Monotone relations which preserve arcs and acyclicity

by

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This paper is dedicated
to Professor A. D. Wallace
on the occasion of his sixtieth birthday,
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1. Introduction. A relation from X to Y is a multifunction from X to Y: precisely, R is a *relation* from X to Y iff $R \subset X \times Y$ and the first projection of R is all of X. The following are some long-known theorems [9] about monotone functions: if $f\colon X \to Y$ is a non-constant continuous monotone function and if Y is Hausdorff, then X an arc (simple closed curve, dendrite, unicoherent space) implies $f(X)$ an arc (simple closed curve, dendrite, unicoherent space). In this paper we present similar theorems about monotone relations.

2. Definitions and preliminaries. Let $R \subset X \times Y$; if $A \subset X$, AR will denote $\{y \in Y \mid (a, y) \in R$ for some $a \in A\}$, and AR will be called the *image of A under R*; if $B \subset Y$, RB will denote $\{x \in X \mid (x, b) \in R$ for some $b \in B\}$, and RB will be called the *inverse image of B under R*. We will use $\square$ to denote the empty set, A^* to denote the closure of A, and $A \backslash B$ to denote $\{a \in A \mid a \notin B\}$.

A relation R from X to Y is defined to be *monotone* iff Ry is connected for each $y \in Y$ and *noninclusive* iff $Ry \not\subset Ry'$ whenever $y \neq y' \in XR$. This latter condition is fairly restrictive and seems to be useful only when X has some order characterization. It generalizes the fact that for a function $f\colon X \to Y$, $f^{-1}(y) \cap f^{-1}(y') = \square$ whenever $y \neq y'$. So far, the only equivalence we have found for noninclusivity is the following simple one.

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LEMMA A. *If R is a relation from X to Y, R is noninclusive iff $(X\backslash Ry)R = XR\backslash y$ for each $y \in Y$.*

Proof. Let R be noninclusive; if $y \in Y$ and $y' \in XR\backslash y$, then $Ry' \cap (X\backslash Ry) \neq \square$ so $y' \in (X\backslash Ry)R$. It is always true that $(X\backslash Ry)R \subset XR\backslash y$, hence we have equality. Conversely, suppose the condition holds and $y \neq y'$ in XR; then $y \in XR\backslash y' = (X\backslash Ry')R$ implies $Ry \cap (X\backslash Ry') \neq \square$, so $Ry \not\subset Ry'$.

We will use the following lemma, which collects two well-known results.

LEMMA B. *Let R be a relation from X to Y which is* U.S.C. (*for each U open in Y, $\{x \in X |\ xR \subset U$ is open in $X\}$). Then*

(i) *If xR is connected for each $x \in X$ and P is connected in X, PR is connected;*

(ii) *If xR is compact for each $x \in X$ and P is compact in X, PR is compact.*

Proof. (i) is Theorem 3.9 in [5], and (ii) appears as Theorem 3.2 in [5] and on page 110 of [1].

By *continuum* we mean a nonnull compact connected Hausdorff space. *In all that follows, X and Y will denote nonnull Hausdorff spaces and R will be a monotone* U.S.C. *relation from X to Y with the property that xR is a continuum for each $x \in X$ and XR is not a point.*

3. Arc, pseudocircle and tree theorems. By *arc* we mean a continuum with exactly two noncutpoints; thus an arc need not be metric as we define it, but if it is, it is of course a homeomorph of the unit interval.

THEOREM 1. *Let X be an arc and let R be noninclusive. Then XR is an arc and further, if X is metric, XR is metric.*

Proof. XR is a nondegenerate continuum by Lemma B and hypothesis, and it is well known that such a continuum has at least two noncutpoints. To prove that XR has exactly two, let 0 and 1 denote the noncutpoints of X; we will prove that $0R$ and $1R$ are single points and that every other point of XR is a cutpoint.

Define a cutpoint order on X as follows: for $x, x' \in X$ let $x \leqslant x'$ iff $x = 0$, $x = x'$ or x separates 0 and x'. $x < x'$ will mean that $x \leqslant x'$ and $x \neq x'$. This is a linear order and since X is an arc, the topology induced by it is the given topology of X ([3], Theorem 2-25). For $a, b \in X$, $(a, b]$, $[a, b)$, (a, b) and $[a, b]$ are defined in the usual way: i.e., $(a, b] = \{x \in X|\ a < x \leqslant b\}$, etc.

To see that $0R$ is a single point, suppose $y \cup y' \subset 0R$; then $0 \in Ry \cap Ry'$ and since R is monotone and X is linearly ordered, either $Ry \subset Ry'$ or $Ry' \subset Ry$. R is noninclusive, so $y = y'$; thus $0R$ is a point and similarly, $1R$ is a point.

