# On the uniqueness of the decomposition of finite-dimensional ANR-s into Cartesian products of at most 1-dimensional spaces

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**1. Introduction.** The objective of this paper is to give a complete proof of the Theorem announced in [9], which is concerned with the problem of the uniqueness of the decompositions of spaces into Cartesian products.

Some information about other results concerning this problem, which have been obtained by several authors, were given in [9]. Here we recall our terminology: A space $X$ containing at least 2 points is said to be *topologically prime* if it is not homeomorphic with any Cartesian product of two spaces containing at least 2 points each. Two decompositions of a given space such that after certain permutations the respective factors are homeomorphic are considered as the same. Throughout this paper, ANR spaces are always assumed to be compact.

Now, our Theorem (it is Theorem 2 in [9]) may be stated as follows:

THEOREM 1. *If a space $X \in$ ANR has a decomposition into the Cartesian product $X \underset{\text{top}}{=} \mathop{P}\limits_{i=1}^{n} X_i$ of topologically prime spaces of dimension $\leqslant 1$, then this decomposition is unique.*

This theorem gives a partial answer to the question raised by K. Borsuk (see [2], p. 140) of whether or not the decomposition of any space into the Cartesian product of topologically prime spaces of dimension $\leqslant 1$ is unique, and it generalizes two earlier theorems in this field: the first of K. Borsuk (see [3]) and the second one given in [8]. It is worth mentioning that if the compactness is not assumed, our theorem is false, as shown by the following example due to Z. Furdzik: If

$$A = [x \in E^1\colon\ 0 \leqslant x \leqslant 1], \qquad B = [x \in E^1\colon\ 0 \leqslant x < 1],$$

$$C = [x \in E^1\colon\ 0 < x < 1],$$

where $E^1$ denotes the set of all real numbers, then

$$A \times B \underset{\text{top}}{=} B \times B \underset{\text{top}}{=} B \times C .$$

On the other hand, a very interesting result recently obtained by R. D. Anderson (see [1]) shows that it is impossible to extend this theorem to countable Cartesian products, because an infinite countable Cartesian product of dendrites always yields the Hilbert cube. However, it is not known whether or not this theorem can be extended to arbitrary compact spaces of finite dimensions.

Here we keep the same notation as in [9], which we now shortly recall.

If $X = \mathop{P}\limits_{i=1}^{n} X_i$, then by $p_j\colon X \to X_j$ we denote the *projection* of the space $X$ onto the factor $X_j$. If $x \in X$, then we set $x_i = p_i(x)$.

$\mathrm{ord}_x X$ denotes the *order* of the point $x$ in the space $X$ in the sense of Menger-Urysohn ([7], p. 200).

By the symbol $Q^n$ (possibly with a subscript) we denote the *n-dimensional cell*, i.e. a homeomorphic image of the set $[x \in E^n\colon |x| \leqslant 1]$, where $E^n$ denotes the Euclidean $n$-space. $Q^{no}$ and $Q^{n\bullet}$ denote respectively the *interior* and the *boundary* of the cell $Q^n$. Similar signs are used for arcs, i.e. the 1-dimensional cells.

The structure of this work is as follows: In Section 2 we give the definition needed for the proof of Theorem 1. Namely, we recall the definitions of the local Betti numbers and others related to it. Four simple lemmas concerning these numbers are also given. In Section 3 these numbers are used to investigate spaces which are Cartesian products of local dendrites and 3 lemmas concerning the properties of those spaces are proved. Such spaces are examined for the reason that—as mentioned at the beginning of Section 8—each connected space satisfying the assumptions of our theorem is a homeomorphic image of such a space. In Section 4, taking a Cartesian product of $n$ local dendrites, we define 2 families, $\boldsymbol{R}$ and $\boldsymbol{S}$, consisting of subsets of that product. Those definitions were already given and illustrated in [9]. Here, it is worth mentioning that the family $\boldsymbol{R}$, by definition dependent on the way in which the space is represented as a Cartesian product of local dendrites, determines in a natural manner the $(n-1)$-dimensional factors of the given product (except some special cases). On the other hand, the family $\boldsymbol{S}$ has a topological character, i.e. it is independent of the way in which the space is represented as a Cartesian product. For this reason, when the identity of those families is proved, we shall be able to determine invariantly the $(n-1)$-dimensional factors of our product, and thus the induction step in the proof of our theorem can be taken. We prove this identity in Sections 5-7. This is the most essential part of the proof of our theorem. The proof of that theorem in the case where the space $X$ is connected is given in Section 8. An easy generalization to arbitrary spaces is obtained in Section 9.



**2. Local Betti numbers. The basic definitions and lemmas relative to those numbers.** First of all, we recall the definition of local Betti numbers which is given for instance in [4] and was already described in [9]. By a theorem of R. L. Wilder (see [10]), these numbers coincide with those defined earlier by E. Čech (see [5]). Only metrizable spaces are considered and Čech's homology groups with rational coefficients are used.

Let $X$ be a compact space and let $x \in X$. Consider the family of all (open) neighbourhoods of the point $x$, which, together with the relation $\supset$, yields a directed set. The *n-th local homology group* $LH_n(x, X)$ of the space $X$ at the point $x$ is defined as the limit of the direct system formed by the groups $H_n(X, X-U)$ with the natural projections $H_n(X, X-U) \to H_n(X, X-V)$, where $U$ and $V$ are neighbourhoods of $x$ such that $U \supset V$.

The *n-th local Betti number* $\beta_n(x, X)$ of the space $X$ at the point $x$ is defined as the rank (the dimension) of the group $LH_n(x, X)$. In the case where, although that rank is infinite, nevertheless there is an arbitrarily small neighbourhood $U$ of $x$ such that the group $H_n(X, X-U)$ has a finite rank we set $\beta_n(x, X) = \omega$.

Further, we recall the definition of the exterior Betti zero-number $a_0(x, X-A)$ of a closed subset $A$ of a locally connected space $X$ at a point $x \in A$, following E. Čech ([5], p. 694).

$a_0(x, X-A) = m$ if and only if $m$ is the smallest number such that there exist arbitrarily small (connected) neighbourhoods of the point $x$ separated by the set $A$ into $m+1$ components.

$a_0(x, X-A) = \omega$ if and only if there exist arbitrarily small neighbourhoods of the point $x$ separated by $A$ into a finite number of components and if this fact does not hold for any fixed natural number.

$a_0(x, X-A) = \infty$ if and only if neither case considered above holds.

Finally, given a compact space of a finite dimension, we shall make use of the following

DEFINITION OF THE SETS $X^{[1]}$, $X^{[2]}$ AND $X^{[3]}$. $X^{[1]} = [x \in X\colon \beta_n(x, X) = 0]$, $X^{[2]} = [x \in X\colon \beta_n(x, X) = 1]$, $X^{[3]} = [x \in X\colon \beta_n(x, X) \geqslant 2]$, where $n = \dim X$.

Remark 1. Evidently, the sets $X^{[1]}$, $X^{[2]}$ and $X^{[3]}$ are disjoint and $X = X^{[1]} \cup X^{[2]} \cup X^{[3]}$.

Remark 2. If $X$ is a local dendrite (for the definition see the beginning of Section 3), then $X^{[1]}$ consists of the end-points, $X^{[2]}$ of the points with the order equal to 2, and $X^{[3]}$ of the ramification points of that space.

Now we shall give 4 lemmas concerning the local Betti numbers introduced above, which will be needed in the sequel. The first and the second one is proved in [5].

LEMMA 1 (The Local Theorem of the type of Mayer-Vietoris). *Let $X$ be a compact space, $A_i = \bar{A}_i \subset X$ for $i = 1, 2$, $X = A_1 \cup A_2$, $x^0 \in A_1 \cap \cap A_2$. Then:*

(i) *If $\beta_n(x^0, A_i) = 0$ for $i = 1, 2$, then $\beta_{n-1}(x^0, A_1 \cap A_2) \geqslant \beta_n(x^0, X)$.*

(ii) *If $\beta_{n-1}(x^0, A_i) = 0$ for $i = 1, 2$, then $\beta_n(x^0, X) \geqslant \beta_{n-1}(x^0, A_1 \cap A_2)$.*

LEMMA 2 (The Local Duality Theorem). *If $A$ is a closed subset of a cell $Q^n$ and $x^0 \in A \cap Q^{n\circ}$, then $a_0(x^0, Q^n - A) = \beta_{n-1}(x^0, A)$.*

LEMMA 3. *Let $A = \bar{A} \subset Q^n$ and $x^0 \in A \cap Q^{n\bullet}$. If there exists a set $B \subset Q^n - A$, both open and closed with respect to a neighbourhood of $x^0$ in $Q^n$ minus $A$, having no common points with a neighbourhood of $x^0$ in $Q^{n\bullet}$ and such that $x^0 \in \bar{B}$, then $\beta_{n-1}(x^0, A) > 0$.*

Proof. By our assumptions, one can find an $(n-1)$-dimensional simplex $\Delta$ of a sufficiently fine triangulation of the sphere $Q^{n\bullet}$ contained in $Q^{n\bullet} - B$ and containing $x^0$ in its interior. Next, take a cell $Q_0^n$ isometric with $Q^n$ form the disjoint union $Q^n \cup Q_0^n$ and identify each point of $\Delta$ with a point of $Q_0^n$ corresponding to it under a given isometry. As the quotient space one obtains a cell $Q_1^n$ such that $x^0 \in Q_1^{n\circ}$; moreover, there exists a neighbourhood $U^1$ of $x^0$ in $Q_1^n$ such that the set $B \subset Q^n - A - \Delta$ is both open and closed in $U^1 - A$. From the assumption that $x^0 \in \bar{B}$ and from the construction of $Q_1^n$ we deduce that, for each neighbourhood $V^1$ of $x^0$ in $Q_1^n$ such that $V^1 \subset U^1$, $(V^1 - A) \cap B \neq \emptyset$ as well as $(V^1 - A) - B \neq \emptyset$, these sets being both open and closed in $V^1 - A$. Consequently $a_0(x^0, Q_1^n - A) > 0$, and therefore, in view of Lemma 2, $\beta_{n-1}(x^0, A) > 0$.

