Imbedding collections of compact 0-dimensional subsets of E^2 in continuous collections of mutually exclusive arcs

by

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If $\mathcal{G}$ and $\mathcal{H}$ are collections of sets, then $\mathcal{G}$ is said to be *imbedded* in $\mathcal{H}$ if there is a one-to-one correspondence f from $\mathcal{G}$ onto $\mathcal{H}$ such that $g \subset f(g)$ for each g in $\mathcal{G}$. M. E. Hamstrom ([2]) investigated the imbedding of upper-semicontinuous collections of continuous curves in continuous collections of continuous curves, and it is proposed here to consider several problems concerning the imbedding of collections of compact 0-dimensional subsets of E^2 in collections of arcs in E^2 or E^3. The author would like to express her gratitude to B. J. Ball for his many constructive suggestions.

If each element of $\mathcal{G}$ is a compact 0-dimensional set in E^2, it follows easily from Theorem 135 in [3] that $\mathcal{G}$ can be imbedded in a collection of arcs in E^2; if, however, the elements of $\mathcal{G}$ are mutually exclusive, or $\mathcal{G}$ is upper-semicontinuous, etc., it is not always possible to imbed $\mathcal{G}$ in a collection of arcs satisfying similar conditions. For example, there is a continuous collection of mutually exclusive finite subsets of E^2 whose decomposition space is an arc that cannot be imbedded in any collection of mutually exclusive arcs in E^2; namely, the collection $\{g(t)\colon 0 \leqslant t \leqslant 2\}$ where $g(t) = \{(t, 0), (2-t, 0)\}$ if $0 \leqslant t < 1$, $g(1) = \{(1, 0)\}$ and if $1 < t \leqslant 2$ then $g(t) = \{(1, 1-t), (1, t-1)\}$.

Suppose that $\mathcal{G}$ is an upper semicontinuous collection of mutually exclusive compact 0-dimensional subsets of E^2. It is shown in Theorem 1 that if $\mathcal{G}^*$ ($\mathcal{G}^*$ denotes the union of the elements of $\mathcal{G}$) is compact and 0-dimensional and the upper semicontinuous decomposition space of $\mathcal{G}$ is a subset of an arc, then there is a continuous and equicontinuous collection of mutually exclusive arcs in E^2 in which $\mathcal{G}$ is imbedded. Recall that a collection $\mathcal{H}$ of arcs is equicontinuous if for each positive number e there is a positive number d such that if p and q are two points of an arc H of $\mathcal{H}$ such that the distance from p to q is less than d, then the diameter of the subarc of H having p and q as its endpoints is less than e. If $\mathcal{G}$ is continuous and its decomposition space is an arc then a sufficient condition is found in Theorem 2 for the existence of a continuous collection of mutually exclusive arcs in E^3 in which $\mathcal{G}$ is imbedded, that is satisfied, in particular, if each element of $\mathcal{G}$ is finite (Theorem 3); and a sufficient and almost necessary condition is found for the existence of a continuous and equicontinuous collection of mutually exclusive arcs in E^3 in which $\mathcal{G}$ is imbedded, in Theorem 4. Some corollaries are stated in terms of extensions of mappings, and some questions are raised.

Tietze's extension theorem as well as theorems in [3] are used frequently without specific mention.

Theorem 1. *Suppose that $\mathcal{G}$ is an upper semicontinuous collection of mutually exclusive, compact 0-dimensional subsets of E^2 whose decomposition space is a subset of an arc. If $\mathcal{G}^*$ is compact and 0-dimensional, then there is a continuous and equicontinuous collection of mutually exclusive arcs in E^2 in which $\mathcal{G}$ is imbedded.*

Proof. Suppose that $\mathcal{G}$ is as in the hypothesis. Since $\mathcal{G}^*$ is compact and 0-dimensional, there is a homeomorphism of E^2 onto itself that takes $\mathcal{G}^*$ into $\{(x,0): x \in [0,1]\}$, so it may be assumed that $\mathcal{G}^* \subset \{(x,0): x \in [0,1]\}$. Denote by f a homeomorphism from $\mathcal{G}$ into $[0,1]$. The continuous function h from $\mathcal{G}^*$ into $[0,1]$ defined by $h(x,0) = t$ if $(x,0) \in f^{-1}(t)$ can be extended, since $\mathcal{G}^*$ is a closed subset of E^1, to a continuous function h' from E^1 into $[0,1]$. The homeomorphism k of E^2 onto itself defined by $k(x,y) = \big(x, h'(x,0)+y\big)$ takes g into the arc $\{(x,y): 0 \leqslant x \leqslant 1,\ y = f(g)\}$ for each g in $\mathcal{G}$. Hence $\mathcal{G}$ is imbedded in the continuous and equicontinuous collection $\{k^{-1}\big(\{(x,y): 0 \leqslant x \leqslant 1,\ y = f(g)\}\big): g \in \mathcal{G}\}$ of mutually exclusive arcs in E^2.

Corollary. *If K is a compact and 0-dimensional subset of the interior of a disk D in E^2, and if f is a continuous function from K into $(0,1)$, then there is a continuous and open extension f' of f from D onto $[0,1]$ such that $f'^{-1}(t)$ is an arc for each $t \in [0,1]$.*

Proof. Suppose that K, D, and f are as in the hypothesis. Since K is compact, there are numbers a and b such that $f(K) \subset [a,b] \subset (0,1)$. Since K is 0-dimensional, it will be assumed that K is contained in $\{(x,0): x \in [a,b]\}$. The collection $\{f^{-1}(t): t \in f(K)\}$ satisfies the hypothesis of Theorem 1, so from the proof of Theorem 1, there is a homeomorphism k of E^2 onto itself that takes $f^{-1}(t)$ into the arc $\{(x,y): a \leqslant x \leqslant b,\ y = t\}$ for each $t \in f(K)$. There is a homeomorphism h of E^2 onto itself that is the identity at points of $k(K)$ and takes $k(D)$ onto the rectangular disk with vertices $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$. If $y'(p)$ denotes the y-coordinate of the point p in E^2, define f' by $f'(x,y) = y'\big(hk(x,y)\big)$ for each $(x,y) \in D$. The function f' has the disired properties since the collection $\{f'^{-1}(t): t \in [0,1]\}$ is a continuous decomposition of D into mutually exclusive arcs and $f'' = f$ at points of K.

