A general theory of structure spaces

by

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Structure spaces have been considered for a great number of different algebraic systems such as rings, Banach algebras, semi-rings, lattices, lattice-ordered groups etc. In a situation like this the problem of unification naturally poses itself. The purpose of this paper is to show that the theory of $\alpha$-ideals as developed in [1] and [2] appears as a natural framework for a general theory of structure-spaces. Some indications in this direction were already given in [1]. We shall here pursue the subject along more general lines. The present development is in fact general enough to cover a large number of special cases. On the other hand a reasonable part of the theory of structure spaces for special algebraic systems may be generalized to our situation.

In § 6 we first give the necessary algebraic background. With exception of Proposition 0.2 this is contained in [1] and [2], to which we refer for proofs, more details and special cases. In § 1 we turn to the theory of structure-spaces for commutative semi-groups with an $\alpha$-system. Theorems 1, 2 and 3 concerning Hausdorff structure-spaces are mainly generalizations of parts of the corresponding theory for rings in [4]. In § 2 we consider the compactness of structure-spaces. The theory there follows the same lines as the theory in [3] and [5]. It is easy to see that if $R$ is a semisimple commutative ring with identity, then $\mathfrak{M}(R)$ is disconnected if and only if $R$ is the direct sum of two of its ideals, $R_1$ and $R_2$. If $\mathfrak{M}(R) = \mathfrak{M}_1 \cup \mathfrak{M}_2$ is a partition of $\mathfrak{M}(R)$ into disjoint open-closed proper subsets, $R_1$ and $R_2$ may be chosen so that $\mathfrak{M}(R_i)$ is homeomorphic to $\mathfrak{M}_i$, $i = 1, 2$. The argument here makes use of the additivity structure of the ring. In § 3 we show how this may nevertheless be transferred to our situation, the additivity being taken care of mainly by the additivity axiom for an $\alpha$-system on a commutative semigroup, introduced in [2]. The representation theorem of § 4 is related to Theorem 14 of [10]. Finally, we illustrate by a simple example a rather interesting procedure: an algebraic problem may be reformulated in terms of structure-spaces, and solved by simple topological reasoning.

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§ 0. Preliminaries. Let $S$ be a commutative semigroup. We shall say that there is defined an $x$-system on $S$ if to every $A \subseteq S$ there corresponds $A_x \subseteq S$ such that

\begin{align*}
A & \subseteq A_x, \\
A \subseteq B_x & \Rightarrow A_x \subseteq B_x, \\
A B_x & \subseteq B_x, \\
A B_x & \subseteq (A B)_x.
\end{align*}

(0.4) is referred to as the continuity axiom. A subset $A$ in $S$ is said to be an $x$-ideal, or shorter an ideal, if $A = A_x$. We may give an equivalent definition of an $x$-system in the following way: let $S$ be a commutative semigroup, and let $X$ be a non-empty family of subsets of $S$, called $x$-ideals, such that the following conditions are satisfied:

(0.5) The intersection of any non-empty family of $x$-ideals is an $x$-ideal.

(0.6) For any $a \in S$ and any $A \in X$, $A : a$ is an $x$-ideal containing $A$.

Let $A$ be any subset of $S$, and put $A_x = \bigcup_{a \in A} B$. Then the correspondence $A \rightarrow A_x$ defines an $x$-system in the sense of (0.1)-(0.4) and the family of $x$-ideals is $X$.

$A_x$ is said to be a proper ideal if $A_x \neq S, \emptyset$. A prime $x$-ideal is an ideal $P_x$ satisfying $a b \in P_x \Rightarrow a \in P_x \vee b \in P_x$, and a maximal $x$-ideal is a proper ideal not properly contained in any proper ideal. A minimal prime ideal is defined correspondingly. The families of prime, maximal and minimal ideals in $S$ are denoted by $\mathfrak{P}$, $\mathfrak{M}$ and $\mathfrak{m}$ respectively.

An $x$-system is said to be of finite character if the set-theoretic union of any chain of $x$-ideals is an $x$-ideal.

If $S$ is a commutative semigroup with an $x$-system, and $T$ is a subsemigroup of $S$, then it is easily verified that the family of all intersections between an $x$-ideal in $S$ and $T$ defines an $x$-system on $T$. This $x$-system is said to be induced on $T$ from $S$. (This definition is not the best one, but will be the relevant definition for our purpose.)

Given a family of $x$-ideals in $S$, $(A_x)^{\infty}$. Put $(\bigcup_{a \in A} A_x^a = (\bigcup_{a \in A} A_x)^a_x$. $X$ is a complete lattice under $\cup$ and $\cap$. $\cup$ will be referred to as $x$-union.

Furthermore, put $A_x \cup B_x = (A B)_x$. This operation is referred to as $x$-multiplication.

The nilpotent radical of an $x$-ideal $A_x$, denoted by $\text{rad} A_x$, is the set of all elements $a$ in $S$ such that for some $n$, $a^n \in A_x$, $A_x$ is said to be half-prime if $\text{rad} A_x = A_x$. If every $x$-ideal in $S$ is half-prime, the $x$-system is said to be half-prime. The element $e \in S$ is called an $x$-identity if $(e) = S$ and $e \in S^x$.