LEMMA 4. *Let $X$ be a compact $n$-dimensional space, where $n > 0$. If $x^0 \in A = \bar{A} \subset X$, then $\beta_n(x^0, A) \leqslant \beta_n(x^0, X)$.*

Proof. Consider the family of all open neighbourhoods of $x^0$ in $X$. Since $\dim X = n$, it follows that the natural map $H_n(A, A-U) \to H_n(X, X-U)$ is a monomorphism (we make use of the well-known property of absolute homology groups and of the fact that $H_n(X, X-U)$ is isomorphic with $H_n(\tilde{X})$, where $\tilde{X}$ is the quotient space obtained by identifying all points of $X-U$). Consequently, applying the operation of taking the direct limit, we obtain the natural monomorphism

$$LH_n(x^0, A) \to LH_n(x^0, X) .$$

The desired inequality immediately follows.

## 3. Some properties of spaces which are Cartesian products of local dendrites.

Let us recall (cf. [7], p. 227) that a local dendrite is a continuum each point of which has a neighbourhood which is a dendrite. It is known that the class of local dendrites coincides with the class of connected 1-dimensional ANR-s.



Throughout this section we shall consider any given space $X$ which is a Cartesian product $\mathop{P}_{i=1}^{n} X_i$, where all the factors $X_i$ are local dendrites and $n = \dim X \geqslant 2$. The convention is adopted that the subscript in a symbol denoting a point (resp. a set, except for the symbol $Q^n$) is the index of the factor $X_i$ that contains this point (resp. this set).

The following definition will be useful for us: An (open) neighbourhood $U$ of a point $x^0$ is called a *regular neighbourhood* provided $U = \mathop{P}_{i=1}^{n} U_i$, where, for each $1 \leqslant i \leqslant n$, $U_i$ is a connected set (different from $X_i$), $\bar{U}_i$ is a dendrite, $\mathrm{Fr}(U_i)$ consists of a finite number of points, none lying in the same component of $\bar{U}_i - x_i^0$ as another one, and $\overline{\overline{\mathrm{Fr}(U_i)}} = \mathrm{ord}_{x_i^0} X_i$ in case $\mathrm{ord}_{x_i^0} X_i < \omega$. It is seen that every point $x^0$ has arbitrarily small regular neighbourhoods.

In Lemmas 5 and 6 we shall make use of the following construction: Given a regular neighbourhood $U$ of a point $x^0$, one constructs a polyhedron $P \subset \bar{U}$ by setting $P = \mathop{P}_{i=1}^{n} P_i$, where $P_i \subset \bar{U}_i$ is the union of the arcs joining the point $x_i^0$ to $\mathrm{Fr}(U_i)$. Moreover, we construct a retraction

$$r\colon \big(\bar{U}, \mathrm{Fr}(U)\big) \to (P, P-U)$$

as follows: For each $1 \leqslant i \leqslant n$ the retraction $r_i\colon \big(\bar{U}_i, \mathrm{Fr}(U_i)\big) \to (P_i, P_i - U_i)$ is defined so as to leave fixed every point $x_i \in P_i$ and to map each component of $\bar{U}_i - P_i$ onto the single point which is its boundary. Next, $r(x)$ is defined as the point with the coordinates $r_i(x_i)$.

The retraction $r$ has the following two properties:

(3.1) *There is a homotopy $h_t\colon i \simeq r$, where $i$ is the identity map and $h_t\colon \big(\bar{U}, \mathrm{Fr}(U)\big) \to \big(\bar{U}, \mathrm{Fr}(U)\big)$ for $t \in I$.*

(3.2) *If $x \in P$ and $\mathrm{ord}_{x_i} X_i \leqslant 2$ for each $1 \leqslant i \leqslant n$, then the set $r^{-1}(x)$ consists only of the point $x$.*

To prove (3.1), let us notice that the fact that the set $\bar{U}_i$ is an absolute retract implies the existence of a homotopy $h_{i,t}\colon \bar{U}_i \to \bar{U}_i$ such that $h_{i,0}(x_i) = x_i$ and $h_{i,1}(x_i) = r_i(x_i)$ for $x_i \in \bar{U}_i$, as well as $h_{i,t}(x_i) = x_i$ for $x_i \in \mathrm{Fr}(U_i)$, $t \in I$. Next, defining $h_t(x)$ as the point with the coordinates $h_{i,t}(x_i)$, we obtain the desired homotopy $h_t$.

To prove (3.2), suppose that the point $x \in P$ satisfies the condition mentioned there and let $y \in r^{-1}(x)$. Then, by the definition of $r$, $y_i \in r_i^{-1}(x_i)$. From the properties of $U_i$ and from the facts that $x_i \in P_i$ and $\mathrm{ord}_{x_i} X_i \leqslant 2$ we infer that the point $x_i$ cannot be the boundary of any component of $\bar{U}_i - P_i$. Consequently, the set $r_i^{-1}(x_i)$ consists only of the point $x_i$, and therefore $y_i = x_i$, which proves that $y = x$.

Now we pass to our lemmas.

**Lemma 5.** *Let $x^0 \in X$. Then:*

(i) *If there is an $1 \leqslant i \leqslant n$ such that $\operatorname{ord}_{x_i^0} X_i = 1$, then $\beta_n(x^0, X) = 0$.*

(ii) *If $\operatorname{ord}_{x_i^0} X_i = 2$ for each $1 \leqslant i \leqslant n$, then $\beta_n(x^0, X) = 1$.*

(iii) *If neither the assumption of (i) nor that of (ii) is satisfied, then $\beta_n(x^0, X) \geqslant 2$.*

Proof. To establish (i), assume for instance that $\operatorname{ord}_{x_1^0} X_1 = 1$. Let $U$ be any regular neighbourhood of $x^0$. The set $\operatorname{Fr}(U)$ may be represented as the union

$$[\operatorname{Fr}(U_1) \times \mathop{P}_{i=2}^{n} \overline{U}_i] \cup [\overline{U}_1 \times \operatorname{Fr}(\mathop{P}_{i=2}^{n} U_i)],$$

where the common part of the ingredients is $\operatorname{Fr}(U_1) \times \operatorname{Fr}(\mathop{P}_{i=2}^{n} U_i)$. Since, in our case, $\operatorname{Fr}(U_1)$ consists of one point, that common part is easily seen to be a strong deformation retract of the second ingredient. Consequently the first ingredient, being homeomorphic to $\mathop{P}_{i=2}^{n} \overline{U}_i$, is a deformation retract of $\operatorname{Fr}(U)$. Therefore $H_{n-1}(\operatorname{Fr}(U)) = H_{n-1}(\mathop{P}_{i=2}^{n} \overline{U}_i) = 0$. Since, on the other hand, $H_n(\overline{U}) = 0$, making use of the Exactness Axiom we conclude that $H_n(\overline{U}, \operatorname{Fr}(U)) = 0$. Finally, by the strong form of the Excision Axiom valid for the Čech homology groups, $H_n(X, X-U) = 0$, and therefore $LH_n(x^0, X) = 0$, because the point $x^0$ has arbitrarily small regular neighbourhoods. Thus $\beta_n(x^0, X) = 0$, which proves (i).

To prove (ii), let us consider any pair $U, V$ of regular neighbourhoods of $x^0$ such that $U \supset V$. Let $P$ denote the polyhedron constructed for $U$ in the way described at the beginning of this section. Consider the following commutative diagram:

$$\begin{array}{ccc} H_n(P, P-U) & \xrightarrow{i} & H_n(P, P-V) \\ \downarrow k & & \downarrow j \\ H_n(X, X-U) & \xrightarrow{l} & H_n(X, X-V), \end{array}$$

where all the maps are natural. In our case, the polyhedron $P$ is an $n$-cell whose boundary is $P-U$. Consequently the group $H_n(P, P-U)$ is isomorphic with the group of all rational numbers. Since, as is easily seen, $P \cap \overline{V}$ is a subcell of $P$ (with respect to a certain triangulation), it follows that $i$ is an isomorphism. From (3.1) and from the strong form of Excision Axiom valid for the Čech homology groups we deduce that $k$ is an isomorphism. For similar reasons, having observed that the cell $P \cap \overline{V}$ coincides with the cell constructed for $V$ in the way described at the beginning of this section, $j$ also is an isomorphism. Hence, by the commutativity of our diagram, we conclude that $l$: $H_n(X, X-U) \to H_n(X, X-V)$ is an isomorphism, all groups being isomorphic with the group of rational numbers. It follows that the group $LH_n(x^0, X)$, obtained by taking the direct limit, also is isomorphic with that group. Thus $\beta_n(x^0, X) = 1$, which proves (ii).