The hypothesis of Theorem 1 is very restrictive, but the author knows of no weaker condition that will insure the existence of a continuous collection of mutually exclusive arcs in E^2 in which a given collection of mutually exclusive compact 0-dimensional subsets of E^2 whose decomposition space is an arc is imbedded. The following example illus-

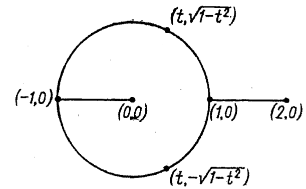

Fig. 1

trates some of the difficulties encountered. This collection can be imbedded in a collection of mutually exclusive arcs in E^2, but not in a continuous collection, even in S^2.

$$\mathcal{G} = \{g(t): -2 \leqslant t \leqslant 2\}$$

where

$$g(t) = \begin{cases} \{(-t-2,\,0)\} & \text{if} \quad -2 \leqslant t \leqslant -1, \\ \{(t,\sqrt{1-t^2}),\,(t,-\sqrt{1-t^2})\} & \text{if} \quad -1 < t < 1, \\ \{(t,0)\} & \text{if} \quad 1 \leqslant t \leqslant 2. \end{cases}$$

The rest of the paper deals with the imbedding of continuous collections of mutually exclusive compact 0-dimensional subsets of E^2 whose decomposition spaces are arcs, in continuous collections of mutually exclusive arcs in E^3.

Definition. Suppose that $\mathcal{G}$ is a continuous collection of mutually exclusive compact 0-dimensional subsets of E^2 whose decomposition is an arc.

1. An arc in $\mathcal{G}^*$ is a *minimal trace* of $\mathcal{G}^*$ if it intersects each element of $\mathcal{G}$ in exactly one point.

2. The element g of $\mathcal{G}$ is said to be a *change element* of $\mathcal{G}$ if g contains a point x that is an endpoint of a component of the intersection of two minimal traces of $\mathcal{G}$, and if x is an endpoint of both of these minimal traces, then the component of their intersection containing x contains no other point.

Remark. It follows from Theorem 2.1, page 186, of [7] that if $\mathcal{G}$ is as in the definitions above and p is a point of $\mathcal{G}^*$, then p is contained in a minimal trace of $\mathcal{G}$.

THEOREM 2. *Suppose that $\mathcal{G}$ is a continuous collection of mutually exclusive compact 0-dimensional subsets of E^2 whose decomposition space is an arc. If the closure (with respect to the decomposition topology) of the collection of change elements of $\mathcal{G}$ is 0-dimensional, then $\mathcal{G}$ can be imbedded in a continuous collection of mutually exclusive arcs in E^3.*

A rough proof of Theorem 2 is given in steps (1) through (5), so that the purpose of the lemmas, which are rather complicated, will be understood.

(1) $\mathcal{G}$ is written $\{g(t)\colon 0 \leqslant t \leqslant 1\}$, where g is a homeomorphism from $[0,1]$ onto $\mathcal{G}$, and a homeomorphism f is constructed of E^3 onto itself that takes $g(t)$ into the plane $\{z=t\}$. Denote by $\mathcal{G}'$ the collection $\{fg(t);\ 0 \leqslant t \leqslant 1\}$.

(2) The closure, $\mathcal{K}$, of the collection of change elements of $\mathcal{G}'$ is imbedded, by means of Theorem 1, in a continuous collection of mutually exclusive arcs so that each arc is in the same z-plane as the element of $\mathcal{G}'$ it contains.

(3) Lemma 2 (whose proof is shortened by Lemma 1) gives a constructive method of imbedding closed intervals of the arc $\mathcal{G}'$ that are strictly between elements of $\mathcal{K}$ in continuous collections of mutually exclusive arcs. Since such an interval contains no change element of $\mathcal{G}'$, the union of its elements is the union of a collection of mutually exclusive arcs (portions of traces) rising from one z-plane to another.

(4) Lemma 3 shows how to imbed all of the elements of $\mathcal{G}$ between (and including) two elements of $\mathcal{K}$ in a continuous collection of mutually exclusive arcs, using the arcs constructed in Lemma 2.

(5) All of these collections are combined to form a continuous collection of mutually exclusive arcs in which (under f^{-1}) the collection $\mathcal{G}$ is imbedded.

