi_m) \]

Apply (2.11) to get the conclusion of the lemmas.

Theorem 2.2. Let \( \psi: \times^* E \to \times E \), \( \times^* E \to \times E \), \( \times^* E \to \times E \) be combinatorial operators. Then

\[ (\psi_0, \ldots, \psi_m) = \psi_0 \otimes (\psi_0, \ldots, \psi_m) \]

Proof. Due to (2.11), \( \phi_0 = (\psi_0 \otimes (\psi_0, \ldots, \psi_m)) \). Thus the last assertion of Theorem 2.1 and (2.9), (2.10) yield

\[ (\psi_0, \ldots, \psi_m) = \psi_0 \otimes (\psi_0, \ldots, \psi_m) \]

If \( \psi = F, \psi = G \), then an application of Lemma 2.6 and also (2.11) yields immediately

\[ \psi_0, \ldots, \psi_m) = \psi_0 \otimes (\psi_0, \ldots, \psi_m) \]

The right-hand side is by definition \( F(G_1, \ldots, G_n, \ldots) \), i.e., it is \( \psi = (\psi_0, \ldots, \psi_0) \).
From now on write \( f < (g', \ldots, g^n) \) as \( f(g', \ldots, g^n) \) when no confusion is possible.

3. Partial recursive combinatorial operators and \( \mathbb{R} \uparrow \) combinatorial functions. Let \( \mathcal{G}_\ast \colon \times^n \mathcal{F}(E) \to E \) be one of the usual fully effective 1-1 onto maps. A combinatorial operator \( \varphi : \times^n \mathcal{F}(E) \to \mathcal{F}(E) \) is called partial recursive if

\[
\varphi(x, \varphi(x, \ldots, \varphi(x, y) \ldots), y) \text{ is a partial recursive operator.}
\]

**Lemma 3.1.** If \( \varphi : \times^n \mathcal{F}(E) \to \mathcal{F}(E) \), \( \varphi' : \times^n \mathcal{F}(E) \to \mathcal{F}(E) \), \( \varphi'' : \times^n \mathcal{F}(E) \to \mathcal{F}(E) \) are partial recursive combinatorial operators, then \( \varphi(\varphi', \ldots, \varphi'') \) is also a partial recursive combinatorial operator.

**Proof.** Theorem 2.1 implies that \( \tau = \varphi(\varphi', \ldots, \varphi'') \) is a combinatorial operator. We describe how to recursively enumerate all pairs \( (x, \mathcal{G}_\ast(x)) \).

Effectively enumerate all \( (x, \mathcal{G}_\ast(x)) \) and all \( (w, \mathcal{G}_\ast(w, w)) \), \( (w, \mathcal{G}_\ast(w, w)) \), \( \ldots \), \( (w, \mathcal{G}_\ast(w, w)) \).

This is possible due to (3.1) for \( \varphi, \varphi', \ldots, \varphi'' \). If at any stage an \( (x, \mathcal{G}_\ast(x)) \) has been generated such that for all \( i = 1, \ldots, n \) and all \( w \in \mathcal{G}_\ast(w) \), \( (w, \mathcal{G}_\ast(w)) \) has been generated, then list \( (x, \mathcal{G}_\ast(w)) \) as an \( (x, \mathcal{G}_\ast(x)) \). Then (2.5) assures that all and only the correct pairs are listed.

It is easy to see that for a combinatorial operator \( \varphi \), \( \varphi \) is partial recursive in the sense of (3.1) if and only if \( \varphi \) is a partial recursive functional. From that point of view the immediately preceding argument can be omitted since partial recursive functionals are closed under composition.

Since the next few theorems concern \( \mathbb{R} \uparrow \) functions, some remarks on these functions are in order. It is an easy exercise that the characteristic function of a subset \( a \) of \( E \), \( a \) on \( a \) and 0 on \( E - a \), is the limit of a monotone increasing recursive sequence of recursive functions if and only if \( a \) is recursively enumerable. Similarly, the characteristic function of \( a \) is the limit of a monotone decreasing recursive sequence of recursive functions if and only if \( a \) is the complement of a recursively enumerable set. Thus the characteristic functions in \( \mathbb{R} \uparrow \) are exactly the characteristic functions of recursively enumerable sets, while the characteristic function of the complement of a recursively enumerable but not recursive set is always outside \( \mathbb{R} \uparrow \). (A function of the latter type can be expressed in both two number quantifier forms in the Kleene-Mostowski hierarchy.) A useful characterization of membership in \( \mathbb{R} \uparrow \) is

\[
\psi(\psi(a, n)) = \psi(a, n) \quad \text{for} \quad a \in \mathcal{F}(E).
\]

**Proof.** Suppose that \( f_1 : \times^3 \mathcal{F}(E) \to E \) is a monotone \( \uparrow \) recursive sequence of recursive functions such that \( \psi(i(j)) = \lim f(t) \) for all \( i \in \times^3 \mathcal{F}(E) \). Define the recursively enumerable family \( \{\beta_i\}_{i \in \mathcal{F}(E)} \) by letting \( \beta_i \) consist of all \( \beta \in E \) such that for some \( j, f(t) > n \). Obviously \( \psi(i(j)) = \lim f(t) \) implies \( \beta \) has \( \psi(i(j)) \) elements. Conversely, suppose that \( \{\beta_i\}_{i \in \mathcal{F}(E)} \) is given. Then \( \beta \) can be enumerated in stages so that at stage \( n \) only a finite number of elements of a finite number of the \( \beta_i \) have been enumerated. Let \( \beta_n \) be the set of elements of \( \beta \) enumerated by the \( n \)th stage. Let \( f_n(\beta) \) be the number of elements of \( \beta_n \).

**Theorem 3.2.** A combinatorial series \( f \in \mathbb{R} \uparrow \) if and only if \( f \neq \psi \) for at least one partial recursive combinatorial operator \( \varphi \).

**Proof.** Let \( f = \sum \psi(\beta_i(n)) \) \( \beta_i \in \mathcal{F}(E) \). Suppose that \( c \in \mathbb{R} \uparrow \). Use (3.2) to produce a family \( \{\beta_i\}_{i \in \mathcal{F}(E)} \) with \( \beta_i \) having \( c(i) \) elements for all \( i \in \times^3 \mathcal{F}(E) \). Use notation as in (1.1), define \( \psi(x) : \times^3 \mathcal{F}(E) \to \mathcal{F}(E) \) by

\[
\psi(a, n) = \psi(a, n) \quad \text{for} \quad a \in \mathcal{F}(E).
\]

Then \( \psi \) is precombinatorial and induces a combinatorial operator \( \varphi \). Obviously \( \psi(a) \) has the same number of elements as \( \beta_n \), i.e., \( c(a, n) \), so \( \psi \neq \varphi \). \( \varphi \) is partial recursive since the assumption that \( \{\beta_i\}_{i \in \mathcal{F}(E)} \) is a recursively enumerable family entails that all \( \psi(a, n) \) can be recursively enumerated for \( n \in \beta_0 \) and \( a \in \times^3 \mathcal{F}(E) \); this is precisely the set of \( (a, \mathcal{G}_\ast(a)) \). Conversely, suppose that \( \psi \) is a partial recursive combinatorial. Choose a recursively enumerable family \( \{\mathcal{G}_\ast(a, n)\}_{i \in \mathcal{F}(E)} \) such that for all \( i, \mathcal{G}_\ast(a, n) \in \mathcal{F}(E) \) and \( \psi(a) \neq \psi(i) \). Then (3.1) for \( \varphi \) implies that \( \psi(i) \in \mathcal{F}(E) \) is a recursively enumerable family of recursively enumerable sets. Of course by (1.1) \( \psi(i) \) has \( c(i) \) elements. Thus (3.2) implies that \( c \) is \( \mathbb{R} \uparrow \).