To prove (iii), assume for instance that $\operatorname{ord}_{x_1^0} X_1 \geqslant 3$ and $\operatorname{ord}_{x_i^0} X_i \geqslant 2$ for $2 \leqslant i \leqslant n$. Then the space $X$ contains the set $A = T \times Q^{n-1}$, where $T$ is a dendrite of the form T, such that $x^0$ lies inside the ramification $(n-1)$-cell of this set. Since $\beta_n(x^0, A) = 2$, applying Lemma 4 we deduce that $\beta_n(x^0, X) \geqslant 2$, which proves (iii).

Remark. The following formulas are easy consequences of the lemma which has been proved:

$$X^{[1]} = [x \in X\colon \bigvee_{1 \leqslant i \leqslant n} \operatorname{ord}_{x_i} X_i = 1]. \tag{3.3}$$

$$X^{[2]} = [x \in X\colon \bigwedge_{1 \leqslant i \leqslant n} \operatorname{ord}_{x_i} X_i = 2]. \tag{3.4}$$

$$X^{[3]} = [x \in X\colon (\bigwedge_{1 \leqslant i \leqslant n} \operatorname{ord}_{x_i} X_i \geqslant 2) \wedge (\bigvee_{1 \leqslant i \leqslant n} \operatorname{ord}_{x_i} X_i \geqslant 3)]. \tag{3.5}$$

Thus we obtain an invariant characterization of the sets appearing on the right-hand sides of these formulas.

**Lemma 6.** *Let $x^0$ be such a point of the space $X$, at most one coordinate of which is a ramification point of the respective factor $X_i$. If $x^0 \in A = \overline{A} \subset X$ and $\beta_n(x^0, A) > 0$, then the set $A$ contains a cell $Q^n$ of the form $\mathop{P}_{i=1}^{n} I_i$, where $I_i \subset X_i$ is an arc such that $x_i^0 \in I_i^0$ for $1 \leqslant i \leqslant n$.*

Proof. The assumption that $\beta_n(x^0, A) > 0$ implies that $LH_n(x^0, A) \neq 0$, and therefore there is a regular neighbourhood $U$ of $x^0$ such that $H_n(A, A-U) \neq 0$. By the strong form of the Excision Axiom valid for the Čech homology groups, we infer that

$$H_n(A \cap \overline{U}, A \cap \operatorname{Fr}(U)) \neq 0. \tag{3.6}$$

Consider the polyhedron $P \subset \overline{U}$ constructed in the way described at the beginning of this section. Since $\beta_n(x^0, A) > 0$, it follows from Lemma 4 that $x^0 \notin X^{[1]}$. Therefore, by (3.3) and by the assumption on the coordinates of $x^0$, the polyhedron $P$ is easily seen to be the union of some $n$-cells $Q_i^n$, where $i = 1, 2, \ldots, m$, $m \geqslant 2$, with exactly one common face of dimension $n-1$, whose interior contains $x^0$. The cells $Q_i^n$ are the Cartesian products of arcs, the set $P-U$ being the union of their remaining (closed) $(n-1)$-dimensional faces.

Let $r$: $(\overline{U}, \operatorname{Fr}(U)) \to (P, P-U)$ be the retraction as defined at the beginning of this section. The diagram

$$\begin{array}{ccc} H_n(A \cap \overline{U}, A \cap \operatorname{Fr}(U)) & \to & H_n(\overline{U}, \operatorname{Fr}(U)) \\ \downarrow (r|A \cap \overline{U})_* & & \downarrow r_* \\ H_n(r(A \cap \overline{U}), r(A \cap \overline{U}) - U) & \to & H_n(P, P-U), \end{array}$$

where the horizontal maps are induced by inclusions, is commutative. Since $\dim X = \dim \overline{U} = n$, it follows that the horizontal maps are monomorphisms. By (3.1), $r_*$ is an isomorphism. Consequently $(r|A \cap \overline{U})_*$ is a monomorphism, and therefore $H_n(r(A \cap \overline{U}), r(A \cap \overline{U}) - U) \neq 0$ with regard to (3.6). This implies that the set $r(A \cap \overline{U})$, being a closed subset of $P$, must contain at least two of the $n$-cells $Q_i^n$. Otherwise there were some points $q^i \in Q_i^{n\circ}$, where for instance $2 \leqslant i \leqslant m$, such that $r(A \cap \overline{U}) \subset P - \bigcup_{i=2}^{m} (q^i)$. This last set can be retracted by deformation onto $P - U$, and therefore we would have $H_n(r(A \cap \overline{U}), r(A \cap \overline{U}) - U) = 0$. The union of two cells $Q_i^n$ contained in $r(A \cap \overline{U})$ yields a cell $Q^n$ of the required form.

It remains to show that $Q^n \subset A$. For this purpose, first observe that the fact that the set of the ramification points of a local dendrite is (at most) countable (cf. [7], p. 227) implies that the set $M = [x \in X: \bigvee_{1 \leqslant i \leqslant n} \operatorname{ord}_{x_i} X_i \geqslant 3]$ is the union of a countable number of closed, $(n-1)$-dimensional sets, and therefore has dimension equal at most to $n-1$. Consequently, the set $Q^n \cap (X - M)$ is dense in $Q^n$. Now, let us notice that (3.2) implies that for each point $x$ of this set one has the equality $r^{-1}(x) = (x)$. This and the inclusion $Q^n \subset r(A \cap \overline{U})$ imply that $Q^n \subset A$, which completes the proof.

LEMMA 7. *Let $A$ be a closed subset of $X$ containing $x^0$ and let $\beta_n(x^0, A) > 0$. Then, given some $1 \leqslant i \leqslant n$, there exists an arc $I \subset [x \in A: \bigwedge_{j \neq i} x_j = x_j^0]$ such that $x^0 \in I^\circ$.*

Proof. First observe that in order to establish this lemma it suffices to show that:

(3.7) *For every number $n \geqslant 2$, for every space $X$ being a Cartesian product of $n$ local dendrites and for every point $x^0 \in X$ the following implication holds: If $x^0 \in A = \overline{A} \subset X$ and $\beta_n(x^0, A) > 0$, then, for each $1 \leqslant i \leqslant n$, $\beta_{n-1}(x^0, [x \in A: x_i = x_i^0]) > 0$.*

Indeed, from this condition and from our assumptions, using the induction on $n$, we can deduce that the first local Betti number of the set $[x \in A: \bigwedge_{j \neq i} x_j = x_j^0]$ at the point $x^0$ is positive. Now observe that this set is equal to $A \cap D$, where $D = [x \in X: \bigwedge_{j \neq i} x_j = x_j^0]$ is a local dendrite homeomorphic with $X_i$. It follows that it must contain the desired arc $I$. Otherwise, if $D' \subset D$ is a dendrite constituting a neighbourhood of $x^0$ in $D$, then the dimension at the point $x^0$ of the intersection of $A$ with the closure of each—except at most one—component of $D' - x^0$ is equal to 0 (cf. [7], p. 112, No. 9). Hence $\operatorname{ord}_{x^0} D \cap A = \operatorname{ord}_{x^0} D' \cap A \leqslant 1$, and therefore $\beta_1(x^0, D \cap A) = 0$, which is a contradiction.



Passing now to the proof of (3.7), first observe that, by the local character of this condition and by the fact that each point of a local dendrite has a neighbourhood which is a dendrite, it suffices to show (3.7) for the spaces which are Cartesian products of dendrites. For simplicity, we fix $i = 1$ and first we establish (3.7) under the assumption that $\operatorname{ord}_{x_1^0} X_1 < \omega$. We proceed by induction on this order.

If $\operatorname{ord}_{x_1^0} X_1 = 1$, then $\beta_n(x^0, X) = 0$ in virtue of Lemma 5 (i). Then Lemma 4 implies that $\beta_n(x^0, A) = 0$ for each set $A = \overline{A} \subset X$ containing $x^0$. Thus (3.7) is satisfied in the vacuum.

Now let $m \geqslant 2$ and suppose (3.7) to be valid for each space $X$ which is a Cartesian product of $n$ dendrites in every point $x^0 \in X$ such that $\operatorname{ord}_{x_1^0} X_1 < m$. Consider a space $X = \mathop{P}_{i=1}^{n} X_i$, where $X_i$ are dendrites, and let $x^0 \in A = \overline{A} \subset X$, where $\operatorname{ord}_{x_1^0} X_1 = m$ and $\beta_n(x^0, A) > 0$. Select one of the $m$ components of $X_1 - x_1^0$ and denote it by $B_1$, and let $B = [x \in X: x_1 \in B_1]$. We shall consider separately two cases: first, when the number $\beta_n(x^0, A \cap (X - B))$ is positive, and, second, when it is equal to 0.

For the first case, let us observe that the inductive hypothesis may be applied to the set $X - B = (X_1 - B_1) \times \mathop{P}_{i=2}^{n} X_i$, which is the Cartesian product of $n$ dendrites, and to our point $x^0$, because $x^0 \in X - B$ and $\operatorname{ord}_{x_1^0}(X_1 - B_1) = m - 1$. Hence our inequality $\beta_n(x^0, A \cap (X - B)) > 0$ implies that $\beta_{n-1}(x^0, [x \in A \cap (X - B): x_1 = x_1^0]) > 0$. Since $[x \in A: x_1 = x_1^0] = [x \in A \cap (X - B): x_1 = x_1^0]$, the desired inequality follows.