LEMMA 1. *Suppose that $A(0)$ is an arc in the plane $\{z=0\}$, $A(1)$ is a straight line segment in the plane $\{z=1\}$, and $\mathcal{B}$ is a collection of vertical straight line segments each having one endpoint on $A(0)$ and the other on $A(1)$. If $B(0)$ and $B(1)$ are elements of $\mathcal{B}$ joining the endpoints of $A(0)$ to those of $A(1)$, then there is a collection of arcs $\{A(t)\colon 0<t<1\}$, each joining a point of $B(0)$ to a point of $B(1)$ such that if $0<t<1$, then $\mathcal{B}^* \cap \{z=t\} \subset A(t) \subset \{z=t\}$, and such that the collection $\{A(t)\colon 0 \leqslant t \leqslant 1\}$ is continuous.*

Proof. Suppose $A(1) = \{(0,y,1)\colon 0 \leqslant y \leqslant 1\}$. For each point $p=(x,y,0)$ of $A(0)$, denote by $j(p)$ either (1) the straight line segment from p to $(0,0,1)$ if $y<0$, (2) the straight line segment from p to $(0,y,1)$ if $0 \leqslant y \leqslant 1$, or (3) the straight line segment from p to $(0,1,1)$ if $y>1$. Define a function f on $A(0)\times[0,1]$ by $f(p,t) = j(p) \cap \{z=t\}$. Clearly f is continuous and f is one-to-one on $A(0)\times[0,1)$. Furthermore, $\{f(p,0)\colon p \in A(0)\} = A(0)$ and $\{f(p,1)\colon p \in A(0)\} = A(1)$. Hence, if for $0<t<1$, $A(t)$ is defined to be $\{f(p,t)\colon p \in A(0)\}$, then $A(t)$ is an arc in $\{z=t\}$ and the collection $\{A(t)\colon 0 \leqslant t \leqslant 1\}$ is continuous. If B is an element of $\mathcal{B}$ and p is the endpoint of B in $A(0)$, then $j(p) = B$. It follows that if $0<t<1$, then $\mathcal{B}^* \cap \{z=t\} \subset A(t)$, and the endpoints of $A(t)$ are in $B(0) \cup B(1)$. Thus $\{A(t)\colon 0<t<1\}$ is the desired collection.

DEFINITION. An *ascending arc* is an arc that intersects each z-plane in at most one point.

LEMMA 2. *Suppose that $\mathcal{B}$ is a continuous collection of mutually exclusive ascending arcs each with one endpoint on the arc $A(0)$ in $\{z=0\}$ and the other on the arc $A(1)$ in $\{z=1\}$ such that* (i) *$\mathcal{B}^*$ is compact,* (ii) *no vertical line contains two points of $\mathcal{B}^*$, and* (iii) *$\mathcal{B}$ contains two arcs $B(0)$ and $B(1)$ joining the endpoints of $A(0)$ to the endpoints of $A(1)$ such that $\mathcal{B}^*-\big(B(0) \cup B(1)\big)$ is closed. If D is a simple domain in $\{z=0\}$ such that D contains the projection of $A(0) \cup A(1) \cup \mathcal{B}^*$ into $\{z=0\}$, then there is a collection $\{A(t)\colon 0<t<1\}$ of arcs having one endpoint on $B(0)$ and the other on $B(1)$ such that* (i) *$\{A(t)\colon 0 \leqslant t \leqslant 1\}$ is continuous, and* (ii) *if $0<t<1$, then $\mathcal{B}^* \cap \{z=t\} \subset A(t) \subset \{(x,y,z)\colon (x,y,0) \in D,\ 0<z<1\}$.*

Proof. Two homeomorphisms, f and g, of E^3 onto itself will be defined so that $A(0)$, $A(1)$, and $\mathcal{B}$, under f, will satisfy the hypothesis of Lemma 1, and the collection of arcs assured by the conclusion of Lemma 1 will, under gf^{-1}, be contained in the cylinder over D as required. For the construction of g it is necessary to trace points directly above $A(1)$ and points directly below $A(0)$ through f. For this purpose, choose a point w directly above a point w' in $A(1)$ as a representative. The case will be similar for points directly below $A(0)$. The homeomorphism f will be defined as the composition of four homeomorphisms of E^3 onto itself, $f = f_1f_2f_3f_4$, whose construction is defined in steps (1), (2), (3), and (4).

(1) Denote by p the projection map of E^3 onto $\{z=0\}$. Define a function h from $p(\mathcal{B}^*)$ onto $[0,1]$ by $h(x,y,0) = z$ if $(x,y,z) \in \mathcal{B}^*$. The function h is well defined since no vertical line intersects two points of $\mathcal{B}^*$ and is continuous since $\mathcal{B}$ is. Since $p(\mathcal{B}^*)$ is closed, there is a continuous extension h' of h from $\{z=0\}$ onto $[0,1]$ such that $h'(q)$ is in $(0,1)$ if q is in $\{z=0\}-p(\mathcal{B}^*)$. Define a homeomorphism f_1 from E^3 onto itself by $f_1(x,y,z) = (x,y,z-h'(x,y,0))$. Note that f_1 has the following properties:

(i) $f_1(\mathcal{B}^*) \subset \{z=0\}$,

(ii) $f_1\big(A(1)-\mathcal{B}^*\big) \subset \{z>0\}$, $f_1\big(A(0)-\mathcal{B}^*\big) \subset \{z<0\}$,

(iii) $f_1(w)$ is above $f_1(w')$, and

(iv) f_1 changes no x or y coordinate.

(2) Since $\{f_1(B)\colon B \in \mathcal{B}\}$ is a continuous and equicontinuous collection of mutually exclusive arcs filling up a compact subset of $\{z=0\}$, it follows from results in [1] that there is a homeomorphism k of $\{z=0\}$ onto itself which lines the arcs up so that (i) $kf_1(B(0))$ is the straight line segment $\{(x,0,0)\colon 0 \leqslant x \leqslant 1\}$, (ii) $kf_1(B(1))$ is $\{(x,1,0)\colon 0\leqslant x \leqslant 1\}$, and (iii) if $B \in \mathcal{B}-\{B(0), B(1)\}$, then there is a number c in $(0,1)$ such that $kf_1(B) = \{(x,c,0)\colon 0 \leqslant x \leqslant 1\}$. Furthermore, k can be constructed so that for each t

$$kf_1(\mathcal{B}^* \cap \{z=t\}) \subset \{(1-t, y, 0)\colon 0 \leqslant y \leqslant 1\}.$$

The homeomorphism f_2 of E^3 onto itself (which will extend k) is defined by $f_2(x,y,z) = k(x,y,0)+(0,0,z)$. Notice that f_2 has the following properties:

(i) The set $f_2f_1(A(1)-\mathcal{B}^*)$ and the set $f_2f_1(A(0)-\mathcal{B}^*)$ are above and below, respectively, the plane $\{z=0\}$, and no vertical line intersects either set in more than one point;

(ii) the point $f_2f_1(w)$ is directly above the point $f_2f_1(w')$; and

(iii) the point set $f_2f_1(\mathcal{B}^*)$ is in $\{(x,y,0)\colon 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$.