**Corollary.** The \( \mathbb{R} \uparrow \) combinatorial series are closed under formal composition.

The \( \mathbb{R} \uparrow \) series of course include the recursive series, but the recursive series are not themselves closed under formal composition. For an example, first choose a recursive relation \( \mathcal{E}(E \times E) \) such that

\[
(i, j) \in \mathcal{E}(E \times E) \quad \text{for} \quad i \neq j.
\]

For each \( i \in \mathcal{E}(E \times E) \), \( \mathcal{E}(E \times E) \) is recursively enumerable and not recursive. Make the obvious notational conventions and substitute the constant \( s_k \) for \( s_k \) in the series

\[
\sum_{\mathcal{E}(E \times E)} (\psi(a)) \psi(i(j)) \quad \text{to get the series} \quad \sum_{\mathcal{E}(E \times E)} (\psi(a)) \psi(i(j)).
\]

Since \( \psi(i(j)) = \psi(i(j)) \), which
is not recursive even though the constant series \( \kappa_0 \) and the series \( \sum a_\gamma \left( \begin{array}{c} \gamma \\ \kappa_0 \end{array} \right) \) are both recursive.

**Theorem 3.3.** Let \( f = \sum a(i_1, \ldots, i_n) \left( \begin{array}{c} i_n \\ \kappa_0 \end{array} \right) \) be an \( R \uparrow \) combinatorial series. Then \( f \) induces a well-defined map \( p_f : \times \times \times \rightarrow \Omega \) such that for any \( e \times \times \times \), \( f(e) = \phi(e) \), where \( e \in \times \times \times \) is such that \( \langle e \rangle = e \) and \( \phi \) is a partial recursive combinatorial operator such that \( \phi(f) = f \).

**Proof.** It must be shown that \( f(e) \) is independent of the choice of \( e \) and \( \phi \). Suppose \( \langle e \rangle = \langle f \rangle = e \). Suppose \( \phi(e) = \phi(f) = e \). Suppose that \( a_0 \in \text{domain } p_1, a_1 \in \text{partial recursive } p_1(a_0) = \beta_0 \). Let \( A \) consist of all \( \gamma \in \times \times \times \text{ such that } \gamma_0 = a_0 \) is a subset of the domain of \( p_1 \) for \( i = 1, \ldots, n \). Let \( B \) consist of all \( \gamma_1, \ldots, p_n(a_0) \) such that \( \gamma \in \times \times \times \). Let \( p_1 : \times \times \times \rightarrow A \) be the 1-1 onto map such that \( (g, \gamma_1, \ldots, p_n(a_0)) \). Since \( p_1, \ldots, p_n \) are partial recursive, certainly \( \exists \langle \gamma \rangle \in \times \times \times \text{ such that } \langle \gamma \rangle \text{ is recursively enumerable.} \)

Thus by (3.1) \( \phi(g) \) can be recursively enumerated without repetitions uniformly in \( \langle \gamma \rangle \) for \( \gamma \in A \). For the same reason \( \langle \gamma \rangle \) is recursively enumerable and \( \phi(g') \) can be recursively enumerated without repetitions uniformly in \( \langle \gamma \rangle \) for \( \gamma \in A \). But since \( p_1 \) is 1-1, for \( \gamma \in A \), we have \( \langle \gamma \rangle = \langle p_1 \gamma \rangle \). Thus by (2.2) and the fact that \( \phi(g) = \phi(f) \), certainly \( \phi(\langle p_1 \gamma \rangle) \) and \( \phi(\langle p_2 \gamma \rangle) \) have the same number of elements (finite or infinite). Map the ith element of \( \phi(\langle p_1 \gamma \rangle) \) onto the ith element of \( \phi(\langle p_2 \gamma \rangle) \) in the enumerations without repetition chosen above, for all \( \gamma \in \times \times \times \).

By (2.3) for \( \phi \) and \( \psi \), we conclude that \( \psi \) is an \( R \uparrow \) partial recursive \( g \) with domain \( \bigcup_i \phi(g_i) \), range \( \bigcup_i \phi(g_i) \) such that for each \( \gamma \in A \), \( \phi(g_i) \) is mapped 1-1 onto \( \phi(g_i) \).

But \( \phi(a) \) is a union of \( \phi(g_i)^a \)'s and \( \phi(b) \) is the union of the corresponding \( \phi(g_i)^b \)'s, so \( \gamma \) maps \( \phi(a) \in A \) onto \( \phi(b) \). Thus \( \phi(a) = \phi(b) \).

**Corollary.** Let \( f = \sum a_0 + \cdots + a_n \) be an \( R \uparrow \) combinatorial series. Then

\[
\begin{align*}
&\sum_{a_0 + \cdots + a_n} \phi(a_0 + \cdots + a_n) \left( \begin{array}{c} a_n \\ \kappa_0 \end{array} \right) \\
&= \phi \left( \sum_{a_0 + \cdots + a_n} \phi(a_0 + \cdots + a_n) \left( \begin{array}{c} a_n \\ \kappa_0 \end{array} \right) \right) \\
&= \phi \left( \sum_{a_0 + \cdots + a_n} \phi(a_0 + \cdots + a_n) \right) \\
&= \phi \left( \sum_{a_0 + \cdots + a_n} \phi(a_0 + \cdots + a_n) \right)
\end{align*}
\]

Finally, Proposition I (§ 1) follows immediately from this Corollary and the fact that the extension of the projection \( p : \times \times \times \rightarrow \Omega, p(a_0, \ldots, a_n) = a_0 \), is the projection \( p \circ \times \times \times \rightarrow \Omega, p(a_0, \ldots, a_n) = a_0 \) (cf. [12], § 10).

**4. Inequalities.** Proposition II (§ 1) will be proved by the aid of the next six lemmas. The category method of [12], § 7 will be employed.

**Lemma 4.1.** Let \( f(u) = \sum a(i_1) \left( \begin{array}{c} i_1 \\ \kappa_0 \end{array} \right) \), \( g(u) = \sum b(i_1) \left( \begin{array}{c} i_1 \\ \kappa_0 \end{array} \right) \) be an \( R \uparrow \) combinatorial series. Then (i), (ii) below are equivalent.

(i) There exists a \( k \in E \) and there exists a \( \kappa \in E \) such that \( f(u + k) = g(u + k) \) for all \( u \in \kappa \).

(ii) There exists a \( k \in E \) and an \( R \uparrow \) function \( e : E \rightarrow E \) such that for all \( i, a(i) + b(i) \left( \begin{array}{c} i \\ \kappa_0 \end{array} \right) + \cdots + a(i + k) \left( \begin{array}{c} i + k \\ \kappa_0 \end{array} \right) = d(i) \left( \begin{array}{c} i \\ \kappa_0 \end{array} \right) + \cdots + d(i + k) \left( \begin{array}{c} i + k \\ \kappa_0 \end{array} \right) \).