Next let $y \in XR\backslash(0R \cup 1R)$. Since $\{y\}$ is closed and R is U.S.C., Ry is closed; R is monotone and 0 and 1 are not in Ry, so $Ry = [p, q] \subset (0, 1)$. Let $P = [0, p]$ and $Q = [q, 1]$, and note that since R is noninclusive, $PR \cup QR = XR$. By Lemma B, PR and QR are compact, and Y is Hausdorff so they are closed. Neither PR nor QR is a point, and $PR \cap QR = y$: since if $y' \in PR \cap QR$, then $Ry' \cap P \neq \square \neq Ry' \cap Q$ and Ry' is connected so $Ry' \supset Ry$. This implies $y = y'$ since R is noninclusive, so we have shown that y is a cutpoint of XR which completes the proof that XR is an arc.

Now suppose that X is also metric. We will define a *trace* for R (a continuous function $t\colon X \to XR$ such that $t(x) \in xR$ for each $x \in X$) which is onto; since X is a compact metric space and XR is Hausdorff, this will imply that $t(X) = XR$ is metric (Theorem 3-23, [3]).

Since XR is an arc, its topology is the same as the order topology gotten by defining $y \leqslant y'$ in XR iff $y = 0R$, $y = y'$ or y separates $0R$ and y'. Define $t\colon X \to XR$ by $t(X) = \inf xR$, which is well defined since xR is compact and linearly ordered. Note that $t(0) = 0R$ and $t(1) = 1R$ so $t(X)$ contains the noncutpoints of X.

To prove that t is continuous, note that $\{[0R, y)$ and $(y, 1R]|\ y \in (0R, 1R)\}$ is a subbasis for XR and, since R is U.S.C., $t^{-1}((y, 1R]) = \{x \in X|\ \inf xR \in (y, 1R]\} = \{x \in X|\ xR \subset (y, 1R]\}$ is clearly open. To see that $t^{-1}([0R, y))$ is open requires more work. Let $y_0 \in [0R, y)$; find y_1 such that $y_0 < y_1 < y$. Let $Ry_0 = [p_0, q_0]$ and $Ry_1 = [p_1, q_1]$. $q_0 \neq q_1$ since R is noninclusive, and $Ry_0 \cap (q, 1] = \square$ since $((q_1, 1])R$ is a connected set which contains $1R$ but not y_1, hence not y_0. Therefore $q_0 < q_1$, so we have $t^{-1}(y_0) \subset Ry_0 \subset [0, q_1)$ which is open in X; further, $[0, q_1) \subset t^{-1}([0R, y))$: for clearly $([0, p_1))R \subset [0R, y)$ and if $x \in [p_1, q_1)$, then $y_1 \in xR \cap [0R, y)$, so $t(x) \in [0R, y)$. This completes the proof that t is continuous, so we can conclude that $t(X) = XR$, since $t(X)$ is connected and contains the noncutpoints of XR.

COROLLARY. *If X is an arc and $f\colon X \to Y$ is a nonconstant continuous monotone function, then $f(X)$ is an arc.*

Let us define X to be a *pseudocircle* iff X is a nondegenerate continuum such that the omission of any two distinct points separates it. One can use Lemma 11.19 of [10] to show that this is equivalent to either of the following: X is a nondegenerate continuum such that for any two distinct points $a, b \in X$ (for two distinct points $a, b \in X$), X is the union of arcs I and J, each having a and b as noncutpoints and having no other points in common. A metric pseudocircle is of course a simple closed curve.

THEOREM 2. *Let X be a pseudocircle; let R be noninclusive, and let there exist $a \neq b$ in X such that aR and bR are distinct points. Then XR is a pseudocircle and if X is metric, XR is also.*

Proof. Let I and J be arcs such that $X = I \cup J$ and $I \cap J = a \cup b$. It is simple to see that $R' = R \cap (I \times Y)$ satisfies all hypotheses of Theorem 1, so that $IR' = IR$ is an arc. Similarly, JR is an arc, and $IR \cap JR = aR \cup bR$ since R is monotone, which completes the proof that XR is a pseudocircle.

If X is metric, then by Theorem 1, IR and JR are metric, hence XR is also.

COROLLARY. *If X is a pseudocircle and $f\colon X \to Y$ is a nonconstant continuous monotone function, then $f(X)$ is a pseudocircle.*

A *tree* is a nondegenerate continuum in which each two distinct points are separated by a third point, and a *branch point* of a tree is a point whose complement has at least three components. We note that a metric tree is a dendrite [9] and a tree is locally connected [8]. The other properties of trees which we will use are all known and are either immediate from the definition or are corollaries to the following lemma. A space is called *hereditarily unicoherent* iff the intersection of each two closed connected sets is connected.