An $x$-ideal $A_x$ is shown to be non-prime if and only if $A_x \supseteq B_x \cup C_x$ for some $B_x$ and $C_x$ properly containing $A_x$. This implies that if $S$ has an $x$-identity, then $\emptyset \subseteq S^x$. A subset $M$ of $S$ which is either empty or closed under multiplication is referred to as an $m$-set. We have, for $x$-systems of finite character:

**Proposition 0.1.** Given an $x$-ideal $A_x$ in $S$. If $M$ is a maximal $m$-set contained in $C A_x$ and $P_x$ is an ideal maximal with respect to the property of containing $A_x$ and being contained in $C M$, then $P_x$ is a minimal prime ideal over $A_x$. ($C A_x$ denotes the complement of $A_x$ in $S$.)

**Corollary 1.** Any prime ideal $P_x$ over $A_x$ contains a least one minimal prime ideal over $A_x$.

**Corollary 2.** For any maximal $m$-set $M$ contained in $C A_x$, $C M$ is a minimal prime ideal over $A_x$.

For a number of algebraic systems, the following condition turns out to be of importance:

(0.7) To every $a \in S$ there exists an idempotent element $[a]$ such that for every $x$-ideal $A_x$ in $S$, $a \in A_x$ implies $[a] \in A_x$.

For a distributive lattice with the $l$-system, we may choose $a = \{a\}$, for a lattice ordered group with the $e$-system and the multiplication $a \cdot b = [a] \cap [b]$, we put $[a] = a$ if $a \neq 0$.

At present, we shall only note the following consequence of (0.7), which will be useful later (see [7], Theorem 0.5):

**Proposition 0.2.** Given an ideal $A_x \neq \emptyset$, in an $x$-system satisfying (0.7). Then an ideal $P_x \supseteq A_x$ is a minimal prime ideal over $A_x$ if and only if it satisfies the following condition:

(0.8) To every idempotent element $c$ in $P_x$ there exists $b \in P_x$ such that $b c \in A_x$.

**Proof.** If $P_x$ satisfies (0.8) and $A_x \subseteq Q_x \subseteq P_x$ for a prime ideal $Q_x$, choose $a \in P_x$, $a \in Q_x$. Then $[a] \in P_x$, $[a] \not\in Q_x$. To $[a]$ there corresponds by (0.8) $b \in P_x$ such that $[a] = b \epsilon A_x$. Then $[a] \in Q_x$, a contradiction. Conversely, if $P_x$ is a minimal prime ideal over $A_x$, $M = C P_x$ is an $m$-set, maximal in $C A_x$ by Corollary 2 above. For an idempotent element $c \in A_x$, put $M(c) = M \cup \{b \in M \mid b \epsilon A_x\}$. Obviously $M(c)$ is an $m$-set properly containing $M$, thus $M(c) \cap A_x \neq \emptyset$ (and (0.8) is satisfied).

**Theorem 0.1** (Kroll-Stone). For an $x$-system of finite character, $\text{rad} A_x$ is equal to the intersection of the minimal prime ideals over $A_x$.

Let $S$ and $T$ be commutative semigroups with $x$-systems which we denote respectively by $y$ and $x$. We shall say that a multiplicative...
homomorphism $\varphi$ of $S$ into $T$ is a $(y, z)$-homomorphism if $\varphi(A_y) \subseteq (\varphi(A_z))$, for all subsets $A$ of $S$, or, equivalently, if the inverse image of a $z$-ideal in $T$ is a $y$-ideal in $S$. If $\varphi$ is a multiplicative homomorphism of a commutative semigroup $S$ onto a semigroup $T$, and $S$ has an $x$-system $y$, then the set of all $B \subseteq T$ such that $\varphi^{-1}(B)$ is a $y$-ideal in $S$ defines an $x$-system $y_\varphi$ in $T$. This makes $\varphi$ a $(y, y_\varphi)$-homomorphism.

For $a, b \in S$, put

$$a \equiv b(A_\varphi) \iff \{a_x, \varphi(A_x) \} = \{a_x, b_x \}.$$

This is a congruence relation in $S$, which we refer to as $x$-congruence. If $\varphi$ denotes the canonical multiplicative homomorphism of $S$ onto $S/A_\varphi$, we shall call $\varphi_x$ the canonical $x$-system on $S/A_\varphi$.

The $x$-system on $S$ is said to be additive if for any $x$-ideals $A_x, B_x$ in $S$ and $x, y \in A_x \cup B_x$, there exists $b \in B_x$ such that $c = b(A_x)$.

**Theorem 0.2.** An $x$-system on $S$ is additive if and only if the canonical mapping $\varphi^{-1} : A_\varphi \cap B_\varphi \to A_x \cup B_x$ is bijective for any $x$-ideals $A_x$ and $B_x$ in $S$.