**Proof.** Assume that (i) holds. Write \( a(u) = \sum b(i_1) \left( \begin{array}{c} i_1 \\ \kappa_0 \end{array} \right) \). The combinatorial identity

\[
\left( \begin{array}{c} i + k \\ \kappa_0 \end{array} \right) = \sum_{i=0}^{k} \left( \begin{array}{c} i \\ \kappa_0 \end{array} \right) + \cdots + \left( \begin{array}{c} i - k \\ \kappa_0 \end{array} \right)
\]

yields without difficulty that

\[
f(u + k) = \sum_{i=0}^{k} \left( \begin{array}{c} i \\ \kappa_0 \end{array} \right) a(u) + \cdots + a(u + k) \left( \begin{array}{c} i + k \\ \kappa_0 \end{array} \right),
\]

\[
g(u + k) = \sum_{i=0}^{k} \left( \begin{array}{c} i \\ \kappa_0 \end{array} \right) b(u) + \cdots + b(u + k) \left( \begin{array}{c} i + k \\ \kappa_0 \end{array} \right).
\]

The identity \( f(u + k) = g(u + k) \) yields (ii) when corresponding coefficients of \( \left( \begin{array}{c} i \\ \kappa_0 \end{array} \right) \) are compared. The converse is similar.

**Lemma 4.2.** Suppose that \( f(u) \), \( g(u) \) are an \( R \uparrow \) combinatorial series and that there is a \( k \) and an \( R \uparrow \) combinatorial series \( h(u) \) such that \( f(u + k) + h(u) = g(u + k) \) is a formal identity. Then \( f(a) = g(a) \) for almost all \( a \in \Omega \).

**Proof.** Due to Proposition I, \( f(a + k) + h(a) = g(a + k) \) for all \( a \in \Omega \). Therefore \( f(a) = g(a) \) for any \( a \in \Omega \) if \( (0, 1, \ldots, k - 1) \), since any such \( y \) is of the form \( y + k \).

Topologize \( \times \times \times \) as follows. Suppose that \( \delta_1, \delta_2 \) are disjoint finite subsets of \( E \). Let \( U(\delta_1, \delta_2) \) consists of all subsets of \( E \) which include \( \delta_1 \), but are disjoint from \( \delta_2 \). Choose a base for open sets in \( \times \times \times \) consisting of all \( U(\delta_1, \delta_2) \) for \( \delta_1, \delta_2 \) disjoint finite subsets of \( E \). Then \( \times \times \times \) is a Cantor space (a homomorphic of the Cantor set).

**Lemma 4.3.** The set \( \times \times \times \) of all \( a \in \times \times \times \) such that \( \langle a \rangle = \langle a \rangle + 1 \) is of the first category.
Lemma 4.4. Let $f(u) = \sum c(i) \left[ \begin{array}{l} n \\ i \end{array} \right]$, $g(u) = \sum d(i) \left[ \begin{array}{l} m \\ i \end{array} \right]$ be $R \uparrow$ combinatorial series. Suppose that $\chi$ is the set of all $\alpha \in E$ such that for some $\gamma \in \Omega$, $f_{\delta}((\alpha)) + \gamma = g_{\delta}((\alpha))$. Then $\chi$ is of the first category unless there exists a neighborhood $U = U(\delta, \delta_0)$, a 1-1 recursive function $p$, and partial recursive combinatorial operators $\varphi, \psi$ with $\varphi^\# = f$ and $\psi^\# = g$ satisfying (4.1) below.

(4.1) For any finite $\delta \in U$, there exists a $\beta \in \mathcal{E}(E)$ such that $\varphi(\beta) \oplus \beta$ is a subset of the domain of $p$ and $p(\varphi(\delta) \oplus \beta) = \psi(\delta)$. 

Proof. Suppose that no such $U$, $\varphi$, $\psi$ exist. Then it must be shown that $\chi$ is of the first category. There are only a countable number of $p, \varphi, \psi$. Therefore it suffices to show that for each choice of $p, \varphi, \psi$, the following set $X(p)$ is nowhere dense: $X(p)$ is the set of all $\alpha \in \mathcal{E}(E)$ such that there exists a $\beta \in \mathcal{E}(E)$ with $\varphi(\alpha) \oplus \beta$ a subset of the domain of $p$ and $p(\varphi(\alpha) \oplus \beta) = \psi(\alpha)$.

Given a neighborhood $U = U(\delta, \delta_0)$, we produce a non-empty subneighborhood $V$ of $U$ disjoint from $X(p)$. According to the negation of (4.1) there must be a finite $\delta_1 \subseteq \delta$ for which no $\beta \in \mathcal{E}(E)$ do we have that $\varphi(\beta) \oplus \beta$ is a subset of the domain of $p$ and $p(\varphi(\beta) \oplus \beta) = \psi(\beta)$.

If there is no $\delta_2 \supseteq \delta_1 \in U$, for which there exists a $\beta$ with $\varphi(\beta) \oplus \beta$ a subset of the domain of $p$ and $p(\varphi(\beta) \oplus \beta) = \psi(\beta)$, then certainly $V = U(\delta, \delta_0)$ will do. Otherwise, there is a smallest $\delta_2 \supseteq \delta$ such that this $\delta_2$ is then the smallest $\delta \supseteq \delta$ such that:

1. If $\alpha \in E$ and $\alpha_0$ is defined and $\alpha_0 \subseteq \alpha$, then $|p(2\alpha)| < \alpha$.
2. If $\alpha \in E$ and $\alpha_0$ is defined and $\alpha_0 \subseteq \alpha$, then $|p(\alpha)| < \alpha$. 

By choice of $\delta$, certainly $\delta \supseteq \delta_0$. The minimality of $\delta_0$ assures that $V = U(\delta, \delta_0) \cup (\omega \in \alpha) \cup (\epsilon)$ will do.

Lemma 4.5. Let $f(u) = \sum c(i) \left[ \begin{array}{l} n \\ i \end{array} \right]$, $g(u) = \sum d(i) \left[ \begin{array}{l} m \\ i \end{array} \right]$ be $R \uparrow$ combinatorial series. Suppose that $f_{\lambda}((\alpha)) < g_{\lambda}((\alpha))$ for almost all $\alpha \in \Omega$. Then there exists a $k$ and an $R \uparrow$ combinatorial series $h(u)$ such that $f(u) + k + h(u) = g(u) + k$ is a formal identity.

Proof. The proof proceeds through Lemma 4.4. Suppose that no choice of $U, \varphi, \psi, p$ satisfies (4.1). Then Lemmas 4.4 and 4.3 imply that $X \cup X_0$ is of the first category. The Baire category theorem assures that there is an infinite $\alpha \in \mathcal{E}(E) \cup (X \cup X_0)$.

Thus $\alpha_0 = (\alpha) \in \lambda \rightarrow E$. The definition of $X$ assures that $f_{\alpha}(\alpha_0) < g_{\alpha}(\alpha_0)$. Thus $f_{\lambda}((\alpha)) < g_{\lambda}((\alpha))$ does not hold for almost all $\alpha \in \Omega$, contrary to hypothesis.

Therefore there is a choice of $U = U(\delta, \delta_0), \varphi, \psi, p$ satisfying (4.1). Let $g : E \rightarrow E$ be a 1-1 recursive function with range a subset of $E \cup (\delta_0 \cup \delta_0)$.

Then $\chi(\alpha) = \delta_0 \cup g(\alpha)$ is a recursive combinatorial operator $\chi : \mathcal{E}(E) \rightarrow \mathcal{E}(E)$. Also $(\varphi, \psi) = (\varphi, \psi)$, where $\delta_0$ is the cardinality of $\delta_0$. Define $\varphi_1, \psi_1 : \mathcal{E}(E) \rightarrow \mathcal{E}(E)$ by $\varphi_1(\alpha) = \varphi(\varphi(\alpha), \varphi_1(\alpha), \psi_1(\alpha)) = \psi_1(\varphi(\alpha))$. Then $\varphi_1, \psi_1$ are partial recursive combinatorial operators such that $\varphi_1(\alpha) = f(\alpha) + k$, $\psi_1(\alpha) = g(\alpha) + k$.

