A general theory of structure spaces

by

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Structure spaces have been considered for a great number of different algebraic systems such as rings, Banach algebras, semi-rings, lattices, lattice-ordered groups etc. In a situation like this the problem of unification naturally poses itself. The purpose of this paper is to show that the theory of x-ideals as developed in [1] and [2] appears as a natural framework for a general theory of structure-spaces. Some indications in this direction were already given in [1]. We shall here pursue the subject along more general lines. The present development is in fact general enough to cover a large number of special cases. On the other hand a reasonable part of the theory of structure spaces for special algebraic systems may be generalized to our situation.

In § 0 we first give the necessary algebraic background. With exception of Proposition 0.2 this is contained in [1] and [2], to which we refer for proofs, more details and special cases. In § 1 we turn to the theory of structure-spaces for commutative semi-groups with an x-system. Theorems 1, 2 and 3 concerning Hausdorff structure-spaces are mainly generalizations of parts of the corresponding theory for rings in [4]. In § 2 we consider the compactness of structure-spaces. The theory there follows the same lines as the theory in [3] and [5]. It is easy to see that if R is a semisimple commutative ring with identity, then $\mathfrak{M}(R)$ is disconnected if and only if R is the direct sum of two of its ideals, R_1 and R_2. If $\mathfrak{M}(R) = \mathfrak{F}_1 \cup \mathfrak{F}_2$ is a partition of $\mathfrak{M}(R)$ into disjoint open-closed proper subsets, R_1 and R_2 may be chosen so that $\mathfrak{M}(R_i)$ is homeomorphic to $\mathfrak{F}_i$, $i = 1, 2$. The argument here makes use of the additive structure of the ring. In § 3 we show how this may nevertheless be transferred to our situation, the additivity being taken care of mainly by the additivity axiom for an x-system on a commutative semigroup, introduced in [2]. The representation theorem of § 4 is related to Theorem 14 of [10]. Finally, we illustrate by a simple example a rather interesting procedure: an algebraic problem may be reformulated in terms of structure-spaces, and solved by simple topological reasoning.

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§ 0. Preliminaries. Let S be a commutative semigroup. We shall say that there is defined an *x-system* on S if to every $A \subseteq S$ there corresponds $A_x \subseteq S$ such that

(0.1) $$A \subseteq A_x\,,$$

(0.2) $$A \subseteq B_x \Rightarrow A_x \subseteq B_x\,,$$

(0.3) $$AB_x \subseteq B_x\,,$$

(0.4) $$AB_x \subseteq (AB)_x\,.$$

(0.4) is referred to as the *continuity axiom*. A subset A in S is said to be an *x-ideal*, or shorter an *ideal* if $A = A_x$.

We may give an equivalent definition of an x-system in the following way: let S be a commutative semigroup, and let $\mathfrak{X}$ be a non-empty family of subsets of S, called x-ideals, such that the following conditions are satisfied:

(0.5) *The intersection of any non-empty family of x-ideals is an x-ideal.*

(0.6) *For any $a \in S$ and any $A \in \mathfrak{X}$, $A:a$ is an x-ideal containing A.*

Let A be any subset of S, and put $A_x = \bigcap_{\substack{B \in \mathfrak{X} \\ A \subseteq B}} B$. Then the correspondence $A \to A_x$ defines an x-system in the sense of (0.1)-(0.4) and the family of x-ideals is $\mathfrak{X}$.

A_x is said to be a *proper* x-ideal if $A_x \neq S, \emptyset$. A *prime* x-ideal is an ideal P_x satisfying $ab \in P_x \Rightarrow a \in P_x \vee b \in P_x$, and a *maximal* x-ideal is a proper ideal not properly contained in any proper ideal. A *minimal prime ideal* is defined correspondingly. The families of prime, maximal and minimal ideals in S are denoted by $\mathfrak{P}$, $\mathfrak{M}$ and $\mathfrak{N}$ respectively.

An x-system is said to be of *finite character* if the set-theoretic union of any chain of x-ideals is an x-ideal.

If S is a commutative semigroup with an x-system, and T is a subsemigroup of S, then it is easily verified that the family of all intersections between an x-ideal in S and T defines an x-system on T. This x-system is said to be *induced on T from S*. (This definition is not the best one, but will be the relevant definition for our purpose.)

Given a family of x-ideals in S, $\{A_x^{(i)}\}_{i \in I}$. Put $\bigcup_{i\in I}{}_x\, A_x^{(i)} = (\bigcup_{i \in I} A_x^{(i)})_x$. $\mathfrak{X}$ is a complete lattice under $\cup_x$ and $\cap$. $\cup_x$ will be referred to as *x-union*. Furthermore, put $A_x \circ_x B_x = (AB)_x$. This operation is referred to as *x-multiplication*.