Given any family $\mathcal{F}$ of $x$-ideals in $S$, we adopt the following notation: For $\mathcal{F} \subseteq \mathcal{I}$, put $k\mathcal{F} = \bigcap_{A_x \in \mathcal{F}} A_x$, for an $x$-ideal $B_x$ in $S$, $kB_x = \{ a_x \in \mathcal{I} : a_x \subseteq B_x \}$. Furthermore, put $kA_\varphi = \{ a_\varphi \in \mathcal{I} : a_\varphi \subseteq A_\varphi \}$.

**§ 1. Structure spaces and their separation properties.**

Let $S$ be a commutative semigroup with an $x$-system, and let $\mathcal{I}$ be a family of proper $x$-ideals such that

$$B_x \cap C_x \subseteq A_x = B_x \cap C_x \cap A_x$$

whenever $A_x \in \mathcal{I}$ and $B_x$ and $C_x$ are arbitrary intersections of ideals from $S$. Under this assumption $\mathcal{I}$ is said to be a structure-family for $S$.

If $\mathcal{U} \subseteq \mathcal{I}$, put $\mathcal{U} = k\mathcal{U}$ if $\mathcal{U} = \mathcal{O}$, $\mathcal{U} = \mathcal{O}$.

The above definitions are justified by

**Proposition 1.** $\mathcal{U} \to \mathcal{U}$ defines a topology on $\mathcal{I}$ if and only if $\mathcal{I}$ is a structure-family for $S$.

Proof. (1.1) is equivalent to $\mathcal{U} \cup \mathcal{V} \subseteq \mathcal{U} \cup \mathcal{V} \subseteq \mathcal{U}$ as $\mathcal{V} \subseteq \mathcal{U}$, $\mathcal{U} \subseteq \mathcal{V}$ and $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{U} \cup \mathcal{V}$ are satisfied, without assuming (1.1), the proposition follows.

This topology is referred to as the Zariski topology and as the Stone topology on $\mathcal{I}$, the corresponding topological space is called a structure-space for $S$. It will be denoted by $\mathcal{I}(S)$.

There is an obvious 1-1 correspondence between the substructures of a structure-family and the subspaces of the corresponding structure-space, in the sense that every subfamily of a structure-family is a structure-family, and all subspaces are obtained in this way.
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LEMMA 1. Let \( S \) be a basis for the neighbourhood system of \( A_s \). Then

\[
N(A_s) = \bigcup_{U \in S} U.
\]

If the \( \varepsilon \)-system is of finite character, we have

\[
N(A_\varepsilon) = \bigcup_{U \in S} U.
\]

Proof. For two bases \( S_1 \) and \( S_2 \) we get \( \bigcup U \subseteq \bigcup U \) and
the first part follows from Proposition 3. To prove the second half of the
lemma, we first observe that \( N(A_\varepsilon) = \emptyset \) implies \( \bigcup U = \emptyset \). For
\( a \in \bigcup U \) we find, by the finite character property, a finite set \( \{a_1, \ldots, a_k\} \subseteq \bigcup U \) such that \( a \in \{a_1, \ldots, a_k\} \varepsilon \). Let \( a \in \bigcup U \), and determine \( A \in S \)
such that \( A \subseteq \bigcup U \). Then

\[
a \in \bigcup U \subseteq \bigcup U = \bigcup U \subseteq \bigcup U \subseteq \bigcup U.
\]

We thus have \( \bigcup U \subseteq \bigcup U \subseteq \bigcup U \), and the lemma is proved.

Obviously \( \exists \subseteq \bigcup U \subseteq \bigcup U \). The equalities are reflected in \( \exists(S) \)
as follows ((1.9) generalizes Theorem 2.7 of [4]).

THEOREM 1.

\[
N(A_s) = A_s \quad \text{for every neighbourhood } U \subseteq A_s.
\]

If the \( \varepsilon \)-system is of finite character, we get

\[
N(A_\varepsilon) = A_\varepsilon \quad \text{for every } A_\varepsilon \subseteq \exists(S) \subseteq \exists(a).
\]

Proof. Clearly, \( N(A_s) = \exists \subseteq \exists \subseteq \exists \) for every neighbourhood \( U \subseteq A_s \), and (1.8) follows. Assume that the \( \varepsilon \)-system of \( S \) is of finite character, and that \( N(A_s) = A_s \) for every \( A_s \subseteq S \). Then, for \( A_s \subseteq S \), \( a \in A_s \subseteq S \), and \( A_s \subseteq S \).

Corollary. Let \( S \) be a commutative semigroup with an \( \varepsilon \)-system
of finite character, satisfying (0.7). Assume also that \( \exists \subseteq \exists \). Then
the structure-space of the minimal prime ideals, \( \mathcal{R}(S) \), is a disconnected Hausdorff space.

Proof. Let \( P_e \subseteq \mathcal{R}(S) \). Then for every \( e \in P_e \) there exists by
Proposition 3.9.3, \( b \in P_e \) such that \( a \in P_e \), \( b \in P_e \), \( a \in P_e \), \( b \in P_e \)
and \( a \in P_e \). We conclude \( P_e = N(P_e) \). As \( \mathcal{R}(S) \) is \( T_1 \), the proof is complete.