Applying (4.1) it follows that for all finite $\alpha$, there exists a $\beta$ with $\varphi_1(\alpha) \oplus \beta$ a subset of the domain of $p$ and $p(\varphi_1(\alpha) \oplus \beta) = \psi_1(\alpha)$. Without difficulty, $\beta$ is defined uniquely by $\alpha$.

Define $\tau : \mathcal{E}(E) \rightarrow \mathcal{E}(E)$ by requiring that $\tau(\alpha)$ consist of all $\alpha \in E$ for which:

1. $p(2\alpha + 1) = \alpha$ is defined;
2. $(\varphi_1(\alpha) \oplus \beta) = \psi_1(\alpha)$.

Then $\tau$ has one of the two properties of a precombinatorial operators, namely for $\alpha \neq \alpha'$ certainly $\tau(\alpha)$ is disjoint from $\tau(\alpha')$. (However, $\tau$ does not necessarily have the other property of precombinatorial operators; it may be for all $\alpha = \alpha'$ but $\tau(\alpha)$ and $\tau(\alpha')$ have different numbers of elements.) For finite $\alpha_0$, let $\tau(\alpha) = \bigcup_{\alpha \in \alpha_0} \tau(\alpha)$.

The point of introducing $\tau(\alpha)$ is that $\tau(\alpha)$ is the unique $\beta$ referred to before that is, it is easily seen that for finite $\alpha \in \mathcal{E}(E)$, $\varphi_1(\alpha) \oplus \tau(\alpha)$ is a subset of the domain of $p$ and $p(\varphi_1(\alpha) \oplus \tau(\alpha)) = \psi_1(\alpha)$.

The fact that $p$ is partial recursive and (3.1) for $\psi_1$ can be applied to show that $\tau(\alpha)$ can be recursively enumerated without repetitions uniformly in $\delta_0(\alpha)$ for $\alpha \in \mathcal{E}(E)$.

If $\epsilon(\alpha)$ is the number of elements in $\tau(\alpha)$ for $\alpha < \alpha_0$), then it follows that $\epsilon(\alpha)$ is $\epsilon$ by (3.2). Further, a simple induction shows that for $\alpha \in \mathcal{E}(E)$, $\varphi_1(\alpha) \oplus \tau(\alpha) = \psi_1(\alpha)$. Therefore if $f(u + k) = \sum c(i) \left[ \begin{array}{l} n \\ i \end{array} \right]$, $g(u + k) = \sum d(i) \left[ \begin{array}{l} m \\ i \end{array} \right]$, we have $c(i) + d(i) = g(i)$. If $h(u) = \sum e(i) \left[ \begin{array}{l} m \\ i \end{array} \right]$, this means that $f(u + k) + h(u) = g(u + k)$.

Lemma 4.6. There exist $\alpha$ and $\beta$ such that $f_{\lambda}(\alpha) < g_{\lambda}(\alpha)$ whenever $f, g$ are $R \uparrow$ combinatorial series for which every choice of $\lambda$ and $u \uparrow$ combinatorial series $h(u)$ yields $f(u + k) + h(u) = g(u + k)$.

Proof. Call the set $X$ produced in Lemma 4.4 $X(f, g)$ to indicate dependence on $f, g$. Let $Y$ be the union of all $X(f, g)$ for $f, g$ ranging over the countably many $m$ pairs mentioned in Lemma 4.6. Then any $\alpha$ in $\mathcal{E}(E) \cup (Y \cup X_0)$ yields a suitable $\alpha = \alpha'$. Since $X \cup X_0$ is of the first category, there are $\alpha, \alpha'$. Since each $\alpha$ contains only a countable number of $\alpha'$, there are $\alpha, \alpha'$.

Theorem 4.2. Let $f(u) = \sum c(i) \left[ \begin{array}{l} n \\ i \end{array} \right]$, $g(u) = \sum d(i) \left[ \begin{array}{l} m \\ i \end{array} \right]$, $R \uparrow$ combinatorial series. Then the following conditions are equivalent.

(i) $f_{\lambda}(\alpha) = g_{\lambda}(\alpha)$ for almost all $\alpha \in \Omega$.

(ii) There exists a $k$ such that $f(u + k) = g(u + k)$.
(iii) There exists a $k$ such that for all $i$,

$$
\alpha_i^{(0)} + \alpha_{i+1}^{(0)} + \ldots + \alpha_{i+k}^{(0)} = d_i^{(0)} + d_{i+1}^{(0)} + \ldots + d_{i+k}^{(0)}.
$$

Moreover, there are $\varepsilon$-tools such that $f_\varepsilon(x) \neq g_\varepsilon(x)$ whenever $f, g$ are $R \uparrow$ series which fail to satisfy (iii).

Proof. Assume that (i) holds. Apply Proposition II. For some $k_1, k_2 \in E$ and some $R \uparrow$ combinatorial series $h(u), h_2(u)$, we have

$$
f(u + k_1) + h_2(u) = g(u + k_2) \quad \text{and} \quad g(u + k_2) + h(u) = f(u + k_1).
$$

Let $k = \max(k_1, k_2)$. Then

$$
f(u + k + h(u + (k - k_2)) = g(u + k_2), \quad g(u + k_2 + h(u + (k - k_2))) = f(u + k_1).
$$

It follows immediately that $f(u + k) = g(u + k)$ upon examination of coefficients. Next, assume that (ii) holds. Apply Proposition I to get $f_\varepsilon(x) = g_\varepsilon(x)$ for any $x \in \Omega - \{0, 1, \ldots, k-1\}$. As for (iii), the same argument applies as applied in the proof of Lemma 4.1 to show that (iii) is equivalent to (ii). The last part follows from the last part of Proposition II similarly.

An example will show that Proposition I fails when applied to a slightly wider class of series than $R \uparrow$ series. An identity between combinatorial functions will be exhibited (with the combinatorial functions expressible in both two number quantifier forms) which fails in $\Omega$.

Let $S$ be a recursively enumerable but not recursive set. Let $M$ be the set of all $x \in E$ such that $x$ is not a perfect square. Let $T = \{2^n \mid n \in S\} \cup M$. Then $T$ is recursively enumerable but not recursive. Further, note that the difference of successive squares $(x+1)^2 - x^2 = 2x + 1$ is monotone increasing and unbounded. This has the consequence that for any $k$, there is an $\varepsilon$ such that for $t \geq \varepsilon$, if $t \in E - M$, then $\varepsilon \in M, \ldots, i + k \in E$.

Let $c_0: E \rightarrow E$ be the characteristic function of $T$, let $d$ be the characteristic function of $E - T$. Let $f(u) = \sum c(i)u^i$, $g(u) = \sum d(i)u^i$, let $f(x), g(x)$ denote the corresponding combinatorial functions. The identity $f = f(x) + g(x)$ holds in $E$ (or equivalently, the formal identity $2f = f(u) + g(u)$ holds in combinatorial series. Suppose that this identity yields an identity in $\Omega$. A fortiori, $2f \neq f_\varepsilon(x)$ for all $x \in \Omega$. Hence Proposition II implies that for some $h, k, 2^{i+k} = f(x + k) + h(u)$ for an $R \uparrow$ series $h(u)$. Obviously $h(u)$ must have finite coefficients, so this yields an identity $2^{i+k} = f(x + k) + h(x)$ in combinatorial functions in $E$. But also $2^{i+k} = f(x + k) + g(x + k)$, so $g(x + k) = h(x)$. Let $h(x) = \sum c(i)u^i$, with $\varepsilon$ an $R \uparrow$ function. As has been remarked, $\varepsilon$ can be chosen so that for $i \geq \varepsilon$ and $i \in E - M$, we have $\varepsilon \in E - M, ..., i + k \in E$. Thus for $i \in E - M$ and $i \geq \varepsilon$,

$$
e(i) = d(i) + d(i+1)u^i + \ldots + d(i+k)u^k = d(i).
$$

By choice $\varepsilon$ is $R \uparrow$, so there exists a monotone increasing recursive sequence of recursive functions $\varepsilon_i$ such that $\lim \varepsilon_i = \varepsilon$. Define a monotone increasing recursive sequence of recursive functions $g_\varepsilon$ as follows.