(3) A homeomorphism f_3 of E^3 onto itself will be constructed so that the arc $f_3f_2f_1(A(1))$ is contained in the union of two upper half planes through the line $\{x=0,\ z=0\}$, the arc $f_3f_2f_1(A(0))$ lies in the union of two lower half planes through the line $\{x=1,\ z=0\}$, and f_3 is the identity on $\{z=0\}$.

Denote by V the union of all vertical lines containing a point of $f_2f_1(A(1)) = A'$. If L is a vertical line intersecting A', then there is exactly one point, $(x(L), y(L), z(L))$, of A' on L. Define a function h_L of E^1 onto itself by: $h_L(r) = r(|x(L)|/z(L))$ if $r \geqslant 0$ and $z(L)>0$, and $h_L(r) = r$ if $r<0$ or $z(L)=0$. (No $z(L)$ is less than 0.) Define a homeomorphism h of V onto itself by $h(x,y,z) = (x,y,h_L(z))$, where L is the vertical line containing (x,y,z). The function h is the identity on $V \cap \{z \leqslant 0\}$, and $h(A') \subset \{(x,y,z)\colon z=|x|\}$.

Since the projection of A' into $\{z=0\}$ is an arc, there is a homeomorphism e of E^3 onto itself that takes V onto the strip $V' = \{(0,y,z)\colon 0 \leqslant y \leqslant 1\}$ such that e does not change the z-coordinate of any point and takes each vertical line onto a vertical line. Since any homeomorphism of V' onto itself which takes each vertical line onto itself and is the identity on $\{z \leqslant 0\}$ can be extended to a homeomorphism of E^3 onto itself with the same properties, it follows that there is an extension of h to a homeomorphism h' of E^3 onto itself which takes each vertical line into itself and is the identity on $\{z \leqslant 0\}$.

Similarly, there is a homeomorphism g' of E^3 onto itself that takes the arc $f_2f_1(A(0))$ into $\{(x,y,z)\colon z=-|x-1|\}$ such that g' takes each vertical line onto itself, and is the identity on $\{z \geqslant 0\}$.

The homeomorphism $f_3 = h'g'$ has the desired properties.

(4) The arcs $f_3f_2f_1(A(1))$ and $f_3f_2f_1(A(0))$ are in the sets $\{z=|x|\}$ and $\{z=-|x-1|\}$ respectively; it is a relatively simple matter to construct a homeomorphism f_4 of E^3 onto itself that takes the arcs into the vertical planes $\{x=0\}$ and $\{x=1\}$ respectively, is the identity on $\{x,y,0)\colon 0 \leqslant x \leqslant 1\}$, and takes $f_3f_2f_1(w)$ into a point with negative x-coordinate. (See diagram on Fig. 2.)

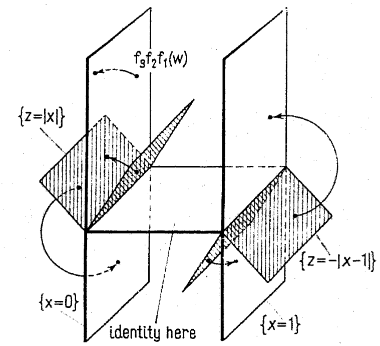

Fig. 2

The homeomorphism $f = f_4f_3f_2f_1$ of E^3 onto itself has the following properties:

(1) $f(A(1)) \subset \{x=0\}$, and $f(A(0)) \subset \{x=1\}$,

(2) $f(B(0)) = \{(x,0,0)\colon 0 \leqslant x \leqslant 1\}$, and $f(B(1)) = \{(x,1,0)\colon 0 \leqslant x \leqslant 1\}$,

(3) if $B \in \mathcal{B}$, then for some c in $[0,1]$, $f(B) = \{(x,c,0)\colon 0 \leqslant x \leqslant 1\}$,

(4) if $0 \leqslant t \leqslant 1$, then $f(\mathcal{B}^* \cap \{z=t\}) = f(\mathcal{B}^*) \cap \{x=1-t\}$,

(5) $f(w)$ is in $\{x<0\}$.

With the arcs so positioned it follows from Lemma 1 that there is a continuous collection $\mathcal{H} = \{H(t)\colon 0 \leqslant t \leqslant 1\}$ of mutually exclusive arcs such that (i) $H(0) = f(A(0))$ and $H(1) = f(A(1))$, (ii) if $0 \leqslant t \leqslant 1$, $\{x=1-t\} \cap f(\mathcal{B}^*) \subset H(t)$, (iii) the endpoints of each $H(t)$ are on $f(B(0))$ and $fB(1)$, and (iv) $f(w)$ is not in $\mathcal{H}^*$.