LEMMA C. *If X is a tree, then*

(i) *X is hereditarily unicoherent, and*

(ii) *if $a \neq b$ in X, there is a unique minimal subcontinuum $C(a, b)$ joining a and b, and $C(a, b)$ is an arc.*

Proof. (i) Suppose that X is not hereditarily unicoherent: then there are H and K, subcontinua of X, such that $H \cap K = E \cup F$, where E and F are separated. Since $E \cup F \subset H$, there is a minimal subcontinuum C in H which joins E and F; then it can be shown that $C \backslash (E \cup F)$ is connected, and if $e \in C \cap E$ and $f \in C \cap F$, $e \cup f \cup [C \backslash (E \cup F)]$ is connected. There is some $x \in X$ which separates e and f since X is a tree, so clearly $x \in C \backslash (E \cup F)$. However, $e \cup f \subset K \subset X \backslash x$ and K is connected, which is a contradiction. Therefore $H \cap K$ must be connected.

(ii) Since X is a continuum, there is a minimal subcontinuum $C(a, b)$ joining a and b; $C(a, b)$ is unique since X is hereditarily unicoherent. To see that $C(a, b)$ has only a and b as noncutpoints, let $x \in C(a, b) \backslash (a \cup b)$ and suppose that x does not separate a and b. Since X is locally connected, components of $X \backslash x$ are open and locally connected, and there is a component J of $X \backslash x$ such that $a \cup b \subset J$. Each point of J has a connected neighborhood with closure in J, so there is a finite chain of connected open sets, $V_1, \ldots, V_n$, such that $a \in V_1$, $b \in V_n$, $V_i \cap V_{i+1} \neq \square$ and $K = \bigcup_{i=1}^{n} V_i^* \subset J$. Then K is a continuum containing a and b, but $C(a, b) \not\subset K$, which is a contradiction. Therefore x must separate a and b in X and hence in $C(a, b)$, and the noncutpoints of $C(a, b)$ are just a and b.

THEOREM 3. *Let X be a tree; let R be noninclusive, and for each $b \in B = \{x \mid x$ is a branch point of $X\}$, let bR be a point. Then XR is a tree, and if also X is metric and R is* L.S.C. *(RU is open for each U open in Y), then XR is metric.*

Proof. XR is a nondegenerate continuum by Lemma B and hypothesis. To prove that XR is a tree, let $y_1 \neq y_2$ in XR; we will find y_0 in XR which separates y_1 and y_2.

Case 1: $Ry_1 \cap Ry_2 = \square$. Since Ry_1 and Ry_2 are disjoint continua in the tree X, there is an $x_0 \in X$ which separates Ry_1 and Ry_2. Let $y_0 \in x_0 R$ and note that Ry_0 separates $Ry_1 \backslash Ry_0$ and $Ry_2 \backslash Ry_0$, and neither of these sets is empty since R is noninclusive. Therefore $X \backslash Ry_0 = P_1 \cup P_2$, separated sets such that $y_1 \in P_1 R$ and $y_2 \in P_2 R$, and $XR \backslash y_0 = P_1 R \cup P_2 R$. Next we will show that P_1R and P_2R are separated. Suppose $y \in (P_1R)^* \cap P_2R$; by Lemma B, $(P_1R^*) \subset P_1^* R$, so $y \in P_1^* R \cap P_2 R$, which is to say that $Ry \cap P_1^* \neq \square \neq Ry \cap P_2$. If there is any $b \in B \cap Ry_0$, then b separates P_1 and P_2 and Ry is connected, so $b \in Ry$. Then $b \in Ry \cap Ry_0$ and bR is a point, so $bR = y = y_0$; however, $y \in P_2R$ and $y_0 \notin P_2 R$, which is a contradiction. Hence $Ry_0 \cap B$ must be empty, so we can say that Ry_0 is a point or an arc. But then $Ry_0 \subset Ry$, which implies $y = y_0$ since R is noninclusive, and again we have a contradiction. Therefore $(P_1R)^* \cap P_2R = \square$, and dually, $P_1R \cap (P_2R)^* = \square$, so P_1R and P_2R are separated.

Case 2: $Ry_1 \cap Ry_2 \neq \square$. Since R is single-valued on B, $Ry_1 \cap Ry_2 \cap B = \square$; since a tree is hereditarily unicoherent, $Ry_1 \cap Ry_2$ is connected, so $Ry_1 \cap Ry_2$ is a point or an arc. Let $x_0 \in Ry_1 \cap Ry_2$ and let $y_0 \in x_0 R \backslash (y_1 \cup y_2)$, which is nonnull since $x_0 R$ is connected. Since R is monotone, noninclusive and single-valued on B, one can show that $Ry_0 \subset Ry_1 \cup Ry_2$, hence $Ry_0 \subset Ry_1 \cap Ry_2$ and $Ry_0 \cap (Ry_1 \backslash Ry_2) \neq \square \neq Ry_0 \cap (Ry_2 \backslash Ry_1)$. Thus we again have $X \backslash Ry_0 = P_1 \cup P_2$, separated sets such that $y_1 \in P_1 R$ and $y_2 \in P_2 R$, and $XR \backslash y_0 = P_1 R \cup P_2 R$. It now follows just as in Case 1 that P_1R and P_2R are separated, and this completes the proof that XR is a tree.