1. $g_\varepsilon(0) = d(0)$ for $i \leq \varepsilon$.
2. $g_\varepsilon(i) = e_\varepsilon(i)$ for $i \geq \varepsilon$ and $i \in E - M$.
3. $g_\varepsilon(i) = 0$ for $i > \varepsilon$ and $i \in E - M$.

Then for $i \leq \varepsilon$, obviously $\lim g_\varepsilon(i) = d(i)$. For $i > \varepsilon$, if $i \in E - M$, certainly $\lim g_\varepsilon(i) = \lim e_\varepsilon(i) = e(i) = d(i)$. For $i > \varepsilon$, if $i \in M$, then $\lim g_\varepsilon(i) = 0 = d(i)$. Thus $d$ is $R \uparrow$. This is impossible since $d$ is the characteristic function of the complement of a recursively enumerable but not recursive set.

5. Applications to $\Omega - \Lambda$. Throughout this section $f(u) = \sum c(i)u^i$ will be an $R \uparrow$ combinatorial series.

THEOREM 5.1. $f_\varepsilon(u) = f(u) + 1$ for almost all $x \in \Omega$ if and only if some $\varepsilon = \kappa$.

Proof. By Theorem 4.2 and Proposition I, the hypothesis $f_\varepsilon(u) = f(u) + 1$ for almost all $u$ implies that for some $\varepsilon$,

$$
c(0)u^0 + c(1)u^1 + \ldots + c(k)u^k = (1 + c(0))u^0 + c(1)u^1 + \ldots + c(k)u^k.
$$

Thus each side is $\kappa$ and one of $c(0), \ldots, c(k)$ is $\kappa$. The converse is similar.

This shows that any combinatorial series with an $\kappa$ coefficient (i.e., which does not correspond to a combinatorial function) has almost all values in $\Omega - \Lambda$. This is what is meant by saying that combinatorial series are slanted toward $\Omega - \Lambda$.

THEOREM 5.2. The following conditions are equivalent.

(i) $f_\varepsilon(u) = f(u) + 1$ for almost all $u \in \Omega$.

(ii) There exists a $k$ such that $f(u + k) = \sum d(i)u^i$, then all $d_i$ have value 0 or $\kappa$.

(iii) There exists a $k$ such that for all $i$, if one of $\varepsilon_i, \ldots, \varepsilon_{i+k}$ is non-zero, then one of $c(i), \ldots, c(i+k)$ is $\kappa$.

Proof. Apply Proposition I and Theorem 4.2 to see that (ii) is equivalent to the assertion that there exists a $k$ such that for all $i$, $c(i)u^i + \ldots + c(i+k)u^k$ has value 0 or $\kappa$, i.e., to the assertion that if one of $c(i), \ldots, c(i+k)$ is non-zero, then one of this list is $\kappa$. Thus (ii) and (iii) are equivalent. Now assume that (i) holds and apply Propositions...
sition I and Theorem 4.2 to obtain a k such that \( f(u+k) = 2f(u+k) \).

If \( f(u+k) = 2d(i) \), this means that \( d(i) = 2d(i) \) for all \( i \), i.e., all \( d(i) \) are 0 or \( \nu_k \). Thus (ii) holds. Conversely, suppose that (ii) holds. Then \( 2d(u+k) = 2d(i) \), since Proposition I assures that \( 2d(u+k) = f(u+k) \) for all \( u \in \Omega \). This shows that (i) holds.

Call \( f \) non-constant if some \( c(i) > 0 \) with \( i > 0 \).

**Theorem 5.3.** Suppose that \( f \) is non-constant. Then the following are equivalent. (i) \( f(\alpha) = (f_d(\alpha))_d \) for almost all \( \alpha \in \Omega \). (ii) There is a \( k \) such that \( f(u+k) = \sum_{i=0}^{\nu_k} \nu_k(i) \). (iii) There is a \( k \) such that for all \( i \), one of \( c(i) \), ... , \( c(i+k) \) is \( \nu_k \).

Proof. Suppose that (ii) holds. By Theorem 4.2, this means that for all \( i \), \( c(i)_d + \ldots + c(i+k)_d = \nu_k - i, \) one of \( c(i), \ldots, c(i+k) \) is \( \nu_k \).

Conversely if (ii) holds, then Theorem 4.2 implies \( f(u+k) = \sum_{i=0}^{\nu_k} \nu_k(i) \); so (ii) holds. Now assume that (i) holds. Then Proposition I and Theorem 4.2 imply that there is a \( k \) for which \( f(u+k) = f(u+k)^2 \). Since \( f \) is non-constant, there is an \( i_0 \) with \( c(i_0) > 0 \). Let \( g(u) = f(u+k+i_0+1) \). By computing coefficients it is seen that \( g(u) \neq 0 \) implies \( c(i_0) = 0 \), \( c(i_0+1) = 0 \), ..., \( c(0) = 0 \). Further, \( e(0) \) is a sum of terms, one of which is \( \binom{k+i_0+1}{i_0} \). Thus \( e(0) > 0 \). Thus there is a combinatorial series \( h(u) \) such that \( g(u) = 2h(u) \).

Since \( g(u)^2 = g(u) \), \( 2h(u) = g(u) \). It follows immediately from this equation that \( 2g(u) = g(u) \). Thus each \( c(i) \) is \( 0 \) or \( \nu_k \). Since \( e(0), \ldots, e(i) \) are non-zero, \( c(0) = \ldots = c(i_0) = \nu_k \). We claim that \( c(i_0) = \nu_k \) for all \( i \). If not, there is a least \( i_0 \) with \( c(i_0) = 0 \). Obviously \( i_0 > i_0 \), so \( i_0 > i_0 > i \). By assumption, \( e(i_0-1) = \nu_k \). Now \( g(u) = (g(u))^2 = \sum_{i_0}^{\nu_k} e(i)(i)_d(i)_d(i)^{\nu_k} \). When expanded as a combinatorial series, there is a summand \( e(i)(i)_d(i)_d(i)^{\nu_k} \). But \( (i)_d(i)_d(i)_d(i)^{\nu_k} = \sum_{i=0}^{\nu_k} e(i)(i)_d(i)_d(i)^{\nu_k} \). Therefore \( e(i) = \nu_k \), a contradiction. Thus (ii) holds.

Conversely, suppose that (ii) holds. Then \( f(u+k) = \sum_{i=0}^{\nu_k} \nu_k(i) \). But \( \sum_{i=0}^{\nu_k} \nu_k(i)^2 = \sum_{i=0}^{\nu_k} \nu_k(i) \), so \( f(u+k)^2 = f(u+k) \). Apply Proposition I, get \( f(u+k) = f(\alpha(u+k))^2 \) for all \( \alpha \in \Omega \). Thus (i) holds.