For $0 \leqslant t \leqslant 1$, define $A'(t)$ to be $f^{-1}(H(t))$ and denote by $\mathcal{A}$ the collection $\{A'(t)\colon 0 \leqslant t \leqslant 1\}$. Then $\mathcal{A}$ is a continuous collection of mutually exclusive arcs in which the collection $\{\mathcal{B}^* \cap \{z = t\}\colon 0 \leqslant t \leqslant 1\}$ is imbedded such that $A(0) = A'(0)$, $A'(1) = A(1)$, and no point of $\mathcal{A}^*$ is directly above a point of $A(1)$ or directly below a point of $A(0)$. The arcs of $\mathcal{A}$ are not, however, necessarily contained in the cylinder over D (see statement of Lemma), so that another homeomorphism, g, is required.

Recall that p denotes the projection map of E^3 onto $\{z = 0\}$. Since $p(A(1) \cup A(0) \cup \mathcal{B}^*)$ is a closed subset of the simple domain D, there is a simple domain D'' in $\{z = 0\}$ containing it whose closure is contained in D. Since $\mathcal{A}^*$ is bounded, there is a simple domain D' in $\{z = 0\}$ containing $\bar{D} \cup p(\mathcal{A}^*)$. There is a homeomorphism h of $\{z = 0\}$ onto itself which is the identity on $\bar{D}''$ and takes D' onto D. Denote by g' the homeomorphism of E^3 onto itself defined by $g'(x, y, z) = h(x, y, 0) + (0, 0, z)$. The collection $\{g'(A'(t))\colon 0 \leqslant t \leqslant 1\}$ has all of the listed properties of $\mathcal{A}$ with the additional property that the projection of each arc is in D.

The arcs still need not be in $\{0 \leqslant z \leqslant 1\}$ as desired, so a homeomorphism g'' of E^3 onto itself will be defined with the property that g'' is the identity on $A(1) \cup A(0) \cup \mathcal{B}^*$, g'' changes no x or y coordinate, and for t strictly between 0 and 1, $g''g'(A'(t))$ is in $\{0 < z < 1\}$. The homeomorphism g will be defined as $g''g'$ and the collection $\{A(t)\colon 0 \leqslant t \leqslant 1\}$, where $A(t) = g(A'(t))$, will be the desired collection of arcs.

Define two functions f'' and f' on $\{z = 0\}$ by:

$$f''(x, y, 0) = \operatorname{lub}\{z\colon (x, y, z) \in \mathcal{B}^* \cup A(1) \cup \{z = \tfrac{1}{2}\}\},$$
$$f'(x, y, 0) = 1 \quad \text{for each} \quad (x, y, 0) \in \{z = 0\}.$$

For $(x, y, 0)$ in $\{z = 0\} - p(A(1)) = K$, $f''(x, y, 0) < f'(x, y, 0)$; f'' is upper semicontinuous; and f' is lower semicontinuous. By Theorem 2 of [5], there is a continuous function f_0 from K into $(\frac{1}{2}, 1)$ such that $f''(x, y, 0) < f_0(x, y, 0) < f'(x, y, 0)$ on K and such that the function f defined to be f_0 on K and the constant function 1 on $p(A(1))$ is a continuous extension of f_0 to $\{z = 0\}$. Define the continuous function e on $\{z = 0\}$ by $e(x, y, 0) = \frac{1}{2} + \frac{1}{2}f(x, y, 0)$.

There is a number m such that if (x, y, z) is in $A(1) \cup g'(\mathcal{A}^*)$, then $z < m$. Define two functions h' and h'' on $\{z = 0\}$ by:

$$h'(x, y, 0) = \begin{cases} 1 & \text{if} \quad (x, y, 0) \in p(A(1)), \\ m, & \text{otherwise;} \end{cases}$$

$$h''(x, y, 0) = \operatorname{lub}\{z\colon (x, y, z) \in g'(\mathcal{A}^*) \cup \{z = 1\}\}.$$

Since h' is lower semicontinuous, h'' is upper semicontinuous, and $h'' \leqslant h'$ (since no point of $g'(\mathcal{A}^*)$ is directly above $A(1)$), by Theorem 1 of [5] there is a continuous function h on $\{z = 0\}$ such that $h'' \leqslant h \leqslant h'$.

Define a homeomorphism k of E^3 onto itself by $k(x, y, z) = (x, y, r)$, where $r = e(x, y, 0) + [(z - h(x, y, 0))(e(x, y, 0) - f(x, y, 0))]/(h(x, y, 0) - f(x, y, 0))$ if $(x, y, 0)$ is in K and $z \geqslant f(x, y, 0)$, and $r = z$ if $(x, y, 0) \in p(A(1))$ or $z \leqslant f(x, y, 0)$.

In particular, $k(x, y, h(x, y, 0)) = (x, y, e(x, y, 0))$, so that if (x, y, z) is in $g'(\mathcal{A}^*) - A(1)$, then $k(x, y, z)$ is in $\{0 < z < 1\}$.

Similarly, there is a homeomorphism k' of E^3 onto itself such that k' is the identity at points of $\{z \geqslant \frac{1}{2}\} \cup A(0) \cup \mathcal{B}^*$, k' changes no x or y coordinate, and if (x, y, z) is in $g'(\mathcal{A}^*) - A(0)$, then $k'(x, y, z)$ is in $\{0 < z < 1\}$.

Define g'' to be kk', and g'' is the desired homeomorphism of E^3 onto itself. This completes the proof of Lemma 2.

