The global dimension of the group rings of abelian groups II

by

S. Balcerzyk (Toruń)

In the present paper we compute the global dimensions of group rings $R(\Pi)$ where Π is an abelian torsion-free group and R is a commutative Noetherian ring. We show that $R(\Pi)$ regarded as an R-algebra satisfies

$$(*)\qquad \text{gl.}\dim R(\Pi) = \text{gl.}\dim R + \dim R(\Pi)$$

for commutative Noetherian rings R and

$$\dim R(\Pi) = \begin{cases} r(\Pi)+1 & \text{if } \Pi \text{ is a torsion-free not finitely generated abelian group,} \\ r(\Pi) & \text{if } \Pi \text{ is finitely generated torsion-free abelian group} \end{cases}$$

for an arbitrary commutative ring R ($r(\Pi) = \text{rank}(\Pi)$).

The above theorem generalizes the author's former results, published in [1] and concerning the case of R being the ring of rational integers.

Formula (*) does not hold for arbitrary (non-Noetherian) rings; for such rings we have the inequalities

$$\text{gl.}\dim R + r(\Pi) \leqslant \text{gl.}\dim R(\Pi) \leqslant \text{gl.}\dim R + r(\Pi)+1\,.$$

1. In this section we prove some preliminary lemmas. All rings and groups are assumed to be commutative.

LEMMA 1. *If A is an R-module and*

$$0 \leftarrow A \leftarrow Q_0 \leftarrow \ldots \leftarrow Q_s \leftarrow 0$$

is an R-projective resolution of A and $\text{Im}(Q_s \to Q_{s-1})$ is not a direct summand of Q_{s-1}, then $\dim_R A = s$.

Proof. If we had $\dim_R A = k < s$, then $\text{Im}(Q_k \to Q_{k-1})$ would be R-projective and Q_i, $i = k, k+1, \ldots, (s-1)$, would admit direct summands $\text{Im}(Q_{i+1} \to Q_i)$, contrary to our assumption.

LEMMA 2. *If R is a commutative Noetherian ring, B is a finitely generated R-module and $\dim_R B = 1$, then there exist finitely generated R-projective module Q and an R-free resolution*

$$0 \leftarrow B \oplus Q \leftarrow F_0 \overset{a}{\leftarrow} F_1 \leftarrow 0 \tag{1}$$

such that free generators $w_1, \ldots, w_m$ of F_0 and free generators $v_1, \ldots, v_n$ of F_1 satisfy

$$a(v_i) = \sum_{j=1}^{m} r_{ij} w_j \qquad (i = 1, 2, \ldots, n)$$

and all ideals

$$\mathfrak{a}_i = (r_{i1}, \ldots, r_{im})$$

are different from R.

Proof. Since $\dim_R B = 1$, then there exists a resolution

$$0 \leftarrow B \leftarrow P_0 \leftarrow P_1 \leftarrow 0 \tag{2}$$

and P_0, P_1 are finitely generated R-projective modules; thus there exist finitely generated R-projective modules Q, Q_1 such that $P_1 \oplus Q_1$, $P_0 \oplus Q_1 \oplus Q$ are R-free. If we add the exact sequences (2) and

$$0 \leftarrow 0 \leftarrow Q_1 \leftarrow Q_1 \leftarrow 0$$

$$0 \leftarrow Q \leftarrow Q \leftarrow 0 \leftarrow 0,$$

then we get an R-free resolution of type (1). Now we subject Q to the additional condition that the rank $r(F_1)$ is minimal. We show that such free resolution satisfies all conditions of our lemma. In fact, if one of ideals, say $\mathfrak{a}_1$, were equal to R, then $a(v_1)$ would generate a direct summand of F_0 and we would get an R-resolution

$$0 \leftarrow B \oplus Q \leftarrow F_0/Ra(v_1) \leftarrow F_1/Rv_1 \leftarrow 0$$

with projective $F_0/Ra(v_1)$. Adding an appropriate projective module to the first and the second terms, we get an exact sequence

$$0 \leftarrow B \oplus Q' \leftarrow F_0' \leftarrow Rv_2 \oplus \ldots \oplus Rv_n \leftarrow 0$$

with Q' projective and F_0' free, contrary to our choice of F_1 of minimal rank.