The connection between the Hausdorff-property of \( \mathcal{R}(S) \) and algebraic properties of \( S \) is of the same kind as in the case where \( S \) is a ring with the \( \varepsilon \)-system. The following theorem is a generalization of Theorem 3.1 of [4].

THEOREM 2. (A), (B), (C) and (D) are equivalent.

(A) \( \exists \subseteq \exists \) is a Hausdorff-space.

(B) For \( A_s \subseteq B_s \) different elements of \( \exists(S) \), there exists \( a \in A_s \), \( b \in B_s \) such that for every \( c \in \exists(S) \), \( a \in c \) or \( b \in c \).

(C) For \( A_s \subseteq B_s \) different elements of \( \exists(S) \), \( N(A_s) \subseteq B_s \).

(D) For every \( A_s \subseteq B_s \) contained in exactly one ideal from \( \exists \).

(E) If \( \exists \subseteq \exists \) and the \( \varepsilon \)-system is of finite character, (A) is also equivalent to

(F) For \( A_s \subseteq B_s \) different ideals from \( \exists \), \( N(A_s) \cap N(B_s) = S \).

Proof. Throughout the proof, let \( A_s, B_s \) denote ideals from \( \exists \). We first verify (A) \( \Rightarrow \) (B) \( \Rightarrow \) (C) \( \Rightarrow \) (D) \( \Rightarrow \) (A).

By Proposition 3.9, (A) \( \Rightarrow \) (B). Assume (B). Let \( A_s \neq B_s \), and let \( a, b \) satisfy (B). Then \( b \in A_s \subseteq B_s \). \( a \subseteq B_s \). (C) follows. Obviously (C) \( \Rightarrow \) (D). Assume (D). Then, for \( A_s \neq B_s \), Lemma 1 (1.6) gives that \( \bigcup U \subseteq B_s \), where \( B_s \) is the

neighbourhood-system of \( A_s \). Choose \( a \in \bigcup U \), \( a \notin B_s \). Then \( a \in A_s \).

If every proper ideal in \( S \) is contained in an ideal from \( \exists \), (A) is also equivalent to

(F) For \( A_s \subseteq B_s \) different ideals from \( \exists \), \( N(A_s) \cong N(B_s) = S \).

Now, assume that the \( \varepsilon \)-system is of finite character, and let \( \exists \subseteq \exists \).

If (A) is satisfied, and \( A_s \neq B_s \), then for \( a \in S \) we get two possibilities:

For \( a \in B_s \), choose \( b \in N(A_s) \cap B_s \). This is possible, for by Theorem 1.

Corollary 3.1. \( N(A_s) = O \). Then \( a \notin B_s \), \( a \in B_s \).

On the other hand, if \( a \notin B_s \), \( a \in B_s \), \( a \in B_s \).

Choose \( a \notin B_s \). For \( b \in B_s \) given by (B), \( b \in N(A_s) \cap B_s \).

Here \( a \subseteq B_s \subseteq S \), and \( b \subseteq B_s \). Assume (E). Let \( A_s \neq B_s \). Choose \( a \subseteq B_s \). For \( b \) given by (B), \( b \in N(A_s) \cap B_s \).

By Lemma 1 (1.7), \( b \subseteq N(A_s) \subseteq B_s \).

Thus (A) follows.

To prove the last part of the theorem, assume that every proper ideal in \( S \) is contained in an ideal from \( \exists \). If \( \exists(S) \) is a Hausdorff-space
and \( A_s \neq B_s \), (D) implies that \( N(A_s) \cap N(B_s) = \emptyset \). If (F) is satisfied,
and for some \( A_s \neq B_s \), \( N(A_s) \subseteq B_s \), we find \( B_s = S \), in contradiction to the definition of a structure-family.
Lemma 2. Assume that the $x$-system is of finite character and that $\mathfrak{I} \subseteq \mathfrak{J}$. Then $\mathcal{N}(\mathfrak{N})$ is half prime for every $\mathfrak{A}_x \in \mathfrak{I}$.

Proof. Let $a \in \mathfrak{N}(\mathfrak{A}_x)$, i.e., $a^* \in \mathfrak{N}(\mathfrak{A}_x)$. For some neighbourhood $U_a$ of $\mathfrak{A}_x$, $a^* \in kU_a$ by Lemma 1. For every $R_x \in U_a$, $a^* \in kR_x$, and since $kR_x \subseteq \mathfrak{N}(\mathfrak{A}_x)$, we have $a \in kR_x$. Thus $a \in kU_x \subseteq \mathcal{N}(\mathfrak{A}_x)$.

The following is a generalization of Corollary 3.8 of [4].

Theorem 3. Assume that the $x$-system is of finite character. Then the following statements are equivalent:

(A) $\mathfrak{B}(\mathfrak{S})$ is a Hausdorff-space.

(B) $\mathfrak{B}(\mathfrak{S})$ is totally disconnected.