LEMMA 3. *If, in the statement of Lemma 2, the requirement that the elements of $\mathcal{B}$ be mutually exclusive is replaced by the hypothesis that the intersection of any pair of elements of $\mathcal{B}$ lies on the lower arc $A(0)$, and the requirement that $\mathcal{B}$ be continuous is replaced by the hypothesis that if $z'' > z' > 0$, then the collection $\{B \cap \{z'' \geqslant z \geqslant z'\}\colon B \in \mathcal{B}\}$ is continuous, then the resulting statement is true.*

Proof. For each positive integer n, denote by $P(n)$ the set $A(0) \cup p(\mathcal{B}^* \cap \{z \leqslant 1/n\})$. Since $P(n)$ is a continuum in $\{z = 0\}$, there is a decreasing sequence $\{D(i)\}$ of simple domains in $\{z = 0\}$ such that $D(1) = D$ and if $i > 1$ then $P(i) \subset D(i)$ and the boundary of $D(i)$ lies in the $1/i$-neighborhood of $P(i)$. For each n let $C(n)$ denote the cylinder $\{(x, y, z)\colon (x, y, 0) \in \overline{D(n)},\ 1/(n+1) \leqslant z \leqslant 1/n\}$, and let $A(1/n+1)$ denote an arc in the simple domain $C(n) \cap C(n+1)$ with endpoints on $B(0)$ and $B(1)$ which contains the compact, 0-dimensional set $\mathcal{B}^* \cap \{z = 1/(n+1)\}$. Since $\mathcal{B}$ is continuous and $\mathcal{B}^* \cap \{z = 0\} \subset A(0)$, the sequence $\{D(i)\}$, and hence the sequence $\{C(i)\}$ converges to $A(0)$.

It follows from Lemma 2 that for each n, there exists a collection $\{A(t)\colon 1/(n+1) < t < 1/n\}$ of arcs each with endpoints on $B(0)$ and $B(1)$ such that (i) $\{A(t)\colon 1/(n+1) \leqslant t \leqslant 1/n\}$ is a continuous collection of mutually exclusive arcs and (ii) if $1/(n+1) < t < 1/n$, then $\mathcal{B}^* \cap \{z = t\}$ is contained in $A(t)$ which is contained in the interior of $C(n)$.

The only possible discontinuity of the collection $\mathcal{A} = \{A(t)\colon 0 \leqslant t \leqslant 1\}$ of mutually exclusive arcs is at the arc $A(0)$. Since the cylinders $\{C(n)\}$ converge to $A(0)$, the collection $\mathcal{A}$ is upper-semicontinuous at $A(0)$, and if a sequence of elements of A converges to a subset of $A(0)$, the subset must be connected and must contain the endpoints of $A(0)$ (since the endpoints of each $A(t)$ are on the arcs $B(0)$ and $B(1)$) and hence must be equal to $A(0)$, so that A is also lower-semicontinuous at $A(0)$. It follows that the collection A is continuous and it clearly satisfies the other conditions.

It should be noted that if the irregularities that occur at the lower arc $A(0)$ also occur at the upper arc $A(1)$, then only a slight modification of the above construction is needed.

Proof of Theorem 2. It may be assumed that the compact set $\mathcal{G}^*$ is contained in the interior of the unit square $\{(x, y, 0): 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$. Let g denote a homeomorphism from $[0, 1]$ onto the upper semicontinuous decomposition space of $\mathcal{G}$ (which is an arc), and for each t, define $g'(t)$ to be $g(t) \cup \{(t, 0, 0), (t, 1, 0)\}$. Denote by $\mathcal{G}'$ the collection $\{g'(t): 0 \leqslant t \leqslant 1\}$.

The closure, $\mathcal{M}$, of the collection of change elements of $\mathcal{G}'$ is 0-dimensional, so by a modification of Theorem 1, $\mathcal{M}$ can be imbedded in a continuous collection $\mathcal{H}'$ of mutually exclusive arcs in $\{z = 0\}$ such that if $H'(t)$ denotes the element of $\mathcal{H}'$ containing the element $g'(t)$ of $\mathcal{M}$, then the endpoints of $H'(t)$ are $(t, 0, 0)$ and $(t, 1, 0)$.

The function f' from $\mathcal{G}'^*$ onto $[0, 1]$ defined by $f'(p) = t$ if p is an element of $g'(t)$ is continuous, and $\mathcal{G}'^*$ is closed, so there is a continuous extension f'' of f' from $\{z = 0\}$ into the real numbers. Denote by f the homeomorphism of E^3 onto itself defined by $f(x, y, z) = \big(x, y, z + f''(x, y, 0)\big)$. Note that $fg'(t) \subset \{z = t\}$ for each t so that if B is a trace of $\mathcal{G}'$, then $f(B)$ is an ascending arc. In particular, $f\big(\{(t, 0, 0): 0 \leqslant t \leqslant 1\}\big) = B(0)$ and $f\big(\{(t, 1, 0): 0 \leqslant t \leqslant 1\}\big) = B(1)$ are ascending straight line segments.

For each $g'(t)$ in $\mathcal{M}$ define $H(t)$ to be the arc $\{(x, y, t): (x, y, 0) \in H'(t)\}$ in $\{z = t\}$, and denote by $\mathcal{H}$ the collection $\{H(t): g'(t) \in \mathcal{M}\}$. Then $\mathcal{H}$ is a continuous collection of mutually exclusive arcs such that for each $H(t)$ of $\mathcal{H}$, $fg'(t) \subset H(t)$ and the endpoints of $H(t)$ are on $B(0)$ and $B(1)$.

If $W = \{t: 0 \leqslant t \leqslant 1, g'(t) \notin \mathcal{M}\}$, then W is the union of a countable (or finite) number of mutually exclusive open intervals $R(1), R(2), \ldots, R(i), \ldots$ For each i, (1) denote by e_i and d_i the right and left endpoints, respectively, of $R(i)$, (2) denote by $L(i)$ the projection of $\{fg'(t): t \in R(i)\}^* \cup H(d_i) \cup H(e_i)$ into the plane $\{z = d_i\}$, (3) denote by $D(i)$ a simple domain in $\{z = d_i\}$ containing $L(i)$ whose boundary is in the $1/i$-neighborhood of $L(i)$, and (4) denote by $C(i)$ the cylinder $\{(x, y, z): (x, y, d_i) \in \overline{D(i)}, d_i \leqslant z \leqslant e_i\}$.