Using the notation of [1], we have

LEMMA 3. *If R is a ring and Π is a non cyclic torsion-free group of rank 1 generated by elements $\sigma_1, \sigma_2, \ldots$ such that $\sigma_{n+1}^{t_{n+1}} = \sigma_n$ ($n = 1, 2, \ldots$; $t_{n+1} > 1$), then the sequence*

$$0 \leftarrow R \overset{\varepsilon}{\leftarrow} P_0 \overset{d_1'}{\leftarrow} P_1 \overset{d_2'}{\leftarrow} P_2 \leftarrow 0,$$

where

$P_0 = R(\Pi)$,

P_1 *is an $R(\Pi)$-free module on free generators $x_1, x_2, \ldots$,*

P_2 *is an $R(\Pi)$-free module on free generators $y_1, y_2, \ldots$,*

ε *is the unit augmentation,*

$d_1'(x_n) = 1 - \sigma_n$,

$d_2'(y_n) = x_n - s_{n+1}x_{n+1}$ *with* $s_{n+1} = 1 + \sigma_{n+1} + \sigma_{n+1}^2 + \ldots + \sigma_{n+1}^{t_{n+1}-1}$,

is an $R(\Pi)$-free resolution of R (Π operates trivially on R) and $\dim_{R(\Pi)} R = 2$.

This lemma was proved in [1], p. 298 for $R = Z$, but the same proof applies to an arbitrary ring R.

LEMMA 4. *If R is a Noetherian ring, B is a finitely generated R-module, $\dim_R B = 1$ and Π is a non cyclic torsion-free group of rank 1, then $\dim_{R(\Pi)} B = 3$ (Π operates trivially on B).*

Proof. By Lemma 3 it follows that for any R-projective module Q we have $\dim_{R(\Pi)} Q \leqslant 2$; then to prove the lemma it is sufficient to prove $\dim_{R(\Pi)} (B \oplus Q) = 3$. Let Q satisfy all the conditions of Lemma 2; we then have an R-free resolution F

$$0 \leftarrow B' \leftarrow F_0 \overset{d_1''}{\leftarrow} F_1 \leftarrow 0$$

of $B' = B \oplus Q$.

Let $S = P \otimes_R F$ be the tensor product of the $R(\Pi)$-resolution P of R and the R-resolution F of B'. The complex S is $R(\Pi)$-free and acyclic because $H_0(P) = R$ is an R-free module. By Lemma 1 it is sufficient to prove that $\mathrm{Im}(S_3 \to S_2)$ is not a direct summand of S_2.

Let us assume that $\mathrm{Im}(S_3 \overset{d_3}{\to} S_2)$ is a direct summand of S_2; then there exists a homomorphism ϱ: $S_2 \to S_3$ such that ϱd_3 is the identity on S_3. If we write $z_n = y_n \otimes v_1$, then the module

$$W = R(\Pi)\{z_1, z_2, \ldots\}$$

is $R(\Pi)$-free and is a direct summand of S_3. Let π be the natural projection of S_3 onto W. Thus we have

$$\begin{aligned} z_n &= \pi(z_n) \\ &= \pi\varrho d_3(y_n \otimes v_1) \\ &= \pi\varrho[(d_2' y_n) \otimes v_1 + y_n \otimes (d_1'' v_1)] \\ &= \pi\varrho[(x_n - s_{n+1}x_{n+1}) \otimes v_1 + y_n \otimes (d_1'' v_1)], \end{aligned}$$

and for the elements $\xi_n = \pi\varrho(x_n \otimes v_1) \in W$ we get the relations

$$z_n = \xi_n - s_{n+1}\xi_{n+1} + \pi\varrho[y_n \otimes (d_1'' v_1)].$$

Since $d_1''v_1 \in \mathfrak{a}_1 F_0$, we have $\pi\varrho[y_n \otimes (d_1''v_1)] \in R(\Pi)\mathfrak{a}_1 W$, and writing

$$\bar{R} = R/\mathfrak{a}_1, \qquad \overline{W} = W/R(\Pi)\mathfrak{a}_1 W$$

we can easily see that $\overline{W}$ is an $\bar{R}(\Pi)$-free module on free generators $\bar{z}_1, \bar{z}_2, \ldots$ and that the elements $\bar{\xi}_n \in \overline{W}$ satisfy the system of equations