(C) For every $\mathfrak{A}_x \in \mathfrak{B}$, $\mathcal{N}(\mathfrak{A}_x) = \mathfrak{A}_x$.

Proof. We verify (C) $\Rightarrow$ (B) $\Rightarrow$ (A) $\Rightarrow$ (C), $(C) \Rightarrow (B)$ follows from Theorem 1 (B), $(B) \Rightarrow (A)$ is obvious. Assume (A). By the Krull-Stein theorem, Lemma 2 gives

$$\mathfrak{N}(\mathfrak{A}_x) = \bigcap_{P_x \in \mathcal{P}(\mathcal{N}(\mathfrak{A}_x))} P_x.$$ 

If $\mathcal{N}(\mathfrak{A}_x) \neq \mathfrak{A}_x$, then for some $P_x \neq P_y \supseteq \mathcal{N}(\mathfrak{A}_x)$, contradicting (A).

§ 2. Compactness. We now turn to the compactness of structure-spaces, and make the following observation:

Proposition 4. If $(a_i)_{i \in I}$ is an open covering of $\mathfrak{N}(\mathfrak{S})$ if and only if $E \subseteq \mathfrak{A}_x$ for every $\mathfrak{A}_x \in \mathfrak{I}$.

Theorem 4. $\mathfrak{N}(\mathfrak{S})$ is compact if and only if every subset $R$ of $S$ with the property $E \subseteq \mathfrak{A}_x$ for every $\mathfrak{A}_x \in \mathfrak{I}$, contains a finite subset $N$ with the property $N \subseteq \mathfrak{A}_x$ for every $\mathfrak{A}_x \in \mathfrak{I}$.

Proof. The theorem follows from Proposition 3 and Proposition 4.

Theorem 5. If the $x$-system is of finite character, $\mathfrak{N}(\mathfrak{S})$ is compact if and only if every ideal $R_x$ satisfies $R_x \subseteq \mathfrak{A}_x$ for every $\mathfrak{A}_x \in \mathfrak{I}$. Assume the condition follows from Theorem 4. Assume the condition follows from Theorem 4. Assume the condition follows from Theorem 4.

Theorem 7. Let $S$ be a commutative semigroup with an $x$-system of finite character and $x$-identity. Then $\mathfrak{B}(\mathfrak{S})$ is disconnected if and only if there exist ideals $A_x, B_x$ in $S$ different from $S$ and $kS$, such that $A_x \cap B_x = k\mathfrak{S}$.

If the $x$-system is additive, and if $k\mathfrak{S} = k\mathfrak{S} = A_x$, then for every partition of $\mathfrak{S}$ into disjoint open-closed subsets $\mathfrak{A}$, $\mathfrak{B}$, we may determine $A_x$ and $B_x$ such that $\mathfrak{A}(\mathfrak{A}) = k\mathfrak{S}$, $\mathfrak{B}(\mathfrak{B})$ homeomorphic to $\mathfrak{A}$, $\mathfrak{B}(\mathfrak{B})$ homeomorphic to $\mathfrak{B}$, where $A_x$ and $B_x$ are equipped with the $x$-systems induced from $S$.

Proof. Let $\mathfrak{B}(\mathfrak{S}) = \mathfrak{A} \cup \mathfrak{B}$ be a partition of $\mathfrak{S}$ into proper, disjoint open-closed subsets and put $A_x = k\mathfrak{S}$, $B_x = k\mathfrak{S}$. Then $A_x \cap B_x = k\mathfrak{S}$. Furthermore, $A_x \cup B_x = S$, for if $A_x \cup B_x$ were a proper ideal, it would be contained in some maximal ideal $M_x$. (This is proved in the usual way by the existence of an $x$-identity.) Then $M_x \subseteq \mathfrak{A} \cup \mathfrak{B}$, a contradiction. Clearly $A_x$ and $B_x$ are different from $S$ and $k\mathfrak{S}$. 

Proof. Under the assumptions of the corollary, $N_x \not\subseteq A_x$ for every $A_x \in \mathfrak{I}$, if and only if $N_x = S$.

The compactness of $\mathfrak{I}(\mathfrak{S})$ thus amounts to a finiteness condition on $S$. On the other hand, we may take a somewhat different point of view:

A proper subset $V$ of $S$ is said to be an $f$-set for $\mathfrak{I}$ if for every finite subset $\mathfrak{N}$ of $V$ there exists an $A_x \in \mathfrak{I}$ such that $N \subseteq A_x$ (see [5]). By Zorn’s Lemma we find that every $f$-set for $\mathfrak{I}$ in $S$ is contained in a maximal $f$-set for $\mathfrak{I}$ in $S$.

Theorem 6. $\mathfrak{I}(\mathfrak{S})$ is compact if and only if every maximal $f$-set for $\mathfrak{I}$ is a member of $\mathfrak{I}$.