In the second case, represent $A$ as the union $(A \cap \overline{B}) \cup (A \cap (X - B))$. Since $\overline{B} = \overline{B}_1 \times \mathop{P}_{i=2}^{n} X_i$ and $\operatorname{ord}_{x_1^0} \overline{B}_1 = 1$, it follows from Lemmas 4 and 5 that $\beta_n(x^0, A \cap \overline{B}) \leqslant \beta_n(x^0, \overline{B}) = 0$. Hence, in our case, the $n$-th local Betti number at the point $x^0$ of either ingredient of the union under consideration is equal to 0. Since $\beta_n(x^0, A) > 0$ as assumed, using Lemma 1 (i) we conclude that $\beta_{n-1}(x^0, A \cap \overline{B} \cap (X - B)) > 0$. Finally, observing that $A \cap \overline{B} \cap (X - B) = [x \in A: x_1 = x_1^0]$, one obtains the desired inequality again.

It remains to establish (3.7) in the case where $\operatorname{ord}_{x_1^0} X_1 = \omega$. Proceeding by *reductio ad absurdum*, suppose that with the assumptions satisfied we have $\beta_{n-1}(x^0, [x \in A: x_1 = x_1^0]) = 0$. Selecting a finite number of the components of $X_1 - x_1^0$, denote by $X_1'$ the closure of their union and let $X' = [x \in X: x_1 \in X_1']$.

Next, let $r: X \to X'$ be the retraction which carries $x \in X - X'$ into the point $r(x)$, the first coordinate of which is $x_1^0$ and the remaining ones equal to those of $x$. In addition, let us set $U = r^{-1}(U')$, where $U'$ is a fixed neighbourhood of $x^0$ in $X'$. Clearly, if $X_1'$ contains a sufficiently great number of the components of $X_1 - x_1^0$ and if $U'$ is sufficiently small, then $U$ is an arbitrarily small neighbourhood of $x^0$ in $X$.

Now set $A' = A \cap X'$ and consider the commutative diagram

$$\begin{array}{ccc} H_n(A', A'-U) & \xrightarrow{\quad k \quad} & H_n(A, A-U) \\ \downarrow i & & \downarrow l \\ H_n\big(A' \cup \mathrm{Fr}(X'), (A' \cup \mathrm{Fr}(X')) - U\big) & \xrightarrow{j} & H_n\big(A' \cup \overline{X - X'}, (A' \cup \overline{X-X'}) - U\big), \end{array}$$

where all the maps are induced by inclusions. First, we shall prove that $i$ is an isomorphism. For this purpose, adopting the convention that, for any $B = \bar{B} \subset X$, $\tilde{H}_n(B)$ denotes the group $H_n(B, B-U)$ (or, which is the same, the group $H_n(B^*)$, where $B^*$ is the quotient space obtained by identifying all points of $B-U$), consider the Mayer-Vietoris sequence (cf. [6], p. 39):

$$\ldots \to \tilde{H}_n\big(A' \cap \mathrm{Fr}(X')\big) \to \tilde{H}_n(A') \oplus \tilde{H}_n\big(\mathrm{Fr}(X')\big) \to \tilde{H}_n\big(A' \cup \mathrm{Fr}(X')\big) \to$$
$$\to \tilde{H}_{n-1}\big(A' \cap \mathrm{Fr}(X')\big) \to \tilde{H}_{n-1}(A') \oplus \tilde{H}_{n-1}\big(\mathrm{Fr}(X')\big) \to \ldots$$

Since $\dim \mathrm{Fr}(X') = n-1$, it follows that $\tilde{H}_n\big(A' \cap \mathrm{Fr}(X')\big) = 0 = \tilde{H}_n\big(\mathrm{Fr}(X')\big)$ and that the natural map of $\tilde{H}_{n-1}\big(A' \cap \mathrm{Fr}(X')\big)$ into $\tilde{H}_{n-1}\big(\mathrm{Fr}(X')\big)$ (and therefore also into $\tilde{H}_{n-1}(A') \oplus \tilde{H}_{n-1}\big(\mathrm{Fr}(X')\big)$) is a monomorphism. Hence, by exactness, the map $i\colon \tilde{H}_n(A') \to \tilde{H}_n\big(A' \cup \mathrm{Fr}(X')\big)$ is an isomorphism.

The map $j$ is also an isomorphism, because the retraction

$$r|(A' \cup \overline{X-X'})\colon \big(A' \cup \overline{X-X'}, (A' \cup \overline{X-X'}) - U\big) \to$$
$$\to \big(A' \cup \mathrm{Fr}(X'), (A' \cup \mathrm{Fr}(X')) - U\big)$$

after composition with inclusion is easily seen to be homotopic with the identity map. By the commutativity of the diagram in consideration, it follows that $l$ is an epimorphism, being also a monomorphism, because $\dim(A' \cup \overline{X-X'}) \leqslant \dim X = n$. Thus $k$ must also be an isomorphism.

Now let $u \in H_n(A, A-U)$ and let $u'$ be an element of $H_n(A', A'-U)$ such that $k(u') = u$. From our supposition that $\beta_{n-1}(x^0, [x \in A\colon x_1 = x_1^0]) = 0$ and from the special case of (3.7) established above as applied to $A' = A \cap X' \subset X'$, we deduce that $\beta_n(x^0, A') = 0$. Therefore $LH_n(x^0, A') = 0$, because the $n$-th local homology group is a vector space over the field of rational numbers. Consequently, there is a neighbourhood $V$ of $x^0$ in $X$ such that $V \subset U$ and $h'(u') = 0$, where $h'\colon H_n(A', A'-U) \to H_n(A', A'-V)$ denotes the natural homomorphism. Finally, by examining the commutative diagram

$$\begin{array}{ccc} H_n(A', A'-U) & \xrightarrow{h'} & H_n(A', A'-V) \\ \downarrow k & & \downarrow \\ H_n(A, A-U) & \xrightarrow{h} & H_n(A, A-V), \end{array}$$

where all the maps are induced by inclusions, we conclude that $h(u) = 0$.



Thus we have proved that there exists an arbitrarily small neighbourhood $U$ of $x^0$ such that for each $u \in H_n(A, A-U)$ there is a neighbourhood $V$ of $x^0$ such that $V \subset U$ and that the natural homomorphism $h\colon H_n(A, A-U) \to H_n(A, A-V)$ carries $u$ into $0$. It follows that $LH_n(x^0, A) = 0$, and therefore $\beta_n(x^0, A) = 0$, which contradicts the assumption and thereby completes the proof of our lemma.

**4. The definitions of the families *R* and *S*.** Let $X$ be a Cartesian product $\mathop{P}\limits_{i=1}^{n} X_i$ of local dendrites, where $n = \dim X \geqslant 2$. As announced in the Introduction, we shall define here two families, $\boldsymbol{R}$ and $\boldsymbol{S}$, consisting of subsets of the space $X$.

DEFINITION OF THE FAMILY $\boldsymbol{R}$. *A set $M = \bar{M} \subset X$ belongs to the family $\boldsymbol{R}$* if and only if for some $1 \leqslant i \leqslant n$ there is a point $x_i^0 \in X_i$ such that $M = [x \in X\colon x_i = x_i^0]$ and, in addition, $\mathrm{ord}_{x_i^0} X_i \geqslant 3$.

Accordingly, family $\boldsymbol{R}$ consists of certain $(n-1)$-dimensional sections of the space $X$ obtained by fixing a coordinate. Since that coordinate is assumed to be a ramification point, it follows that, in the case where all the factors are manifolds, family $\boldsymbol{R}$ is empty. In another case, when exactly one factor $X_i$ has some ramification points, this family consists of disjoint sets homeomorphic with the manifold which is the Cartesian product of the remaining factors.

DEFINITION OF THE FAMILY $\boldsymbol{S}$. *A set $\bar{A} = A \subset X$ belongs to the family $\boldsymbol{S}$* if and only if:

A1. $A$ is an $(n-1)$-dimensional ANR homeomorphic with the Cartesian product of $n-1$ local dendrites.

A2. $$A - A^{[1]} \subset X^{[3]}.$$

A3. There exists a set $F = \bar{F} \subset X$ such that:

(i) $F$ is a neighbourhood of $A$ in $X$.

(ii) $F$ is irreducibly separated by $A$.

(iii) If $x^0 \in A$ and $G$ is an arbitrary component of $F - A$, then $\beta_n(x^0, \bar{G}) = 0$.

(iv) If $Q^{n-1} \subset A$ and $G$ is an arbitrary component of $F-A$, then there exists a cell $Q^n \subset \bar{G}$ such that $Q^n \cap A = Q^{n-1} \subset Q^{n\bullet}$.

(v) Let $x^0 \in A$ and let $\{G^m\}$ denote the sequence of all the components of $F-A$. Then there exists a sequence $\mathfrak{I} = \{I^m\}$ of arcs such that $x^0 \in I^{m\bullet}$, $I^m - x^0 \subset G^m$ for every $m$ and having the following property: If $Q^n \subset X$, $x^0 \in Q^{n\bullet}$, and if there exists a neighbourhood of $x^0$ in $Q^{n\bullet}$ contained in $A$, then there is a subarc $I$ of some arc $I^m$ such that $I \subset Q^n$ and $x^0 \in I^\bullet$.