(3) $$\bar{z}_n = \bar{\xi}_n - \bar{s}_{n+1}\bar{\xi}_{n+1} \qquad (n = 1, 2, \ldots).$$

The elements $\bar{s}_{n+1}$ are neither units nor zero divisors in $\bar{R}(\Pi)$; we then get a contradiction with (1.4) of [1], which states that system (3) has no solutions in $\overline{W}$.

2. In this section we prove

THEOREM 1. *If R is a commutative Noetherian ring and Π is an abelian torsion-free group which is not finitely generated, then*

$$\mathrm{gl.dim}\, R(\Pi) = \mathrm{gl.dim}\, R + r(\Pi) + 1,$$

and if A is such an R-module that $\dim_R A = \mathrm{gl.dim}\, R$, *then*

$$\mathrm{gl.dim}\, R(\Pi) = \dim_{R(\Pi)} A$$

(Π operates trivially on A).

If R is a commutative ring and Π is an abelian finitely generated torsion free group, then

$$\mathrm{gl.dim}\, R(\Pi) = \mathrm{gl.dim}\, R + r(\Pi),$$

and if A is such an R-module that $\dim_R A = \mathrm{gl.dim}\, R$, *then*

$$\mathrm{gl.dim}\, R(\Pi) = \dim_{R(\Pi)} A$$

(Π operates trivially on A).

Proof. The second part of the theorem was proved in [1].

If $\mathrm{gl.dim}\, R = \infty$ or $r(\Pi) = \infty$, then the theorem is obvious.

To prove the first part of our theorem let us start with a non-cyclic group Π of rank 1 and a commutative Noetherian ring with $\mathrm{gl.dim}\, R = s < \infty$.

If $s = 0$, then R is a direct product of a finite number of fields, and we can consider the case where R is a field. By Lemma 3 we have $\mathrm{gl.dim}\, R(\Pi) \geqslant \dim_{R(\Pi)} R = 2$ and $R(\Pi)$ is a union of an increasing sequence of rings of global dimension 1; then $\mathrm{gl.dim}\, R(\Pi) \leqslant 2$.

If $s > 0$ and $\dim_R A = s$, then there exists an R-projective resolution

$$0 \leftarrow A \leftarrow Q_0 \leftarrow Q_1 \leftarrow \ldots \leftarrow Q_s \leftarrow 0.$$

Let us write $B_{-1} = A$, $B_i = \mathrm{Im}(Q_{i+1} \to Q_i)$, $i = 0, 1, \ldots, (s-1)$; we then have exact sequences of R-modules

(4) $$0 \leftarrow B_i \leftarrow Q_{i+1} \leftarrow B_{i+1} \leftarrow 0 \qquad (i = -1, 0, 1, \ldots, (s-1)).$$

We can consider these sequences as exact sequences of $R(\Pi)$-modules with trivial Π-operators. Since $\dim_{R(\Pi)} R = 2$, we have $\dim_{R(\Pi)} Q_i \leqslant 2$ and for any $m \geqslant 3$ we have

$$\mathrm{Ext}^m_{R(\Pi)}(B_{i+1}, X) \approx \mathrm{Ext}^{m+1}_{R(\Pi)}(B_i, X) \quad \text{for} \quad i = -1, 0, 1, \ldots, (s-1).$$

Consequently

$$\mathrm{Ext}^{s+2}_{R(\Pi)}(A, X) = \mathrm{Ext}^{s+2}_{R(\Pi)}(B_{-1}, X) \approx \mathrm{Ext}^{3}_{R(\Pi)}(B_{s-2}, X).$$