Proof. Assume that $\mathfrak{I}(\mathfrak{S})$ is compact, and let $W$ be a maximal $f$-set. We contend that $W \subseteq \mathfrak{I}$, as every $A_x \in \mathfrak{I}$ is an $f$-set, it is sufficient to find $A_x \in \mathfrak{I}$ with $A_x \supseteq W$. Assume that $A_x \not\subseteq W$ for every $A_x \in \mathfrak{I}$. By Theorem 4 we find a finite subset $N$ of $W$ with $N \subseteq k\mathfrak{S}$ for every $A_x \in \mathfrak{I}$, a contradiction. Conversely, assume that every maximal $f$-set is an element of $\mathfrak{I}$. If $\mathfrak{I}(\mathfrak{S})$ were not compact, we could find an $f$-set $F$ for $\mathfrak{I}$ with $F \not\subseteq A_x$, in contradiction to the assumption.

§ 3. Disconnected $\mathfrak{B}(\mathfrak{S})$. Let $E$ be a commutative ring with identity. If $E$ is semisimple, then $\mathfrak{B}(\mathfrak{B})$ is disconnected if and only if $E$ is the direct sum of two of its proper ideals $R_1$ and $R_2$. If $\mathfrak{B}(\mathfrak{B}) = B_1 \oplus B_2$ is a partition of $\mathfrak{B}(\mathfrak{B})$ into disjoint open-closed proper subsets, we may determine $R_i$ and $R_2$ so that $\mathfrak{B}(\mathfrak{B})$ is homeomorphic to $B_1$, $B_2$.

We observe that if $E$ is the direct sum of the ideals $R_1$ and $R_2$, then $R = R_1 \cup R_2$, $R_1 \cap R_2 = \{0\}$ and every $d$-ideal $A$ of $R_1$ is a $d$-ideal of $R$. In view of this the following theorem generalizes the above-mentioned theorem for rings.

Theorem 7. Let $S$ be a commutative semigroup with an $x$-system of finite character and $x$-identity. Then $\mathfrak{B}(\mathfrak{S})$ is disconnected if and only if there exist ideals $R_x$, $S_x$ in $S$ different from $S$ and $k\mathfrak{S}$ such that $A_x \cap B_x = k\mathfrak{S}$.

If the $x$-system is additive, and if $k\mathfrak{S} = k\mathfrak{S} = A_x$, then for every partition of $\mathfrak{S}$ into disjoint open-closed proper subsets $\mathfrak{A}$, $\mathfrak{B}$, we may determine $A_x$ and $B_x$ such that $\mathfrak{A}(\mathfrak{A}) = k\mathfrak{S}$, $\mathfrak{B}(\mathfrak{B})$ homeomorphic to $\mathfrak{A}$, $\mathfrak{B}(\mathfrak{B})$ homeomorphic to $\mathfrak{B}$, where $A_x$ and $B_x$ are equipped with the $x$-systems induced from $S$.

Proof. Let $\mathfrak{B}(\mathfrak{S}) = \mathfrak{A} \cup \mathfrak{B}$ be a partition of $\mathfrak{S}$ into proper, disjoint open-closed subsets and put $A_x = k\mathfrak{S}$, $B_x = k\mathfrak{S}$. Then $A_x \cap B_x = k\mathfrak{S}$. Furthermore, $A_x \cup B_x = S$, for if $A_x \cup B_x$ were a proper ideal, it would be contained in some maximal ideal $M_x$. (This is proved in the usual way by the existence of an $x$-identity.) Then $M_x \subseteq \mathfrak{A} \cup \mathfrak{B}$, a contradiction. Clearly $A_x$ and $B_x$ are different from $S$ and $k\mathfrak{S}$.
Conversely, assume the existence of $A_2$ and $B_3$ satisfying the condition of the theorem. Put $\mathcal{U} = \mathcal{H}_2$, $\mathcal{B} = \mathcal{H}_2$. Clearly, $\mathcal{U}$ and $\mathcal{B}$ are closed. Since $\mathcal{R} \subseteq \mathcal{B}$ and $\mathcal{A}_2 \nsubseteq B_2 = \mathcal{H}_2$, we have $\mathcal{U} \nsubseteq \mathcal{B}$ and $\mathcal{R}$ is not in $\mathcal{B}$, which implies $\mathcal{U} \nsubseteq \mathcal{B}$ and the first part of the theorem is proved.

Finally assume the condition of the last part of the theorem. Define $A_2$ and $B_3$ as above, $A_2 = \mathcal{H}_2$, $B_3 = \mathcal{H}_2$, where $\mathcal{U} \nsubseteq \mathcal{B}$ is a partition of $\mathcal{R}(S)$ into proper, disjoint open closed subsets. By symmetry it is sufficient to prove that $\mathcal{R}(A_2)$ is homeomorphic to $\mathcal{R}$. We denote the $\mathcal{R}$-system induced on $A_2$ by $\mathcal{A}_2$, and prove first that $A_2\mathcal{R}A_0$ is isomorphic to $S/B_3$. Denote the canonical mapping of $A_2$ onto $A_2\mathcal{R}A_0$ by $\psi$, of $S$ onto $S/B_3$ by $\varphi$; and denote the canonical $\mathcal{R}$-system of $S/B_3$ by $\mathcal{R}$, of $A_2\mathcal{R}A_0$ by $\mathcal{A}_2$. Now define