The collection of ascending arcs in $f(\mathcal{G}'^*)$ from $H(d_i)$ to $H(e_i)$, the simple domain $D(i)$, and the cylinder $C(i)$ satisfy the hypothesis of Lemma 3 so there exists a collection $\{H(t): d_i < t < e_i\}$ of arcs such that if $d_i < t < e_i$, then $fg'(t) \subset H(t)$, the endpoints of $H(t)$ are on $B(0)$ and $B(1)$, and $H(t)$ is contained in the interior of $C(i)$; and the collection $\{H(t): d_i \leqslant t \leqslant e_i\}$ of mutually exclusive arcs is continuous.

Denote by $\mathcal{N}$ the collection $\{H(t): 0 \leqslant t \leqslant 1\}$ of mutually exclusive arcs in which the collection $\{fg'(t): 0 \leqslant t \leqslant 1\}$ is imbedded. In order to

show that $\mathcal{N}$ is upper-semicontinuous, it is sufficient to prove that if $\{t_i\}$ is a sequence of numbers converging to the element t of $[0, 1] - W$, and for each i, t_i belongs to the interval $R(i')$, then the sequence $\{H(t_i)\}$ converges to a subset of $H(t)$. It may be assumed that $e_i - d_i < 1/i$ for each positive integer i. The sequence $\{d_{i'}\}$ converges to t, and since the collection $\{g'(t): t \in W\} \cup \{H(t): t \notin W\}$ is upper-semicontinuous, the sequence $\{L(i')\}$ converges to a subset of $H(t)$. Each $H(t_i)$ is contained in the cylinder $C(i')$ whose height, $e_{i'} - d_{i'}$, is less than $1/i'$. Since the distance from the boundary of $D(i')$ to $L(i')$ is less than $1/i'$, it follows that since the sequence $\{L(i')\}$ converges to a subset of $H(t)$, then so does the sequence $\{C(i')\}$, and therefore also the sequence $\{H(t_i)\}$. Hence $\mathcal{N}$ is upper-semicontinuous. Since the subset of $H(t)$ to which the sequence $\{H(t_i)\}$ converges is connected and contains the endpoints of $H(t)$, it is equal to $H(t)$, so that $\mathcal{N}$ is continuous.

Thus the collection $\{f^{-1}(H(t)): 0 \leqslant t \leqslant 1\}$ is a continuous collection of mutually exclusive arcs in E^3 in which $\mathcal{G}'$, and hence $\mathcal{G}$, is imbedded.

THEOREM 3. *If $\mathcal{G}$ is a continuous collection of mutually exclusive finite subsets of E^2 whose decomposition space is an arc, then $\mathcal{G}$ can be imbedded in a continuous collection of mutually exclusive arcs in E^3.*

Proof. Define c to be the function from $\mathcal{G}$ into the positive integers defined by $c(g) = n$ if g has exactly n elements. It will be shown that the collection of change elements of $\mathcal{G}$ is a subset of $\mathcal{K}$, the points of discontinuity of c, and that $\mathcal{K}$ is a closed and 0-dimensional subset of $\mathcal{G}$ in the decomposition topology.

Suppose that there is a change element g of $\mathcal{G}$ at which c is continuous. Then there is an interval $\mathcal{J}'$ of the arc $\mathcal{G}$ containing g such that $c(h) = c(g)$ if $h \in \mathcal{J}'$. Since g has $c(g)$ elements, it can be properly covered by $c(g)$ mutually exclusive open sets such that the set of elements of $\mathcal{G}$ that are properly covered by these open sets is a subinterval $\mathcal{J}$ of $\mathcal{J}'$. (A cover of a set M is *proper* if each element of the cover intersects M.) Since g is a change element, there is a point x in g that is an endpoint of a component of the intersection of two minimal traces of $\mathcal{G}$. Hence the open set containing x contains two points of some element h of $\mathcal{J}$, which implies that $c(h) \geqslant c(g) + 1$. This involves a contradiction, so that the collection of change elements of $\mathcal{G}$ is a subset of $\mathcal{K}$.

Since the set of points of continuity of any function into the integers is an open set, $\mathcal{K}$ is closed. For each positive integer i, denote by $\mathcal{K}(i)$ the set $\{g: g \in \mathcal{K}, c(g) \leqslant i\}$. If each $\mathcal{K}(i)$ is closed and 0-dimensional, then $\mathcal{K} = \bigcup_{i=1}^{\infty} \mathcal{K}(i)$ is 0-dimensional. Suppose that g is an element of $\mathcal{K} - \mathcal{K}(i)$. Since $c(g) > i$, there is a collection $\mathcal{H}$ of more than i mutually exclusive domains in E^2 properly covering g such that the set $\mathcal{U}$

of elements of $\mathcal{G}$ which are properly covered by $\mathcal{H}$ is open in $\mathcal{G}$ in the decomposition topology. Since each element of $\mathcal{U}$ has at least $i+1$ points, $\mathcal{U} \cap \mathcal{K}(i)$ is empty. Hence $\mathcal{K}(i)$ is closed.