**Theorem 5.4.** The following conditions are equivalent.

(i) \( (f_d(\alpha))_d = 2(f_d(\alpha))_d \) for almost all \( \alpha \in \Omega \).

(ii) There exists a \( k \) such that \( f(u+k)^2 = \sum_{i=0}^{\nu_k} e(i)(i)^{\nu_k} \) with each \( e(i) \) having value \( 0 \) or \( \nu_k \).

(iii) There exists a \( k \) such that for all \( i \), \( c(i) \) is \( \nu_k \).

Proof. Propositions I, II, and Theorem 5.2 show that (i) and (ii) are equivalent. The equivalence of (ii) and (iii) follows from a close examination of \( \sum_{i=0}^{\nu_k} e(i)(i)_d(i)_d(i)^{\nu_k} \). Observe that \( \sum_{i=0}^{\nu_k} e(i)(i)_d(i)_d(i)^{\nu_k} = \sum_{i=0}^{\nu_k} e(i)_d(i)_d(i)^{\nu_k} \), where the latter summation extends over all \( i \) satisfying \( \max(i_1, \ldots, i_n) < i < i_1 + \ldots + i_n \) and for each such \( i \), \( c(i_1, \ldots, i_n) > 0 \). Thus

\[
\sum_{i=0}^{\nu_k} e(i)(i)_d(i)_d(i)^{\nu_k} = \sum_{i=0}^{\nu_k} \sum_{i_1, \ldots, i_n} e(i, i_1, \ldots, i_n) (i_1)_d(i_1)_d(i_1)^{\nu_k} \]

where the asterisk indicates summation over all \( (i_1, \ldots, i_n) \) such that \( \max(i_1, \ldots, i_n) < i < i_1 + \ldots + i_n \). From this we derive

\[
\sum_{i=0}^{\nu_k} e(i, i_1, \ldots, i_n) (i_1)_d(i_1)_d(i_1)^{\nu_k} \geq 0 \text{ if and only if there is } c(i) > 0 \text{ such that } i < i_1 + \ldots + i_n.
\]

Proof of (5.1). Suppose that such a \( j \) exists. Then \( \max(j, \ldots, j) < i < j + \ldots + j \) and \( e(i) = 0 \), so

\[
\sum_{i=0}^{\nu_k} e(i, i_1, \ldots, i_n) (i_1)_d(i_1)_d(i_1)^{\nu_k} \geq 0 \text{ if and only if there is } c(i) = 0 \text{ such that } i < i_1 + \ldots + i_n.
\]

Conversely, if \( \sum_{i=0}^{\nu_k} e(i, i_1, \ldots, i_n) (i_1)_d(i_1)_d(i_1)^{\nu_k} > 0 \), then for some \( i_1, \ldots, i_n \), \( e(i_1) > 0 \). We have \( e(i_1) > 0 \), \( e(i_1) > 0 \). If \( i = \max(i_1, \ldots, i_n) \), then \( i < i < i_1 \) and \( e(i) = 0 \). Thus (5.1) and the expansion of \( \sum_{i=0}^{\nu_k} e(i) \) yield

\[
(5.2) \quad \sum_{i=0}^{\nu_k} e(i)(i)_d(i)_d(i)^{\nu_k} = 0.
\]

A similar argument shows that

\[
\sum_{i=0}^{\nu_k} e(i)(i)_d(i)_d(i)^{\nu_k} = 0.
\]
(5.3) \[ \sum c(i, k_1, \ldots, k_n) e(i, j) \cdots e(i, k_n) = \kappa_n \text{ if and only if there exist } c(j_1) \neq 0, \ldots, c(j_n) \neq 0 \text{ such that some one of } c(j_1), \ldots, c(j_n) \text{ is } \kappa_n \text{ and } \max_j (j_1, \ldots, j_n) \leq i < j_1 + \ldots + j_n. \]

Combine this with the expansion of \( \binom{n+k}{k} \) to get

(5.4) \[ f(u + k) = \sum \binom{k}{i} e(i) \text{ if and only if there exist } c(j_1) \neq 0, \ldots, c(j_n) \neq 0 \text{ with some one of } c(j_1), \ldots, c(j_n) \text{ being } \kappa_n, \text{ such that max } (j_1, \ldots, j_n) < i' < j_1 + \ldots + j_n \text{ for some } i' \text{ such that } i < i' < i + k. \]

Then (5.2) and (5.4) show that (ii) and (iii) are equivalent.

**Theorem 5.5.** The following conditions are equivalent for non-constant \( f \).

(i) \( (f Biomass^m)^{n+1} \) for almost all \( x \in \Omega \).

(ii) There exists a \( k \) such that \( \sum \kappa_n \binom{n+k}{k} = \kappa_n \binom{n+k}{k} \).

(iii) There exists a \( k \) such that for all \( t \) there exist \( \kappa_n \) with one of \( c(t_{j_1}), \ldots, c(t_{j_n}) = \kappa_n \) such that \( \max (j_1, \ldots, j_n) < i' < j_1 + \ldots + j_n \) for some \( i' \) with \( i < i' < i + k \).

**Proof.** Suppose that (ii) holds. Then \( f \) non-constant implies \( f(u + k) \) non-constant. Thus

\[ f(u + k) \left( \sum \kappa_n \binom{n+k}{k} \right) = \sum \kappa_n \binom{n+k}{k} = f(u + k) \left( \sum \kappa_n \binom{n+k}{k} \right)^{n+1}. \]

By Proposition 1, \( (f Biomass^m)^{n+1} \) for all \( x \in \Omega \), i.e., (i) follows. Suppose that (i) holds. Then

\[ (f Biomass^m)^{n+1} \] for all \( x \in \Omega \). Apply Proposition 1 and Theorem 5.3 to obtain

a \( k \) such that \( \sum \kappa_n \binom{n+k}{k} = \kappa_n \binom{n+k}{k} \). This is (ii). To see that (ii) is equivalent to (iii) apply (ii) and (iii) of Theorem 5.4.

**Corollary.** Suppose that all \( \kappa(i) \) are zero or \( \kappa_n \). Then the following are equivalent.

(i) \( (f Biomass^m)^{n+1} \) for almost all \( x \in \Omega \).

(ii) There exists a \( k \) such that for all \( i \) there exists an \( j \), \( j - h < i < j \), such that \( \kappa(i) = \kappa_n \).

**Example.** Idempotency \( m \), idempotency \( m \). Let \( f(u) = u + k \).

Certainly for no \( n > 1 \) and \( k > 0 \) is \( (f Biomass^m)^{n+1} \) for no \( m > 1 \) and \( k > 0 \) is \( (f Biomass^m)^{n+1} \). Therefore Theorems 4.2, 5.4, 5.5 show that there are \( c \) is \( x \) for which \( (f Biomass^m) \neq (f Biomass^m)^{n+1} \) for all \( n > 1 \); and simultaneously \( f Biomass^m = (f Biomass^m)^{n+1} \) for all \( m > 1 \).

**Example.** Idempotency \( 1 \), idempotency \( 1 < n < \infty \). Let \( f(u) = \sum \kappa_n \binom{n+k}{k} \). It is easily seen (see Theorem 5.5) (ii), (iii) or directly that

\[ (f Biomass^m)^{n+1} \] and hence that \( (f Biomass^m)^{n+1} \) for all \( x \in \Omega \). Obviously \( f Biomass^m \) has all coefficients \( \kappa_n \) or \( 0, \) \( f Biomass^m = 2f Biomass^m \) for all \( x \in \Omega \). Note that

\[ (f Biomass^m)^{n+1} = \sum \kappa_n \binom{n+k}{k} \]

therefore \( v(n - 1) u \) = \( \kappa_n \) and \( v(j) = 0 \) for \( (n - 1) u < j < n^2 + 1 \). Since \( n^2 + 1 < (n - 1) u \leq n \) as \( i \to \infty \), condition (ii) in the corollary to Theorem 5.5 is not satisfied. Therefore Theorem 4.2 assures that there are \( c \) is \( x \) for which \( f Biomass^m = f Biomass^m \).