$$\varphi: A_2\mathcal{R}A_0 \to S/B_3$$

by $\varphi(\psi(s)) = \psi(s)$ for $s \in A_2$. By definition $A_2\mathcal{R}A_0 = A_2/A_2 \nsubseteq B_3$, $S/B_3 = A_2 \nsubseteq B_3$, $B_3 = A_2 \nsubseteq B_3$, and $\varphi$ is bijective by Theorem 0.2. Clearly $\mathcal{R}$ is multiplicative, and every $\mathcal{R}$-ideal in $A_2\mathcal{R}A_0$ may be written in the form $\mathcal{R}(C_n)$, where $C_n$ is some $\mathcal{R}$-ideal in $A_2$, i.e., an $\mathcal{R}$-ideal contained in $A_2$. Now $\varphi(\psi(C_n)) = \psi(C_n)$, showing that the image by $\psi$ of every $\mathcal{R}$-ideal in $A_2\mathcal{R}A_0$ is an $\mathcal{R}$-ideal in $S/B_3$. Conversely, every $\mathcal{R}$-ideal in $S/B_3$ may be written as $\psi(D_n)$, where $D_n \subseteq B_3$. Now

$$\varphi^{-1}(\psi(D_n)) = \psi(s)$$

for $s \in A_2$ and $\psi(s) \in \mathcal{R}(D_n)$

$$= (\psi(s); s \in A_2$$

and $s \in D_n) = \psi(A_2 \nsubseteq D_n)$

showing that $\varphi$ is an isomorphism. This shows in particular that $A_2\mathcal{R}A_0$ has an $\mathcal{R}$-identity, so that the maximal ideals is a structure-family of $A_2\mathcal{R}A_0$. Furthermore,

$$(3.1) \quad \mathcal{R}(A_2\mathcal{R}A_0) \text{ is homeomorphic to } \mathcal{R}(S/B_3).$$

By (1.2) and (1.3) the family of maximal ideals in $A_2$ is a structure-family, and

$$(3.2) \quad \mathcal{R}(A_2) \text{ is homeomorphic to } \mathcal{R}(A_2\mathcal{R}A_0).$$

Finally, the corollary of Proposition 2 gives

$$(3.3) \quad \mathcal{R}(S/B_3) \text{ is homeomorphic to } \mathcal{R}(S/$$)

(3.1), (3.2) and (3.3) implies that $\mathcal{R}(A_2)$ is homeomorphic to $\mathcal{R}$, and the theorem is proved.

§ 4. A representation theorem. The $\mathcal{R}$-system of a distributive lattice $L$ is a half prime $\mathcal{R}$-system of finite character, when $L$ is considered as a semi-group under $\wedge$. This $\mathcal{R}$-system has the property that every finitely generated ideal is a principal ideal. We shall now prove a theorem closely related to a converse of this statement. In fact, let $S$ be a commutative semigroup with a half-prime $\mathcal{R}$-system of finite character, where every finitely generated ideal is a principal ideal. Denote the canonical $\mathcal{R}$-system of $S/O_1$ by $\mathcal{R}$. Then we have:

**Theorem 8.** There exists a family $\mathcal{S}$ of open sets in $\mathcal{R}(S)$ such that $\mathcal{S}$ is a lattice under $\wedge$ and $\mathcal{S}$, and such that $S/O_1$ is isomorphic to $\mathcal{S}$, where $\mathcal{S}$ is considered as a semi-group under $\wedge$.

**Proof.** Let $\mathcal{S} = \{U(s); s \in S\}$. For $s_1, s_2 \in S$ there exists $s \in S$ such that $s_1 = (s, s_2)$. Then