We know that $\dim_R B_{s-2} = 1$ and B_{s-2} is a finitely generated R-module; then by Lemma 4 it follows that there exists an $R(\Pi)$-module X such that $\mathrm{Ext}^3_{R(\Pi)}(B_{s-2}, X) \neq 0$ and thus $\dim_{R(\Pi)} A \geqslant s+2$. On the other hand,

$$\mathrm{gl.dim}\, R(\Pi) \leqslant 1 + \mathrm{gl.dim}\, R(Z) = s+2$$

because the ring $R(\Pi)$ is a union of an increasing sequence of rings isomorphic to $R(Z)$ (see (1.3) of [1]). Consequently

$$\mathrm{gl.dim}\, R(\Pi) = \dim_{R(\Pi)} A = \mathrm{gl.dim}\, R + r(\Pi) + 1$$

for groups Π of rank 1.

Let us assume that the theorem holds for groups of rank $< r$ and let Π be non-finitely generated torsion-free group of rank r. It is easy to see that the group Π contains a subgroup Π_0 of rank r which is not finitely generated and is an extension of a group $\Pi_0' \approx Z$ by a torsion-free group Π_0'' of rank $r-1$. By (1.3) of [1] we can deduce that

$$s + r \leqslant \mathrm{gl.dim}\, R(\Pi) \leqslant s + r + 1$$

and it is sufficient to prove that $\mathrm{gl.dim}\, R(\Pi) \geqslant s+r+1$.

For any $R(\Pi_0'')$-module A and an $R(\Pi_0)$-module C we have a spectral sequence

(5) $$\mathrm{Ext}^p_{R(\Pi_0'')}(A, \mathrm{Ext}^q_{R(\Pi_0')}(R, C)) \underset{p}{\Rightarrow} \mathrm{Ext}^n_{R(\Pi_0)}(A, C)$$

(see [2], Chapter XVI, Theorem 6.1). We take for A such an R-module with trivial Π_0''-operators that $\dim_R A = \mathrm{gl.dim}\, R$ and for C such an $R(\Pi_0)$-module with trivial Π_0'-operators that $\mathrm{Ext}^{s+r}_{R(\Pi_0'')}(A, C) \neq 0$. For an $R(\Pi_0')$-module R we have an $R(\Pi_0')$-free resolution

$$0 \leftarrow R \leftarrow R(\Pi_0') \xleftarrow{1-\sigma_0'} R(\Pi_0') \leftarrow 0,$$

where σ_0' is a generator of Π_0'. Thus we have

$$\mathrm{Ext}^q_{R(\Pi_0')}(R, C) = \begin{cases} 0 & \text{for} \quad q > 1, \\ C & \text{for} \quad q = 1 \end{cases}$$

and

$$\mathrm{Ext}^k_{R(\Pi_0'')} = 0 \quad \text{for} \quad k > s + r.$$

The "maximum term principle" of spectral sequences yields

$$\mathrm{Ext}^{s+r+1}_{R(\Pi_0)}(A, C) \approx \mathrm{Ext}^{s+r}_{R(\Pi_0'')}\big(A, \mathrm{Ext}^{1}_{R(\Pi_0')}(R, C)\big) = \mathrm{Ext}^{s+r}_{R(\Pi_0'')}(A, C) \neq 0 ;$$

thus

$$\mathrm{gl.dim}\, R(\Pi_0) \geqslant \dim_{R(\Pi_0)} A \geqslant s+r+1$$

and the theorem follows.