It is seen that the family $\boldsymbol{S}$ has been defined in a topological manner by giving some conditions concerning the form and the position in $X$

of the sets belonging to it. These conditions were illustrated in a way in [9]. The identity of $\boldsymbol{R}$ and $\boldsymbol{S}$, which will be proved in sections 5-7, shows that these conditions express the characteristic properties of the sets belonging to $\boldsymbol{R}$. It is worth noticing that, if the space $X$ is a manifold, the family $\boldsymbol{S}$, like the family $\boldsymbol{R}$, is empty. Indeed, in this case, the set $X^{[3]}$ is empty by definition and therefore no set $A$ satisfying A1 can satisfy A2.

Remark. As noticed by Z. Furdzik, if one considers a space $X$, not necessarily compact but homeomorphic with a Cartesian product of connected spaces, in which each point has a neighbourhood which is a dendrite, then, retaining the definition of $R$ and appropriately modifying in the definition of $\boldsymbol{S}$ the condition A1 only, one carries over our proof of the identity of these families to such spaces. Indeed, as may be verified, only local properties of the space $X$ intervene in an essential manner in this proof.

Finally, we give two simple properties of the family $\boldsymbol{R}$ which will be useful for us in proving the inclusion $\boldsymbol{S}\subset\boldsymbol{R}$.

(4.1) *The family $\boldsymbol{R}$ is at most countable and covers the set $X^{[3]}$.*

(4.2) *If $M, N\in\boldsymbol{R}$ and $M\neq N$, then either $M\cap N=\emptyset$ or $\dim(M\cap N) = n-2$.*

Property (4.1) follows from the fact that the set of all ramification points of local dendrite is at most countable and, with respect to the second part of this condition, from formula (3.5). Property (4.2) is an immediate consequence of the definition of $\boldsymbol{R}$.

**5. Proof of the inclusion $\boldsymbol{R}\subset \boldsymbol{S}$.** Let $M$ be a given set of the family $\boldsymbol{R}$. For convenience, we assume that

$$M=[x\in X:\ x_1=x_1^0],$$

where $x_1^0$ is a fixed ramification point of the local dendrite $X_1$. To prove $M\in\boldsymbol{S}$ we must verify the conditions M1, M2 and M3, obtained by substituting $M$ for $A$ in the definition of $\boldsymbol{S}$.

The truth of M1 is a consequence of the fact that $M$ is homeomorphic with the Cartesian product $\mathop{P}\limits_{i=2}^{n} X_i$. The inclusion M2 follows from the definition of $M$ and from formulas (3.3)-(3.5).

To prove M3, let us define the set $F$ by the formula

$$F=F_1\times \mathop{P}\limits_{i=2}^{n} X_i,$$

where $F_1\subset X_1$ is a dendrite constituting a neighbourhood of $x_1^0$ in $X_1$. We must verify conditions (i)-(v).



The truth of (i) is obvious. Since $\mathrm{ord}_{x_1^0} X_1=\mathrm{ord}_{x_1^0} F_1\geqslant 3$, it follows that the point $x_1^0$ irreducibly separates the dendrite $F_1$. This fact and the formula $F-M=(F_1-x_1^0)\times\mathop{P}\limits_{i=2}^{n} X_i$ imply that:

(5.1) *Each component of $F-M$ has the form $G=G_1\times\mathop{P}\limits_{i=2}^{n} X_i$, where $G_1$ is a component of $F_1-x_1^0$.*

This implies (ii). To establish (iii), it suffices to apply Lemma 5 (i) to the set $\bar{G}=\bar{G}_1\times\mathop{P}\limits_{i=2}^{n} X_i$, because $\mathrm{ord}_{x_1^0}\bar{G}_1=1$.

Now, passing to the proof of (iv), consider any given cell $Q^{n-1}\subset M$ and a component $G$ of $F-M$ of the form described in (5.1). The desired cell $Q^n\subset\bar{G}$ is defined by $Q^n=I_1\times Q_0^{n-1}$, where $I_1\subset\bar{G}_1$ is an arc with $x_1^0$ as one of the end-points and $Q_0^{n-1}$ is the homeomorphic image of $Q^{n-1}$ under the natural projection of $X$ onto the factor $\mathop{P}\limits_{i=2}^{n} X_i$.

It remains to prove (v). For this purpose, given a point $x^0\in M$ and a component $G^m=G_1^m\times\mathop{P}\limits_{i=2}^{n} X_i$ of $F-M$, select an arc $I_1^m\subset\bar{G}_1^m$ with $x_1^0$ as one of the end-points and set

$$I^m=[x\in X:\ x_1\in I_1^m\wedge\bigwedge_{2\leqslant i\leqslant n} x_i=x_i^0].$$

We shall prove that the family $\mathfrak{I}=\{I^m\}$ satisfies the required conditions.

Evidently $x^0\in I^{m\circ}$ and $I^m-x^0\subset G^m$, because $I_1^m-x_1^0\subset G_1^m$. Now, given any cell $Q^n\subset X$ whose boundary contains a cell $Q^{n-1}$ such that $Q^{n-1}\subset M$ and $x^0\in Q^{n-1\circ}$, we may assume that the cell $Q^n$ contains no subarc of the arc $I^1\in\mathfrak{I}$ such that $x^0$ is one of its end-points. Under this assumption, beside $Q^n$, we shall also consider the cell

$$Q_1^n=I_1^1\times Q_0^{n-1},$$

where $Q_0^{n-1}$ is the homeomorphic image of the cell $Q^{n-1}\subset M$ under the natural projection of $X$ onto $\mathop{P}\limits_{i=2}^{n} X_i$. Since $Q^{n-1}\subset Q^n\cap Q_1^n$, $I^1\subset Q_1^n$ and $x^0\in\overline{I^1-Q^n}$, it follows that the set $Q_1^n-Q^n$ is disjoint from $Q^{n-1}$ and its closure contains $x^0$. Consequently, by Lemma 3, we have $\beta_{n-1}(x^0, Q^n\cap Q_1^n)>0$. Hence, in view of Lemma 1 (ii), we infer that $\beta_n(x^0, Q^n\cup Q_1^n)>0$, because $\beta_{n-1}(x^0, Q^n)=0=\beta_{n-1}(x^0, Q_1^n)$. Finally, applying Lemma 7 we conclude that the set $[x\in Q^n\cup Q_1^n:\ \bigwedge_{2\leqslant i\leqslant n} x_i=x_i^0]$ contains an arc $L$ such that $x^0\in L^\circ$. It follows from the definition of $\mathfrak{I}$ that this arc must contain two subarcs $L^1, L^2$, either with $x^0$ as one of the end-points, contained in two distinct arcs belonging to $\mathfrak{I}$. Then at least one of these subarcs must be contained in $Q^n$, because the cell $Q_1^n$ has been defined so that its only one common point with any arc of $\mathfrak{I}$ different from $I^1$ is the point $x^0$. This concludes the proof of (v), and thus the inclusion

$$R \subset S \tag{5.2}$$

has been completely proved.

**6. The non-existence of cells $Q^{n-1} \subset A \in S$ such that none of the projections $p_i(Q^{n-1})$ reduces to a single point.** In order to prove this fact, we proceed as follows: Given a set $A$ of the family $S$, first we define (this was already done in [9]) a special kind of $(n-1)$-dimensional cells contained in $A$, called *bent cells*. Next we establish that every cell $Q^{n-1} \subset A$ such that none of the projections $p_i$: $X \to X_i$ carries it into a single point contains a bent cell (see Lemma 8). Finally, we prove that the set $A$ cannot contain any bent cell (see Lemma 9). This proof depends on showing that a contradiction holds between the possession by $A$ of a bent cell and condition A3, which describes the position of $A$ in $X$.

DEFINITION OF BENT CELL. A cell $Q^{n-1} \subset A \in S$ is said to be *bent at a point* $x^0 \in Q^{n-1\circ}$ if there exist two indices $j, k$, $1 \leqslant j < k \leqslant n$, such that $Q^{n-1} = Q_1^{n-1} \cup Q_2^{n-1}$, where

$$Q_1^{n-1} = (\mathop{P}_{i=1}^{j-1} I_i) \times (x_j^0) \times (\mathop{P}_{i=j+1}^{n} I_i), \qquad Q_2^{n-1} = (\mathop{P}_{i=1}^{k-1} I_i) \times (x_k^0) \times (\mathop{P}_{i=k+1}^{n} I_i),$$

$$I_i \subset X_i, \quad x_i^0 \in I_i^\bullet \quad \text{for} \quad i = j, k, \quad x_i^0 \in I_i^\circ \quad \text{for} \quad i \neq j, k;$$

and if the point $x^0$ satisfies the following two conditions:

(i) $\mathrm{ord}_{x_i^0} X_i \geqslant 3$ for $i = j, k$ and $\mathrm{ord}_{x_i^0} X_i = 2$ for $i \neq j, k$.

(ii) There exists such a decomposition of $A$ into the Cartesian product of $n-1$ local dendrites that all coordinates of $x^0$ (except at most one) in this decomposition have the order equal to 2.

Hence, a bent cell is a union of two $(n-1)$-dimensional cells intersecting on a common face of dimension $n-2$; on either cell a coordinate (but not the same one) is fixed. In addition, what is an important fact, both these coordinates are ramification points.

Now we pass to our lemmas.