$$(4.1) \quad U(s_1) \cap U(s_2) = U(s),$$

since $\mathcal{R}(S)$ consists of prime ideals,

$$(4.2) \quad U(s_1) \cup U(s_2) = U(s_1 s_2)$$

and $\mathcal{S}$ is a lattice under $\wedge$ and $\vee$. For $s \in S/O_1$, put $\mathcal{R}(s) = U(s)$. Now, $s_1 = s_2$ if and only if $U(s_1) = U(s_2)$. The $\mathcal{R}$-system is half prime, so this is equivalent to $U(s_1) = U(s_2)$, which is again equivalent to $U(s_1) = U(s_2)$. To sum up, $s_1 = s_2$ means $U(s_1) = U(s_2)$. This shows that $\mathcal{R}$ is well defined and injective. Clearly $\mathcal{R}$ is surjective; that $\mathcal{R}$ is multiplicative follows by (4.2), and it remains to be shown that $\mathcal{R}$ establishes a 1-1 correspondence between the ideals in $S/O_1$ and $\mathcal{S}$. Every half-ideal $A_1$ in $S/O_1$ there corresponds to an $\mathcal{R}$-ideal $A_1$ in $S$ such that $A_1 = \mathcal{R}(A_1)$, where $\mathcal{R}$ is the canonical mapping of $S$ onto $S/O_1$. Now $\mathcal{R}(A_1) = U(s)$, for $s \in A_1$. For $s \in A_1$, we find $U(s) = U(s) = \mathcal{R}(A_1)$. For $s_1, s_2 \in A_1$ we find $s \in A_1$ such that $s_1 = (s_1, s_2)$, and $U(s_1) \cup U(s_2) = U(s) \quad \mathcal{R}(A_1)$, since we clearly have $s \in A_1$. Thus $\mathcal{R}(A_1)$ is an $\mathcal{R}$-ideal. Conversely, every $\mathcal{R}$-ideal in $\mathcal{S}$ may be written in the form $\{U(a); a \in A\}$ for some $A \subseteq S$. As $\mathcal{R}^{-1}(\{U(a); a \in A\}) = \mathcal{R}(A)$, the proof is complete if we show that $A_1 = A$. It is sufficient to show that for any finite subset $A_1 \subseteq S$. As $\mathcal{R}^{-1}\{U(a); a \in A_1\} = \mathcal{R}(A_1)$, the proof is complete.

$\S$ 5. A characteristic property for Boolean algebras. Finally we give an application of the previous theory to a simple algebraic problem. Let $L$ be a distributive lattice with the $\mathcal{R}$-system.

**Theorem 9.** $\mathcal{R}(L)$ is compact if and only if $L$ has a greatest element.

**Proof.** Since the $\mathcal{R}$-system is half-prime of finite character, and every finitely generated ideal is a principal ideal, the theorem follows from the corollary of Theorem 5.
THEOREM 10. Let $L$ be a distributive lattice with a greatest element $1$ and a least element $0$. Then $L$ is a Boolean algebra if and only if $\mathcal{P}(L)$ is Hausdorff.

Proof. If $L$ is a Boolean algebra, then for any $a \in L$, $\mathcal{U}(a) = \mathcal{P}(Ca)$, and $\mathcal{P}(L)$ is Hausdorff by Theorem 1 and Theorem 3 (C). Conversely, assume that $\mathcal{P}(L)$ is Hausdorff and let $a \neq 0, 1$. Let $P_1 \in \mathcal{P}(a)$. For every $Q_i \in \mathcal{P}(a)$ there exists an element $d$ such that $Q_i \subseteq \mathcal{U}(d)$ and $P_i \in \mathcal{U}(d)$. These $\mathcal{U}(d)$ form an open covering of the compact set $\mathcal{P}(a)$ (Theorem 9), and we can find $d_1, d_2, \ldots, d_n$ such that

$$
\mathcal{P}(a) \subseteq \mathcal{U}(d_1) \cup \mathcal{U}(d_2) \cup \cdots \cup \mathcal{U}(d_n) = \mathcal{U}(\bigvee_{i=1}^{n} d_i).
$$

Now put $b = \bigwedge_{i=1}^{n} d_i$. By Theorem 3 (C) and Theorem 1, $\mathcal{P}(b)$ is open, and $P_1 \in \mathcal{P}(b)$. On the other hand $\mathcal{U}(a)$ is closed and therefore compact. Thus the sets $\mathcal{P}(b)$ form an open covering of the compact set $\mathcal{U}(a)$, and we can find $b_1, b_2, \ldots, b_m$ such that

$$
\mathcal{U}(a) \subseteq \mathcal{P}(b_1) \cup \mathcal{P}(b_2) \cup \cdots \cup \mathcal{P}(b_m) = \bigvee_{i=1}^{m} b_i.
$$

Now, for every $P_1 \in \mathcal{P}$, $a \wedge \bigvee_{i=1}^{m} b_i \in P_1$, and since $\mathcal{P}(0) = 0$ we have $a \wedge \bigvee_{i=1}^{m} b_i = 0$. On the other hand, $a \vee \bigvee_{i=1}^{m} b_i = \bigvee_{i=1}^{m} (a \vee b_i)$ is contained in no proper prime ideal. In fact, if $\bigvee_{i=1}^{m} (a \vee b_i) \in P_1$, then for some $i_1$, $a \vee b_{i_1} \in P_1$, consequently $a \in P_1$ and $b_{i_1} \in P_1$, i.e., $P_1 \in \mathcal{P}(a)$ and $P_1 \in \mathcal{U}(b_{i_1})$, which is impossible. This gives that $a \vee \bigvee_{i=1}^{m} b_i = 1$, and $\bigvee_{i=1}^{m} b_i$ is a complement of $a$.

COROLLARY (Nachbin). A distributive lattice with 0 and 1 is a Boolean algebra if and only if every proper prime ideal is a maximal ideal.

Proof. For a distributive lattice we always have $\mathcal{P}(L)$ is Hausdorff, we therefore conclude $\mathcal{P}(L) = \mathcal{P}$. On the other hand, if $\mathcal{P}(L) = \mathcal{P}$, then clearly $\mathcal{P}(L) = \mathcal{P}$, and by the corollary of Theorem 1, $\mathcal{P}(L)$ is Hausdorff.

References