It is easy to see that Theorem 1 does not hold for arbitrary non-Noetherian rings. In fact, if we put $R = K(\Pi)$ where K is a Noetherian ring and Π is an abelian torsion-free group of finite rank which is not finitely generated, then

$$\mathrm{gl.dim}\, R(\Pi) = \mathrm{gl.dim}\, K(\Pi\times\Pi) = \mathrm{gl.dim}\, K + 2r(\Pi)+1 ,$$

$$\mathrm{gl.dim}\, R + r(\Pi)+1 = \mathrm{gl.dim}\, K(\Pi) + r(\Pi)+1 = \mathrm{gl.dim}\, K + 2r(\Pi)+2 .$$

In general we have the inequalities

$$\mathrm{gl.dim}\, R + r(\Pi) \leqslant \mathrm{gl.dim}\, R(\Pi) \leqslant \mathrm{gl.dim}\, R + r(\Pi)+1 .$$

If R is a Noetherian ring, then

$$\mathrm{gl.dim}\, R = \mathrm{w.gl.dim}\, R ;$$

thus for Noetherian rings the first formula of Theorem 1 takes the form

$$\mathrm{gl.dim}\, R(\Pi) = \mathrm{w.gl.dim}\, R + r(\Pi)+1 .$$

This formula does not hold for arbitrary non-Noetherian rings (take R with $\mathrm{w.gl.dim}\, R = 0$ and $\mathrm{gl.dim}\, R > 1$).

3. It is known (see [2], Chapter X, § 6) that for any commutative ring R we have

$$\dim R(\Pi) = \dim_{R(\Pi)} R .$$

Using the spectral sequence (5) we get by similar arguments (starting with Lemma 3).

THEOREM 2. *If R is a commutative ring and Π is an abelian torsion-free group, then*

$$\dim R(\Pi) = \begin{cases} r(\Pi)+1 & \text{if } \Pi \text{ is not finitely generated group,} \\ r(\Pi) & \text{if } \Pi \text{ is finitely generated group,} \end{cases}$$

where $R(\Pi)$ is considered as R-algebra.

In paper [3] the following properties of an R-algebra Γ were studied (R is a commutative ring):

(P_1) *for every R-algebra Λ*

$$\mathrm{f.l.gl.dim}\, \Lambda\otimes\Gamma = \dim\Gamma + \mathrm{f.l.gl.dim}\, \Lambda$$

and

$$\mathrm{l.gl.dim}\, \Lambda\otimes\Gamma = \dim\Gamma + \mathrm{l.gl.dim}\, \Lambda ;$$

(P_2) *for every R-algebra Λ*

$$\dim \Lambda\otimes\Gamma = \dim\Lambda + \dim\Gamma ;$$

(P_3) *if R is a K-algebra, then*

$$K\text{-}\dim\Gamma = R\text{-}\dim\Gamma + K\text{-}\dim R ;$$

and for commutative Γ

(P_4) *if Λ is Γ-algebra satisfying*

$$H^R_r(\Gamma, \Lambda\otimes\Lambda^*) = 0 \text{ for } r>0, \qquad \Gamma\text{-}\dim\Lambda < \infty ,$$

then

$$R\text{-}\dim\Lambda = R\text{-}\dim\Gamma + \Gamma\text{-}\dim\Lambda .$$

It was proved in [3] that the R-algebra $\Gamma = R[x_1, \ldots, x_n]$ of polynomials in n indeterminates has properties (P_1), (P_2), (P_3) and (P_4).

It is easy to check that if we put $\Gamma = R(\Pi)$ and Π is an abelian torsion-free group of finite rank which is not finitely generated and if we take for Λ an R-algebra $R(\Pi')$ with Π' of the same type as Π, then Theorems 1 and 2 imply that all the left side terms are smaller by one than the right side terms. Consequently no property (P_i), $i = 1, 2, 3, 4$, holds for $\Gamma = R(\Pi)$.

References

[1] S. Balcerzyk, *The global dimension of the group rings of abelian groups*, Fund. Math. 55 (1964), pp. 293-301.

[2] H. Cartan, S. Eilenberg, *Homological algebra*, Princeton 1956.

[3] S. Eilenberg, A. Rosenberg, D. Zelinsky, *On the dimension of modules and algebras* (VIII), Nagoya Math. Journal 12 (1957), pp. 71-93.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
UNIWERSYTET MIKOŁAJA KOPERNIKA W TORUNIU
INSTITUTE OF MATHEMATICS, THE POLISH ACADEMY OF SCIENCES
NICOLAS COPERNICUS UNIVERSITY IN TORUŃ

Reçu par la Rédaction le 31. 12. 1964