LEMMA 8. *Every cell $Q_0^{n-1} \subset A \in S$ such that none of the projections $p_i(Q_0^{n-1})$ for $1 \leqslant i \leqslant n$ reduces to a single point contains a bent cell.*

Proof. Clearly, the cell $Q_0^{n-1}$ may be covered by a finite number of $(n-1)$-cells, each contained in the Cartesian product of $n$ dendrites which are subsets of the factors $X_i$, in such a way that these cells can be ordered to yield a sequence such that the intersection of any pair of adjacent cells has dimension equal to $n-1$. Having observed that at



least one of such cells must satisfy the assumptions of our lemma with respect to $Q^{n-1}$, we may assume that:

$$Q_0^{n-1} \subset \mathop{P}_{i=1}^{n} D_i, \tag{6.1}$$

where $D_i \subset X_i$ is a fixed dendrite.

If $x \in Q_0^{n-1\circ}$, then, by Lemma 4, $\beta_{n-1}(x, A) > 0$. Hence $x \in A - A^{[1]}$, and therefore, in virtue of A2, $Q_0^{n-1\circ} \subset X^{[3]}$. Reducing again the cell $Q_0^{n-1}$ when necessary, we may assume that $Q_0^{n-1} \subset X^{[3]}$. Then, in view of (4.1), the family $R$ constitutes a countable covering of $Q_0^{n-1}$ by closed sets. Let $G$ denote the set of those points of $Q_0^{n-1}$ which have a neighbourhood in $Q_0^{n-1}$ contained in a set belonging to $R$. By the Baire Theorem, that set is both open and dense in $Q_0^{n-1}$. It follows from (4.2) that:

(6.2) *For every component $H$ of $G$ there exists a set $M \in R$ such that $\overline{H} \subset M$.*

Let $F = Q_0^{n-1} - G$. From (6.2) and from the assumption that none of the sets $p_i(Q_0^{n-1})$ reduces to a single point we infer that $F \cap Q_0^{n-1\circ} \neq \emptyset$.

Now our immediate objective is to find a cell $Q_1^{n-1} \subset Q_0^{n-1}$ such that the sets $Q_1^{n-1} \cap G$ and $Q_1^{n-1} \cap F$ would have some properties simplifying further consideration. For this purpose, apply the Baire Theorem to the covering of $F$ by the family $R$. Thus we obtain a set $U^1$ open in $Q_0^{n-1}$ and a set $M^1 \in R$ such that $\emptyset \neq U^1 \cap F \cap Q_0^{n-1\circ} \subset M^1$. For simplicity, we assume that the points of $M^1$ have the coordinate $x_1$ fixed. Hence there exist a cell $Q_1^{n-1} \subset U^1$ and a point $x_1^0 \in X_1$ such that

(6.3) $Q_1^{n-1\circ} \cap F \neq \emptyset$ *and* $Q_1^{n-1} \cap F \subset M^1 = [x \in X\colon x_1 = x_1^0]$, *where* $\mathrm{ord}_{x_1^0} X_1 \geqslant 3$.

Now we prove the following property of $Q_1^{n-1}$:

(6.4) *If a sequence of points $\{p^m\}$ selected from distinct components of $Q_1^{n-1} - M^1$ converges to $p$, then $p \in Q_1^{n-1\bullet}$.*

Given some $m$, suppose first that $p^m \notin \dot{Q}_1^{n-1\bullet}$ and let $H^m$ be the component of $Q_1^{n-1} - M^1$ containing that point. By (6.2), $\overline{H^m}$ is contained in a set of the family $R$. This set consists of points with a coordinate $x_i$ fixed, and $i \geqslant 2$ because $H^m \subset Q_1^{n-1} - M^1$ while $\emptyset \neq \overline{H^m} - H^m \subset M^1$. Therefore, $\overline{H^m}$ may be regarded as a subset of the Cartesian product of $n-1$ local dendrites and Lemma 7 applies. Consequently, if

$$C^m = [x \in \overline{H^m}\colon \bigwedge_{2 \leqslant i \leqslant n} x_i = p_i^m],$$

then for each point $x \in C^m \cap (H^m - Q_1^{n-1\bullet})$ there is an arc contained in $C^m$ with $x$ as an interior point. In particular, this holds for the point $p^m$.

On the other hand, since $C^m$ is homeomorphic with the projection $p_1(C^m)$ of itself, and therefore, by (6.1), with a closed subset of the dendrite $D_1$, it follows that each of its components must be a dendrite. Consequently, it must possess two end-points at least, and therefore there is a point $q^m \in C^m$ such that $\mathrm{ord}_{q^m} C^m = 1$ and $q_1^m \neq x_1^0$. Hence $q^m \in (\overline{H^m}-H^m) \cup Q_1^{n-1\bullet}$. However, since $\overline{H^m}-H^m \subset [x \in X\colon x_1 = x_1^0]$ in virtue of (6.3), it follows that the point $q^m$ must belong to the set $Q_1^{n-1\bullet}$.

In case where $p^m \in Q_1^{n-1\bullet}$, we set $q^m = p^m$. Clearly, we may assume that $\lim_{m\to\infty} q^m = q$. This implies that $q \in Q_1^{n-1\bullet}$. Since the points $p^m$, and consequently $q^m$, belong to the closures of distinct components of $Q_1^{n-1}-M^1$, follows that $p, q \in M^1$, whence $p_1 = x_1^0 = q_1$. From the definition of $C^m$ we infer that $q_i^m = p_i^m$ for $2 \leqslant i \leqslant n$, because $q^m \in C^m$. Thus $p_i = q_i$ for $1 \leqslant i \leqslant n$, so that $p = q \in Q_1^{n-1\bullet}$, which proves (6.4).

Now we can already find a point $x^0$, at which some cell contained in $Q_1^{n-1}$ will be bent. For this purpose, let us observe that the countability of the set of the ramification points of a local dendrite implies that the set of such points of $Q_1^{n-1}$ whose at least 3 coordinates (in the Cartesian product $X = \mathop{P}_{i=1}^{n} X_i$) are ramification points is a countable union of closed sets with dimension equal to $n-3$. The same may be said about the set of such points of this cell whose at least two coordinates in a fixed decomposition of $A$ into the Cartesian product $\mathop{P}_{i=1}^{n-1} A_i$ of local dendrites (cf. A1) are ramification points. Hence the union of these sets has dimension less than or equal to $n-3$. On the other hand, let us notice that the set $F$, being non-dense in $Q_1^{n-1}$, must separate this cell. Otherwise, (6.2) and the definition of $G$ imply that $Q_1^{n-1\circ} \subset G$, which contradicts (6.3). Hence $\dim(F \cap Q_1^{n-1\circ}) = n-2$, and therefore there exists a point $x^0 \in F \cap Q_1^{n-1\circ}$, whose at most two coordinates in the Cartesian product $\mathop{P}_{i=1}^{n} X_i$ and at most one in the Cartesian product $\mathop{P}_{i=1}^{n-1} A_i$ are ramification points. Observe at once that, by the formula $x^0 \in Q_1^{n-1\circ} \subset A$ and by A2 and (3.3), none of the coordinates of $x^0$ in these Cartesian products can be an end-point.

First we show that for the point $x^0$ which has been found conditions (i) and (ii) appearing in the definition of the bent cell are satisfied; the cell which is bent at the point $x^0$ will be defined later.

(ii) follows directly from the choice of $x^0$. To prove (i), first observe that, by the formula $x^0 \in F \cap Q_1^{n-1}$ and by (6.3), the first coordinate of $x^0$ coincides with the point $x_1^0$ considered above (which justifies our notation), and thereby $\mathrm{ord}_{x_1^0} X_1 \geqslant 3$. Since $x^0 \in F \cap Q_1^{n-1\circ}$, it follows from



(6.3) and (6.4) that there is a neighbourhood $U$ of $x^0$ in $Q_1^{n-1}$ such that every component of $U-M^1$ is a subset of $G$, $x^0$ being contained in its closure. Such components must exist by the definition of $F$, as $x^0 \in F$. From (6.2) and from the fact that, besides $x_1^0$, at most one coordinate of $x^0$ may be a ramification point we infer that all such components must be contained in a set $M^2 \in \boldsymbol{R}$ on which a coordinate $x_i$ with $i \geqslant 2$ is fixed. Therefore, we may assume that

$$M^2 = [x \in X\colon x_2 = x_2^0],$$

where $\mathrm{ord}_{x_2^0} X_2 \geqslant 3$. Thus (i) is satisfied under the substitutions $j = 1$ and $k = 2$. Moreover: $\overline{U} \subset M^1 \cup M^2$. However, any neighbourhood of $x^0$ in $U$ can be contained neither in $M^1$ nor in $M^2$.

Now let us notice that the set $M^1 \cup M^2$ is homeomorphic in a natural way with the Cartesian product $(X_1 \cup X_2) \times \mathop{P}_{i=3}^{n} X_i$, where the points $x_1^0$ and $x_2^0$ have been identified in the (disjoint) union $X_1 \cup X_2$. Clearly, in this decomposition only the first coordinate of $x^0$ is a ramification point. Since $x^0 \in Q_1^{n-1\circ}$, it follows that $\beta_{n-1}(x^0, \overline{U}) > 0$ and Lemma 6 applies to $\overline{U}$. Therefore there exists a cell $Q_2^{n-1} \subset \overline{U}$ constituting the Cartesian product of arcs in the above decomposition of $M^1 \cup M^2$ in which the respective coordinates of $x^0$ are interior points. Since the cell $Q_2^{n-1}$ can be contained neither in $M^1$ nor in $M^2$, it follows that it must have the form described in the definition of the bent cell. Thus we have proved that our cell $Q_0^{n-1}$ contains the cell $Q_2^{n-1}$ bent at the point $x^0$, which completes the proof of Lemma 8.