**Example.** Idempotency \( m \), idempotency \( m \), with \( 1 < m < n < \infty \). The argument above shows that it suffices to produce a recursive combinatorial series \( f(u) \) such that: (i) \( (f Biomass^m)^{n+1} \) for almost all \( x \in \Omega \), (ii) \( (f Biomass^m)^{n+1} = \sum \kappa_n \binom{n+k}{k} \), (iii) \( (f Biomass^m)^{n+1} = \sum \kappa_n \binom{n+k}{k} \) with each \( \kappa(i) \) either \( \kappa_n \) or 0. (iii) If \( m > 1 \) and \( (f Biomass^m)^{n+1} = \sum \kappa_n \binom{n+k}{k} \), then for all \( k \) there exists a \( t \) with \( 0 < p(t) < \kappa_n \) and \( p(t+1) < \kappa_n \), \( p(t+2) < \kappa_n \), \( p(t+3) < \kappa_n \), etc.

p(t+1) \leq \kappa_n. (iv) If \( n > 1 \) and \( (f Biomass^m)^{n+1} = \sum q(i) \binom{n+k}{k} \), then for all \( k \) there exists an \( t \) with \( q(i) \neq \kappa_n \). Such an \( f Bio \) is

\[ \sum \kappa_n \binom{n+k}{k} \in \binom{n+k}{k} \binom{n+k}{k} \in \binom{n+k}{k} \]
Observe that
\[
(f(u))^{m-1} = \sum_{i=0}^{m-1} \left( \sum_{s \in E \cap i} a_s^{(i)} \right) u_i^{(i)} + \sum_{j, i_0 \neq i} p(j) u_i^{(i_0)} + \cdots + \sum_{s \in E \cap (m-1)} a_s^{(m-1)} u_i^{(m-1)}.
\]
Then, \( \alpha_{i+1} > p[(m-1) \alpha_{i+1}] > 0 \), while for \((m-1) \alpha_{i+1} < j < \alpha_{i+2}, p(j) = 0\). But \( \alpha_{i+1} - (m-1) \alpha_{i+1} = \alpha_{i+2} \), and \( \alpha_{i+2} \to \infty \) as \( i \to \infty \). Thus (iii) holds.

Finally, observe that \( f^{-1}(u) \) is
\[
\sum_{i=0}^{m-1} \left( \sum_{s \in E \cap i} a_s^{(i)} \right) g(j) u_i^{(i)} + \alpha_i^{(i)} u_i^{(m-1)}.
\]
Then, \( g[(n-1) \alpha_{n+1}] = \alpha_n \), while for \((n-1) \alpha_{n+1} < j < \alpha_{n+2}, g(j) = 0\). But \( \alpha_{n+2} - (n-1) \alpha_{n+2} = \alpha_{n+1} \), and \( \alpha_{n+2} \to \infty \) as \( n \to \infty \). Thus (iv) holds.

**Example.** Idempotency \( m < \infty \), idempotence \( \infty \). Examine
\[
f(u) = \sum_{i=0}^{m} \left( \sum_{s \in E \cap i} a_s^{(i)} \right) u_i^{(m)} + \alpha_i^{(m)} u_i^{(m-1)}
\]
by the above method.

6. **Applications to isetic integers.** The only result of preceding sections that is relevant is Proposition I for series with finite coefficients. This special case can be proved very simply by the method used for recursive combinatorial functions in [12], thus avoiding Lemma 2.2. The gain over [12], [13] arises solely from using \( B \uparrow \) coefficient functions rather than just recursive coefficient functions. Notation and terminology are from [12], [13]. Several elementary lemmas are required. The first two are well known.

**Lemma 6.1.** If \( f : E \to E \) is \( \mathcal{W} \) (relative to recursive predicates), then \( f \) is \( \mathcal{W} \).

**Proof.** If \( f \) is \( \mathcal{W} \), then there exists a recursive predicate \( R(u, v, x, y) \) such that \( f(u) = y \) if and only if \( \exists u \forall E \forall u, v, x, y \). Let \( f : E \times E \to E \) be a \( 1 \)-onto general recursive function, let \( k : E \to E \) be such that \( f[k(u), l(u)] = a \) for all \( a \in E \). Define \( S(u, v, x, y) \) by \( S(u, v, x, y) = k[u, v, x, l(u)] \land (u \neq y). \) Then \( f(u) = y \to \exists u \forall E \forall u, v, x, y \).

**Lemma 6.2.** If \( f : E \to E \) is \( \mathcal{W} \) if and only if there exists a recursive sequence of functions \( f_n : E \to E \) such that \( f = \lim_{n \to \infty} f_n \).

**Proof.** Suppose that \( f \) is such a limit. Then define \( E(u, v, x, y) \) by \( E(u, v, x, y) = \exists u \forall E \forall u, v, x, y \). Then \( f(u) = y \to \exists u \forall E \forall u, v, x, y \). Conversely, suppose that \( f \) is \( \mathcal{W} \). Choose \( R, S \) as in the proof of Lemma 6.1. Define \( u_n(a, y) \) as the least \( u \) such that \( \forall \exists u \forall E \forall u, v, x, y \). Define \( f_n(a) \) as the least \( y \) such that \( \forall \exists u \forall E \forall u, v, x, y \). Suppose \( f_n(a) = y_n \). For \( 0 \leq y < y_n \), define \( v_n(a, y) \) as the least \( v \) such that \( \forall \exists u \forall E \forall u, v, x, y \). Define \( f_n(a) = y_n \). For \( 0 \leq u < u_n(a, y) \), define \( w_n(a, u) \) as the least \( u \) such that \( \forall \exists u \forall E \forall u, v, x, y \). Suppose that \( n \) is larger than \( y_n \) and is also larger than each \( v_n(a, y) \) for \( y < y_n \) and is also larger than each \( w(a_n, y) \) for \( y < y_n \) and \( u < u(a, y) \). Then an easy argument shows \( f_n(a) = y_n \).

**Lemma 6.3.** Let \( f : E \times E \to E \) be the limit of a recursive sequence of functions. Then there exist combinatorial functions
\[
f(a_1, \ldots, a_n) = \sum \epsilon(i_1, \ldots, i_n) \left( a_{i_1} \right)^{[i_1]} \cdots \left( a_{i_n} \right)^{[i_n]},
\]
\[
g(a_1, \ldots, a_n) = \sum \delta(i_1, \ldots, i_n) \left( a_{i_1} \right)^{[i_1]} \cdots \left( a_{i_n} \right)^{[i_n]}
\]
such that \( a, d \) are \( B \uparrow \) and \( f(a_1, a_2, \ldots, a_n - a_{n-1} - a_n) = f(a_1, a_2, \ldots, a_n) - g(a_1, a_2, \ldots, a_n) \).