Suppose that $\mathcal{K}(i)$ has a non-degenerate component. Since $\mathcal{K}(i)$ is a subset of an arc, this component contains an open interval $\mathcal{I}'$ of $\mathcal{G}$. Since c is bounded above by i on $\mathcal{K}(i)$, there is an element g of $\mathcal{I}'$ such that $c(g) \geqslant c(h)$ for each $h \in \mathcal{I}'$. Since $\mathcal{K}(c(g)-1)$ is closed, there is a subinterval $\mathcal{I}$ of $\mathcal{I}'$ containing g which does not intersect $\mathcal{K}(c(g)-1)$. Thus, if h is an element of $\mathcal{I}$, then $c(g) = c(h)$, and c is continuous at g. This is impossible and $\mathcal{K}$ is closed and 0-dimensional.

Since the closure of the collection of change elements of $\mathcal{G}$ is 0-dimensional, it follows from Theorem 2 that $\mathcal{G}$ can be imbedded in a continuous collection of mutually exclusive arcs in E^3.

The author knows of no weaker condition than that stated in Theorem 2 that will insure the existence of a continuous collection of mutually exclusive arcs in E^3 in which a given collection of mutually exclusive compact 0-dimensional subsets of E^2 whose decomposition space is an arc is imbedded, or indeed whether or not every such collection can be so imbedded. Theorem 4 is of interest in deciding which collections can be imbedded in continuous and equicontinuous collections of mutually exclusive arcs in E^3.

THEOREM 4. *Suppose that $\mathcal{G}$ is a continuous collection of mutually exclusive compact 0-dimensional subsets of the plane $\{z=0\}$ whose decomposition space is an arc. If there exists a homeomorphism f of E^3 onto itself such that if $g \in \mathcal{G}$, $f(g) \subset E^2$ and no two of its points have the same x-coordinate, then $\mathcal{G}$ can be imbedded in a continuous and equicontinuous collection of mutually exclusive arcs in E^3.*

Proof. Denote by g a homeomorphism from $[0,1]$ onto the arc $\mathcal{G}$. Since $\mathcal{G}^*$ is compact, it will be assumed that $f(\mathcal{G}^*)$ is contained in $\{(x,y,0)\colon 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$. The function k' from the closed subset $\mathcal{G}^*$ of E^2 into the real numbers, defined by $k'(x,y,0) = t$ if $(x,y,0)$ is in $g(t)$, is continuous, so there is a continuous extension k of k' to all of E^2.

Define a homeomorphism h' of E^3 onto itself by $h'(x,y,z) = (x,y,z+k(x,y,0))$. Then if $0 \leqslant t \leqslant 1$, $h'fg(t) \subset \{z=t\}$ and no two of its points have the same x-coordinate. For each point $(x,0,t)$ in the projection of $h'f(\mathcal{G}^*)$ into the plane $\{y=0\}$, define $e(x,0,t)$ to be that unique number y such that $(x,y,t) \in h'f(\mathcal{G}^*)$. Since e is continuous and the projection is closed, there is a continuous extension e' of e from $\{y=0\}$ into the real numbers. Define a homeomorphism h of E^3 onto itself by $h(x,y,z) = (x, y-e'(x,0,z), z)$. For each $t \in [0,1]$, $hh'fg(t) \subset \{(x,0,t)\colon 0 \leqslant x \leqslant 1\}$. If g' denotes the homeomorphism $hh'f$ then $\mathcal{G}$ is imbedded in the continuous and equicontinuous collection $\{g'^{-1}(\{(x,0,t)\colon 0 \leqslant x \leqslant 1\})\colon 0 \leqslant t \leqslant 1\}$ of mutually exclusive arcs in E^3.

The hypothesis of Theorem 4 is almost necessary in that if the collection $\mathcal{G}$ of subsets of $\{z=0\}$ can be imbedded in a continuous and equicontinuous collection $\mathcal{H}$ of mutually exclusive arcs in E^3 whose decomposition space is an arc, then there is a homeomorphism f of $\mathcal{G}^*$ into $\{z=0\}$ such that if $g \in \mathcal{G}$, then no two points of $f(g)$ have the same y-coordinate. This partial converse is a direct result of the proof of Theorem 14 of [6], which implies that there is a homeomorphism from $\mathcal{H}^*$ onto the union of a collection of vertical straight line segments in E^2.

COROLLARY. *Suppose that K is a compact subset of $\{z=0\}$ and that K is contained in the ball D in E^3. If k is a light continuous mapping from K onto $[0,1]$ such that for each $t \in [0,1]$ no two points of $k^{-1}(t)$ have the same x-coordinate, then there are two continuous and open mappings h and g such that (1) g is a mapping from D onto $[-1,2]$ that extends k and if $t \in [-1,2]$ then $g^{-1}(t)$ is a disk, and (2) h is a mapping from D onto $[-1,2]\times[-1,2]$ such that if $(r,s) \in [-1,2]\times[-1,2]$ then $h^{-1}(r,s)$ is an arc contained in the disk $g^{-1}(s)$ and if $t \in [0,1]$ then $k^{-1}(t) \subset h^{-1}(0,t)$.*

Proof. The collection $\{k^{-1}(t)\colon 0 \leqslant t \leqslant 1\}$ satisfies the hypothesis of Theorem 4, and from its proof there is a homeomorphism v of E^3 onto itself such that if $p \in K$ then $v(p) \subset \{(x,0,k(p))\colon 0 \leqslant x \leqslant 1\}$. There is a homeomorphism w of E^3 onto itself that is the identity at points of $v(K)$ and takes the ball $v(D)$ onto the relatively large cube $\{(x,y,z)\colon -1 \leqslant x \leqslant 2,\ -1 \leqslant y \leqslant 2,\ -1 \leqslant z \leqslant 2\}$. If p is a point of D define $g(p)$ to be the z-coordinate of $wv(p)$ and define $h(p)$ to be the ordered pair the first term of which is the y-coordinate of $wv(p)$ and the second term of which is the z-coordinate of $wv(p)$ and g and h are the desired mappings.

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