Now suppose further that X is metric and R is L.S.C. Let $2^{XR} = \{A \subset XR \mid A = A^* \neq \square\}$, and for $U \subset XR$, let $2^U = \{A \in 2^{XR} \mid A \subset U\}$ and $2_U = \{A \in 2^{XR} \mid A \cap U \neq \square\}$. We will let 2^{XR} have the *finite*, or *neighborhood topology* ([4], [5]), i.e., the topology generated by $\{2^U$ and $2_U \mid U$ open in $XR\}$. Define $g\colon X \to 2^{XR}$ by $g(x) = xR$; since R is U.S.C. and L.S.C., g is continuous, and if $\mathcal{A} = \{A \subset XR \mid A$ is a nonnull continuum$\}$, $g(X) \subset \mathcal{A} \subset 2^{XR}$.

In [8], Ward defines a partial order for a tree which satisfies the hypotheses of Capel and Strother's theorem in [2]; by that theorem, there is a continuous function $f\colon \mathcal{A}\to XR$ such that $f(A)\in A$ for each $A\in\mathcal{A}$. The composite function $fg\colon X\to XR$ is continuous, so $fg(X)$ is a connected metric space; if we can prove that $fg(X)$ contains all the noncutpoints of XR, then $fg(X)=XR$ and our proof will be complete. Let y be a noncutpoint of XR; since R is single-valued on B and R is L.S.C., R is single-valued on B^*. If $y=bR$ for some $b\in B^*$, then $fg(b)=y$ so $y\in fg(X)$. If $y\notin(B^*)R$, then there is a component J of $X\setminus B^*$ such that $y\in JR$; X is locally connected, so J is open in X, hence J is nondegenerate. Then a corollary to Lemma C is that J^* is an arc, and it is not difficult to see that J^*, Y and $R'=R\cap(J^*\times Y)$ satisfy the hypotheses of Theorem 1, so $(J^*)R'=(J^*)R$ is an arc. Also, by the proof of Theorem 1, if a and b are the noncutpoints of J^*, aR and bR are the noncutpoints of $(J^*)R$. By supposition, y does not cut XR so y cannot cut $(J^*)R$; hence either $y=aR$ or $y=bR$, and in either case, $y\in fg(X)$.

Corollary. *If X is a tree and $f\colon X\to Y$ is a nonconstant continuous monotone function, then $f(X)$ is a tree.*

4. Unicoherence and acyclicity. It seems that noninclusivity is irrelevant to the preserving of unicoherence or acyclicity with a relation, and that the key condition needed is point unicoherence or point acyclicity, respectively. In [6], Wallace used point acyclicity to prove an acyclicity theorem which is restated in relation theory terminology in [7]. The following theorem is very similar to the theorem in [7]; we restrict X so that its subcontinua are well behaved and conclude that their images are, whereas there conditions are placed on all the closed subsets of a space and one concludes that their images are acyclic. We omit proof of the following theorem, for after one observes that $A\cap B$ is a continuum, for overlapping subcontinua A and B of a hereditarily unicoherent space X, and that $(A\cap B)R=AR\cap BR$ when R is monotone on a hereditarily unicoherent space X, the proof is just like the proof given for the main result in [6].

$H^n(Y;G)$ represents the n-dimensional Alexander-Kolmogoroff-Wallace cohomology groups of a space Y, with an arbitrary but fixed coefficient group G. A continuum Y is *unicoherent* iff $A\cap B$ is connected for any two closed connected subsets A and B whose union is Y, and is *acyclic* iff $H^n(Y;G)=\{0\}$ for all $n\geqslant 1$ and each G. As is well known, a continuum Y is unicoherent if $H^1(Y;G)=\{0\}$ for each G. Recall the standing hypotheses on R made in Section 1.

Theorem 4. *Let X be hereditarily unicoherent and have the property that each of its nondegenerate subcontinua is decomposable (is the union of two proper subcontinua).*

(i) *If $H^1(xR;G)=\{0\}$ for each G and each $x\in X$, then $H^1(KR;G)=\{0\}$ for each G and hence KR is unicoherent, for each subcontinuum K of X.*

(ii) *If xR is acyclic for each $x\in X$, then KR is acyclic for each subcontinuum K of X.*

Certainly a tree satisfies the hypotheses on X, as does a non-locally connected space such as the $\sin(1/x)$ curve together with the arc to which it converges.

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