Lemma 9. *No set $A \in S$ can contain any bent cell.*

Proof. Suppose otherwise, i.e. let $Q_0^{n-1} \subset A$ be a cell bent at a point $x^0 \in Q_0^{n-1\circ}$. Then, by the definition of the bent cell, where, for convenience, it is assumed that $j = 1$ and $k = 2$, and the arcs $I_j = I_1$ and $I_k = I_2$ are renamed respectively as $I_1'$ and $I_2'$, we have:

$$\mathrm{ord}_{x_i^0} X_i \geqslant 3 \quad \text{for} \quad i = 1, 2, \qquad \mathrm{ord}_{x_i^0} X_i = 2 \quad \text{for} \quad 3 \leqslant i \leqslant n;$$

$$Q_0^{n-1} = [(x_1^0) \times I_2' \cup I_1' \times (x_2^0)] \times \mathop{P}_{i=3}^{n} I_i,$$

where $x_i^0 \in I_i'^{\bullet}$ for $i = 1, 2$, $x_i^0 \in I_i^\circ$ for $3 \leqslant i \leqslant n$.

Let $D = \mathop{P}_{i=1}^{n} D_i$ be the Cartesian product of dendrites constituting a neighbourhood of $x^0$ in $X$ and so small that $D \subset F$, where $F$ is a fixed set with the properties described in A3. Reducing—if necessary—the cell $Q_0^{n-1}$, however leaving its structure unchanged, we may assume that $Q_0^{n-1} \subset D$. Further, for $i = 1, 2$, let $D_i'$ denote the component of $D_i - x_i^0$ such that $I_i' - x_i^0 \subset D_i'$ and let

$$Z = [(x_1^0) \times \overline{D_2'} \cup \overline{D_1'} \times (x_2^0)] \times \mathop{P}_{i=3}^{n} D_i.$$

Since the sets $D_1'\times D_2'$ and $(D_1\times D_2)-(\overline{D_1'}\times\overline{D_2'}) = [D_1\times(D_2-\overline{D_2'})]\cup$ $\cup[(D_1-\overline{D_1'})\times D_2]$ are connected, it follows that:

(6.5) *$Z$ separates $D$ into two components*: $D_1'\times D_2'\times \mathop{P}\limits_{i=3}^{n} D_i$

$$\text{and } D-(\overline{D_1'}\times\overline{D_2'}\times \mathop{P}\limits_{i=3}^{n} D_i) = [(D_1\times D_2)-(\overline{D_1'}\times\overline{D_2'})]\times \mathop{P}\limits_{i=3}^{n} D_i.$$

In the next part of the proof we shall construct an $n$-cell with properties yielding a contradiction. First of all, we shall construct its "base" $Q_1^{n-1}$, satisfying the following conditions:

$$(6.6)\qquad \begin{gathered} Q_1^{n-1}\subset A\cap[D-(D_1'\times D_2'\times \mathop{P}\limits_{i=3}^{n} D_i)], \\ x^0\in Q_1^{n-1\circ}, \quad \dim(Q_1^{n-1}\cap Z)\leqslant n-2. \end{gathered}$$

For this purpose, let us consider the cell

$$Q_1^n = I_1''\times I_2''\times \mathop{P}\limits_{i=3}^{n} I_i,$$

where, for $i=1,2$, $I_i''$ is an arc contained in $D_i-D_i'$ and such that $x_i^0\in I_i''^{\circ}$. Such arcs exist, because $\operatorname{ord}_{x_i^0} X_i\geqslant 3$ for $i=1,2$. Since $x^0\in Q_1^{n\circ}\cap A$, it follows from Lemma 4 and condition A3 (iii) that no neighbourhood of $x^0$ in this cell can be contained in the closure of any component of $F-A$. Consequently $\alpha_0(x^0, Q_1^n-A)>0$, and therefore, by Lemma 2, $\beta_{n-1}(x^0, Q_1^n\cap A)>0$. In view of condition (ii) appearing in the definition of the bent cell, Lemma 6 applies to the set $Q_1^n\cap A\subset A$. Thus there exists a cell $Q_1^{n-1}\subset Q_1^n\cap A$ such that $x^0\in Q_1^{n-1\circ}$. Since the definition of $Q_1^n$ yields $Q_1^n\subset D-(D_1'\times D_2'\times \mathop{P}\limits_{i=3}^{n} D_i)$ and $Q_1^n\cap Z = (\overline{D_1'}\cap I_1'')\times$ $\times(\overline{D_1'}\cap I_2'')\times \mathop{P}\limits_{i=3}^{n} I_i = (x_1^0)\times(x_2^0)\times \mathop{P}\limits_{i=3}^{n} I_i$, it follows that (6.6) is satisfied.

Next we shall construct an arc $L^1$ satisfying the conditions

(6.7) $x^0\in L^{1\bullet}$, $L^1-x^0\subset D_1'\times D_2'\times \mathop{P}\limits_{i=3}^{n} D_i$ *and* $L^1\subset I^0\in\mathfrak{J}$, *where* $\mathfrak{J}$ *is a fixed family of arcs with the properties described in* A3 (v).

For this purpose, let us consider the cell

$$Q_2^n = I_1'\times I_2'\times \mathop{P}\limits_{i=3}^{n} I_i.$$

Since its boundary contains our bent cell $Q_0^{n-1}$ and $x^0\in Q_0^{n-1\circ}\subset A$, it follows from A3 (v) that there is an arc $L^1\subset Q_2^n$ such that $x^0\in L^{1\bullet}$ and $L^1\subset I^0\in\mathfrak{J}$. Consequently $L^1-x^0\subset I^0-x^0\subset X-A$, and therefore the inclusion $Q_0^{n-1}\subset A$ implies that

$$L^1-x^0\subset Q_2^n-Q_0^{n-1} = (I_1'-x_1^0)\times(I_2'-x_2^0)\times \mathop{P}\limits_{i=3}^{n} I_i\subset D_1'\times D_2'\times \mathop{P}\limits_{i=3}^{n} D_i.$$

Thus (6.7) is satisfied.

Now let $G^0$ denote the component of $F-A$ containing $I^0-x^0$, where $I^0\in\mathfrak{J}$ is the arc appearing in (6.7) (cf. A3 (v) to determine this component). Applying A3 (iv) to the cell $Q_1^{n-1}\subset A$ previously constructed (see (6.6)) and to this component $G^0$, we find a cell $Q_3^n$ satisfying the conditions:

$$(6.8)\qquad Q_3^n\subset\overline{G^0} \quad\text{and}\quad Q_3^n\cap A = Q_1^{n-1}\subset Q_3^{n\bullet}.$$

This is just the $n$-cell mentioned above, with properties yielding a contradiction, as we shall proceed to show.

First we prove that $\beta_{n-1}(x^0, Q_3^n\cap Z)>0$. For this purpose, observe that, by the formula $x^0\in Q_1^{n-1\circ}\subset Q_3^{n\bullet}\cap A$ and by A3 (v), there is an arc $L^2\subset Q_3^n$ contained in an arc belonging to $\mathfrak{J}$ and such that $x^0\in L^{2\bullet}$. The inclusion $Q_3^n\subset\overline{G^0}$ and the properties of the family $\mathfrak{J}$ imply that $L^2$ must be a subarc of the arc $I^0$. Therefore, reducing $L^2$ when necessary, we may assume that $L^2\subset L^1$ and, by (6.7), we obtain $L^2-x^0\subset D_1'\times D_2'\times$ $\times \mathop{P}\limits_{i=3}^{n} D_i$. It follows that the closure of the set $Q_3^n\cap(D_1'\times D_2'\times \mathop{P}\limits_{i=3}^{n} D_i)$ contains $x^0$. By (6.5), this set is both open and closed in $Q_3^n\cap D-Z$, being disjoint from $Q_1^{n-1}$ in view of (6.6). Since $x^0\in Z$, $D$ is a neighbourhood of $x^0$ in $X$ and $x^0\in Q_1^{n-1\circ}\subset Q_3^{n\bullet}$, applying lemma 3 we conclude that $\beta_{n-1}(x^0, Q_3^n\cap Z)>0$.

Now, taking into consideration the definition of $Z$, one sees that this set decomposes in a natural way into the Cartesian product of $n-1$ dendrites and that all the coordinates of $x^0$ in this product have the order equal to 2. Hence, lemma 6 applies to the set $Q_3^n\cap Z\subset Z$, whose $(n-1)$-th local Betti number at $x^0$ has been proved to be greater than 0. Therefore there exists a cell $Q_2^{n-1}\subset Q_3^n\cap Z$ constituting the Cartesian product of arcs in this decomposition of $Z$ and such that $x^0\in Q_2^{n-1\circ}$. On the other hand, let us notice that $Z$ contains our bent cell $Q_0^{n-1}$, which also constitutes the Cartesian product of arcs in this decomposition and satisfies the condition $x^0\in Q_0^{n-1\circ}$. It follows that the cell $Q_2^{n-1}$ can be reduced so as to be contained in $Q_0^{n-1}$. Consequently

$$Q_2^{n-1}\subset Q_3^n\cap A\cap Z,$$

because $Q_0^{n-1}\subset A$. Thus we obtain a contradiction, since (6.6) and (6.8) imply that $\dim(Q_3^n\cap Z\cap A) = \dim(Q_1^{n-1}\cap Z)\leqslant n-2$. This proves that the set $A$ cannot contain any bent cell and thereby completes the proof of the lemma.