**Proof.** Consider \( k = 1 \) for convenience. Let \( E(a) \) be a recursive sequence of functions with limit \( f \). We construct two sequences of recursive combinatorial functions
\[
f_n = \sum c_n(i, j) \left( a_{i,j} \right)^{[i,j]}, \quad g_n = \sum d_n(i, j) \left( a_{i,j} \right)^{[i,j]}
\]
such that
\[(1) E_n(a_1, a_2) = f_n(a_1, a_2) - g_n(a_1, a_2) \text{ identically},
(2) c_n(i, j) < c_{n+1}(i, j), \quad d_n(i, j) < d_{n+1}(i, j), \text{ for all } i, j, n.
(3) c_n(i, j) \text{ converges to a } c(i, j) \in E, \quad d_n(i, j) \text{ converges to a } d(i, j) \in E.
(4) E(a_1, a_2) = f(a_1, a_2) - g(a_1, a_2) \text{ identically}, \text{ where } f = \sum c(i, j) \left( a_{i,j} \right)^{[i,j]} \text{ and } g = \sum d(i, j) \left( a_{i,j} \right)^{[i,j]}. \]
Let \((y_1, y_2)\) be the first element in the enumeration of \(S(P_a(x_1, \ldots, x_n))\) such that
\[
y_1 = \sum_{(i, j) \subset [n]} a_{ij} e_{ij} \]
and also
\[
y_2 = \sum_{(i, j) \subset [n]} d_{ij} e_{ij} \]
Then define \(f_a(x_1, x_2) = y_1\) and \(g_a(x_1, x_2) = y_2\).

Suppose now that \(n > 0\) and that \(f_{a_1}, \ldots, f_{a_{n-1}}, g_{a_1}, \ldots, g_{a_{n-1}}\) have been defined. Suppose further that \(f_{a_{n-1}}(i, j)\) and \(g_{a_{n-1}}(i, j)\) have been defined for \((i, j) < (x_1, x_2)\). Also certainly \(e_{a_{n-1}}(x_1, x_2)\) and \(d_{a_{n-1}}(x_1, x_2)\) also have been defined. Let \((y_1, y_2)\) be the first member of \(S(P_a(x_1, \ldots, x_n))\) in the order of enumeration such that
\[
y_1 = \sum_{(i, j) \subset [n]} a_{ij} e_{ij} \]
and also
\[
y_2 = \sum_{(i, j) \subset [n]} d_{ij} e_{ij} \]
Define
\[
f_a(x_1, x_2) = y_1\]
and \(g_a(x_1, x_2) = y_2\).

Requirements (1) and (2) have been established by construction. Requirements (3) and (4) remain to be established by induction on \((x_1, x_2)\) in the partially ordered set \(B \times B\).

Suppose \((x_1, x_2) = (0, 0)\). Since \(F(0) = \emptyset\), there is an \(m_0\) such that
\[F(0) = F(0)\] for \(n > m_0\). A look at definitions shows that certainly for \(n > m_0\) and \(a_{ij} = f_{a_{n-1}}(0, 0) = f_{a_{n-1}}(0, 0) = 0\) and \(g_{ij} = g_{a_{n-1}}(0, 0) = 0\). Now suppose \((x_1, x_2) > (0, 0)\). Then as inductive hypothesis that for \((i, j) < (x_1, x_2)\) it is known that \(a_{ij} = e_{ij}\), \(d_{ij} = f_{a_{n-1}}(i, j)\), and \(F(i, j) = F(0)\). Choose \(m_0\) so large that for \(n > m_0\) we have \(a_{ij} = e_{ij}\), \(d_{ij} = f_{a_{n-1}}(i, j)\), and \(F(i, j) = F(0)\). Under a 1-1 effective correspondence between \(B\) and \(B\), Lemma 6.2 yields a corresponding version for \(F\). Use Lemma 6.2 and Lemma 6.3 to choose combinatorial functions \(r_1, s_1: X^B \to B\) and \(r_2, s_2: X^B \to B\) which have \(B^+\) combinatorial series, and are such that for \(x_1, x_2, x_3, x_4 \in B\),
\[
h(x_1, x_2, x_3, x_4) = r_1(x_1, x_2) - s_1(x_1, x_2),
\]
Under a 1-1 effective correspondence between \(B\) and \(B\), Lemma 6.2 yields a corresponding version for \(F\). Use Lemma 6.2 and Lemma 6.3 to choose combinatorial functions \(r_1, s_1: X^B \to B\) and \(r_2, s_2: X^B \to B\) which have \(B^+\) combinatorial series, and are such that for \((x_1, x_2, x_3, x_4, x_5) \in B\),
\[
h(x_5 - x_4, x_3 - x_2, x_1 - x_0) = r_1(x_1, x_2) - s_1(x_1, x_2),
\]
Let \((X)\) abbreviate the following expression:
\[
(x_1, x_2, r_1(x_1, x_2), r_1(x_1, x_2), s_1(x_1, x_2), s_1(x_1, x_2), f_a(x_1, x_2), f_a(x_1, x_2), g_a(x_1, x_2), g_a(x_1, x_2))
\]
Then (6.1) yields that for all \((x_1, x_2, x_3, x_4, x_5) \in B^+\)
\[
p_a(X) + q(X) = p_a(X) + q(X).
\]
Since all functions involved have \(B^+\) combinatorial series, Proposition I
applies to show that the corresponding identity is true in $A$. But it follows easily from [13] that the corresponding identity in $A$ can be rephrased as asserting that for $a_1, a_2, a_3, a_4$ in $A$, $f(a_1, a_2, a_3, a_4) = f(a_1, a_2, a_3, a_4) = f(a_1, a_2, a_3, a_4)$.

But this shows that $\varphi$ is true in $A^*$ since $a_1 - a_2$ and $a_3 - a_4$ range over $A^*$. (However, it has not been shown that the element $r_0(a_1, a_2) - r_0(a_3, a_4)$ of $A^*$ depends only on $a_1 - a_2$.)

If the quantifier prefix in $\varphi$ is one of the forms $\forall^*, \exists^*, \forall^* \exists^*, \exists^* \forall^*$, there are always Skolem functions in both number quantifier forms. In paper [13] it was observed that for the $\forall^* \exists^* \forall^*$ prefix (and hence for all higher prefixes) there need not be Skolem functions in both two-number quantifier forms. This leaves out the case $\forall^* \forall^*$ which can be disposed of easily as follows. It will follow that there is an $\forall^* \forall^*$ prefix statement $\varphi$ without Skolem functions in $\forall^* \forall^*$ form if there exists an $\forall^* \forall^*$ function $f$: $E^* \rightarrow E^*$ which is not $\forall^* \forall^*$. The reason is simply that if $R$ is a recursive relation such that $f(a) = \forall r \in R \{u, v, x, y\}$, then the statement $\forall u \in X \exists v \in Y \forall w \in Z \exists r \in R \{u, v, x, y\}$ is true in $E^*$ and has $f$ as its Skolem function for the outermost existential quantifier. Without difficulty, it suffices to produce an $f$: $E \rightarrow E$ which is $\forall^* \forall^*$ but not $\forall^* \forall^*$. Myhill has made the comment that if $\beta$ is a retractable set retraced by a general recursive retracting function and $E - \beta$ is recursively enumerable and not recursive, then the function $g$ enumerating $\beta$ in order of magnitude is $\forall$ but not $\forall^*$, i.e., $E \times E - f$ is recursively enumerable but not recursive. Such $\beta$ are exhibited in [5].

Modify this remark as follows. Apply the construction of [5] to obtain a set $\alpha$ which is retraced by a fully defined $\forall^*$ function such that $E - \alpha$ is an $\forall^*$ set which is not an $\forall^*$ set. Then the function $f$ enumerating $\alpha$ in order of magnitude will do.

References