Lemmas 8 and 9 having been proved, it follows immediately that:

(6.9) *For each cell $Q^{n-1} \subset A \in S$ exactly one of the projections $p_i(Q^{n-1})$, where $1 \leqslant i \leqslant n$, reduces to a single point.*

**7. Proof of the inclusion** $S \subset \boldsymbol{R}$. Let $A$ be a given set of the family $S$. In view of A1 we may assume that $A$ is the Cartesian product $\mathop{P}\limits_{i=1}^{n-1} A_i$ of local dendrites. Select a cell $Q_0^{n-1} = \mathop{P}\limits_{i=1}^{n-1} I_i$, where $I_i \subset A_i$. By (6.9), for a fixed $j$, where $1 \leqslant j \leqslant n$, we have $p_j(Q_0^{n-1}) = (x_j^0) \subset X_j$. Then $\operatorname{ord}_{x_j^0} X_j \geqslant 3$. Otherwise, in view of A2 and (3.5), for each point $x \in Q_0^{n-1\circ}$ a coordinate other than $x_j$ would be a ramification point, and therefore the fixation of $x_j$ and the countability of the set of the ramification points of a local dendrite would imply that $\dim(Q_0^{n-1}) \leqslant n-2$. Therefore:

$$Q_0^{n-1} \subset M = [x \in X\colon\ x_j = x_j^0] \in \boldsymbol{R}.$$

Let $a \in A$. For each $i$, where $1 \leqslant i \leqslant n-1$, there is an arc $I_i' \subset A_i$ containing the $i$-th coordinate of $a$ and a subarc of the arc $I_i$. Hence, if $Q^{n-1} = \mathop{P}\limits_{i=1}^{n-1} I_i'$, then $a \in Q^{n-1} \subset A$ and $\dim(Q^{n-1} \cap Q_0^{n-1}) = n-1$. In virtue of (6.9), for some $k$, where $1 \leqslant k \leqslant n$, the set $p_k(Q^{n-1})$ reduces to a single point. If we had $k \neq j$, then on the set $Q_0^{n-1} \cap Q^{n-1}$ the coordinates $x_j$ and $x_k$ would be fixed, which is impossible considering the dimension of this set. Therefore $p_j(Q^{n-1}) = (x_j^0)$, so that $a \in Q^{n-1} \subset M$, which proves that $A \subset M$.

By A3 (i), (ii), each sufficiently small neighbourhood of $A$ in $X$ is separated by the set $A$ itself. In particular, a set homeomorphic (in a natural way) with the Cartesian product of a connected neighbourhood of $A$ in $M$ and a dendrite must be separated by $A$. However, this is impossible when $A \subset M \neq A$, and therefore $A = M \in \boldsymbol{R}$, which proves the inclusion:

(7.1) $$S \subset \boldsymbol{R}.$$

(5.2) and (7.1) immediately imply that

(7.2) *The families $\boldsymbol{R}$ and $S$ coincide.*

**8. Proof of Theorem 1 in the case where the space $X$ is connected.** Let $X$ be a connected space satisfying to the assumptions of our Theorem. Then each (non-trivial) at most 1-dimensional factor of any Cartesian decomposition of $X$, as it is homeomorphic with a retract of $X$, must be a connected 1-dimensional ANR; and therefore a local dendrite (cf. the beginning of Section 3). Hence the space $X$ decomposes into the Cartesian product of local dendrites and the uniqueness of this



decomposition must be proved. For this purpose, we introduce the following definition: Given a Cartesian product $\mathop{P}\limits_{i=1}^{n} X_i$ and a space $D$, the number of the subscripts $i$ such that $X_i$ is homeomorphic with $D$ will be called the *multiplicity* of $D$ in that Cartesian product. It is clear that our theorem results at once from the following proposition:

($\mathrm{T}_m$) *For each $n \geqslant 2$ the following implication holds: If the Cartesian products $\mathop{P}\limits_{i=1}^{n} X_i$ and $\mathop{P}\limits_{i=1}^{n} X_i'$ of local dendrites are homeomorphic and if in the sequence $(X_1, X_2, \ldots, X_n)$ and also in $(X_1', X_2', \ldots, X_n')$ at most $m$ of the terms have ramification points, then, for each local dendrite $D$, the multiplicities of $D$ in those Cartesian products are identical.*

We prove $\mathrm{T}_m$ using induction on $m$.

$\mathrm{T}_1$ follows at once from a theorem of K. Borsuk (see [2], p. 159, Corollary).

Now, given some $m \geqslant 2$, suppose $T_{m-1}$ to be true and consider two Cartesian products, $\mathop{P}\limits_{i=1}^{n} X_i$ and $\mathop{P}\limits_{i=1}^{n} X_i'$, satisfying the assumptions of $\mathrm{T}_m$. We may assume that exactly $k$ of the local dendrites $X_i$, where $2 \leqslant k \leqslant m$, have some ramification points. Consider the family $\boldsymbol{R}$ consisting of subsets of $\mathop{P}\limits_{i=1}^{n} X_i$ as defined in Section 4 and consider the Cartesian decompositions, which have been determined in a natural way, of the sets belonging to $\boldsymbol{R}$. We distinguish 2 cases: when all the sets of the family $\boldsymbol{R}$ have the same decomposition (i.e. when all factors $X_i$ having ramification points are homeomorphic) and when there are 2 sets, $M, N \in \boldsymbol{R}$, having different decompositions. In the first case the multiplicity of a local dendrite $D$ in the product $\mathop{P}\limits_{i=1}^{n} X_i$ is greater by 1 than its multiplicity in the decomposition of any set $M \in \boldsymbol{R}$ if the latter multiplicity is positive and if $D$ has some ramification points, and it is equal to that multiplicity in the remaining case. In the second case the multiplicity of $D$ in the product $\mathop{P}\limits_{i=1}^{n} X_i$ is equal to its maximal multiplicity in the decompositions of the sets belonging to $\boldsymbol{R}$.

Now let $\boldsymbol{R}'$ be the respective family consisting of subsets of $\mathop{P}\limits_{i=1}^{n} X_i'$. By the topological character of the family $S$ as defined in Section 4 and by (7.2), there is a one-to-one correspondence between the sets of the families $\boldsymbol{R}$ and $\boldsymbol{R}'$, the corresponding sets being homeomorphic. Hence $\boldsymbol{R}' \neq \emptyset$ and, since the sets belonging to $\boldsymbol{R}'$ cannot be manifolds, it follows that also at least two of the local dendrites $X_i'$ must have ramification points. By the hypothesis of induction, the multiplicities of each local dendrite $D$ in the decompositions of corresponding sets $M \in \mathbf{R}$ and $M' \in \mathbf{R}'$ must be equal. These multiplicities determine in the same way the multiplicities of $D$ in the products $\mathop{P}\limits_{i=1}^{n} X_i$ and $\mathop{P}\limits_{i=1}^{n} X'_i$; it follows that the latter must also be equal. This completes the proof of $T_m$, and therefore Theorem 1 in the case considered has been proved.

**9. A proof of Theorem 1 in the general case.** K. Borsuk has shown (see [3], p. 148) that the uniqueness of the decomposition of an arbitrary polyhedron into the Cartesian product of topologically prime spaces of dimension $\leqslant 1$ follows from the fact that such decompositions are unique for connected polyhedra (that proof was already utilized in [8]). Since the only property of polyhedra used in that proof is the fact that they have a finite number of components, it follows that the proof of K. Borsuk carries over *mutatis mutandis* to the (compact) ANR-spaces. Thus the proof of our Theorem is complete.

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# Corrections to my paper "On a certain class of abstract algebras"

Fundamenta Mathematicae 54 (1964), pp. 115-124

by

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In the above paper the following changes should be made:

(1) In example (a) on p. 116 it must be assumed that $L$ satisfies the cancellation law, i.e. from $rx = 0$ with $r \in R$, $x \in L$ follows $r = 0$ or $x = 0$.

(2) In the example (b) on the same page condition (iii) is not sufficient and must be replaced by the following two:

"(iii) For any two elements $f, g$ of $S$ there exists such an element $h$ in $S$ that $f = gh$ or $g = fh$.

(iv) For every $g \in S$ from $g(a) = g(b)$ with $a, b \in X$ follows $a = b$".

(3) In the Remark on p. 122 "left-cancellation" must be replaced by "right-cancellation" since it is obviously that fact which is proved there.

(4) In view of (3) theorem IV on p. 122 is false, because there may be no left-cancellation law in $S$. What is really proved there is the following fact: *If $A$ is a $v^{**}$-algebra in which every operation depends on at most one variable, then either $A$ consists of algebraic constants only or there exists a semigroup $S$ of transformations of $X$ such that the identical transformation belongs to $S$, the right-cancellation law holds and for any $f, g, F, G$ in $S$ from $fg = FG$ it follows that with a suitable $H \in S$, $g = HG$ or $G = Hg$. Moreover, every algebraic operation has the form given in* (b), pp. 116-117.

However, not every algebra so constructed must be a $v^{**}$-algebra and so this theorem fails as a representation theorem.

The main results of the paper are unaffected by these changes.

I am indebted to Professor K. Urbanik for calling my attention to these facts.

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