The *nilpotent radical* of an x-ideal A_x, denoted by $\operatorname{rad} A_x$, is the set of all elements a in S such that for some n, $a^n \in A_x$. A_x is said to be *half-prime* if $\operatorname{rad} A_x = A_x$. If every x-ideal in S is half-prime, the x-system is said to be half-prime. The element $e \in S$ is called an *x-identity* if $(e)_x = S$ and $e \in S^2$.

An x-ideal A_x is shown to be non-prime if and only if $A_x \supseteq B_x \circ_x C_x$ for some B_x and C_x properly containing A_x. This implies that if S has an x-identity, then $\mathfrak{M} \subseteq \mathfrak{P}$.

A subset M of S which is either empty or closed under multiplication is referred to as an *m-set*. We have, for x-systems of finite character:

PROPOSITION 0.1. *Given an x-ideal A_x in S. If M is a maximal m-set contained in $\complement A_x$, and P_x is an ideal maximal with respect to the property of containing A_x and being contained in $\complement M$, then P_x is a minimal prime ideal over A_x.* ($\complement A_x$ denotes the complement of A_x in S.)

COROLLARY 1. *Any prime ideal P_x over A_x contains a least one minimal prime ideal over A_x.*

COROLLARY 2. *For any maximal m-set M contained in $\complement A_x$, $\complement M$ is a minimal prime ideal over A_x.*

For a number of algebraic systems, the following condition turns out to be of importance:

(0.7) *To every $a \in S$ there exists an idempotent element $|a|$ such that for every x-ideal A_x in S, $a \in A_x \Leftrightarrow |a| \in A_x$.*

For a distributive lattice with the l-system, we may choose $a = |a|$, for a lattice ordered group with the c-system and the multiplication $a \circ b = |a| \cap |b|$, we put $|a| = a \vee 0 - a \wedge 0$.

At present, we shall only note the following consequence of (0.7), which will be usefull later (see [7], Theorem 6.5):

PROPOSITION 0.2. *Given an ideal $A_x \neq \emptyset$, in an x-system satisfying* (0.7). *Then an ideal $P_x \supseteq A_x$ is a minimal prime ideal over A_x if and only if it satisfies the following condition:*

(0.8) *To every idempotent element c in P_x there exists $b \notin P_x$ such that $cb \in A_x$.*

Proof. If P_x satisfies (0.8) and $A_x \subseteq Q_x \subsetneq P_x$ for a prime ideal Q_x, choose $a \in P_x$, $a \notin Q_x$. Then $|a| \in P_x$, $|a| \notin Q_x$. To $|a|$ there corresponds by (0.8) $b \notin P_x$ such that $|a| \cdot b \in A_x$. Thus $|a| \in Q_x$, a contradiction. Conversely, if P_x is a minimal prime ideal over A_x, $M = \complement P_x$ is an m-set, maximal in $\complement A_x$ by Corollary 2 above. For an idempotent element c in A_x, put $M(c) = M \cup \{bc;\ b \in M\}$. Obviously $M(c)$ is an m-set properly containing M, thus $M(C) \cap A_x \neq \emptyset$ and (0.8) is satisfied.

THEOREM 0.1 (Krull-Stone). *For an x-system of finite character, $\operatorname{rad} A_x$ is equal to the intersection of the minimal prime ideals over A_x.*

Let S and T be commutative semigroups with x-systems which we denote respectively by y and z. We shall say that a multiplicative homomorphism φ of S into T is a (y, z)-*homomorphism* if $\varphi(A_y) \subseteq (\varphi(A))_z$, for all subsets A of S, or, equivalently, if the inverse image of a z-ideal in T is a y-ideal in S. If φ is a multiplicative homomorphism of a commutative semigroup S onto a semigroup T, and S has an x-system y, then the set of all $B \subseteq T$ such that $\varphi^{-1}(B)$ is a y-ideal in S defines an x-system y_φ in T. This makes φ to a (y, y_φ)-homomorphism.

For $a, b \in S$, put

$$a \equiv b(A_x) \Leftrightarrow (A_x, a)_x = (A_x, b)_x .$$

This is a congruence relation in S, which we refer to as *x-congruence*. If φ denotes the canonical multiplicative homomorphism of S onto S/A_x, we shall call φ_x the *canonical x-system on* S/A_x.

The x-system on S is said to be *additive* if for any x-ideals A_x, B_x in S and $c \in A_x \cup_x B_x$, there exists $b \in B_x$ such that $c \equiv b(A_x)$.

THEOREM 0.2. *An x-system on S is additive if and only if the canonical mapping* $A_x|_{A_x \cap B_x} \to A_x \cup_x B_x|_{B_x}$ *is bijective for any x-ideals* A_x *and* B_x *in* S.

Given any family $\mathfrak{I}$ of x-ideals in S, we adopt the following notation: For $\mathfrak{A} \subseteq \mathfrak{I}$, put $k\mathfrak{A} = \bigcap_{A_x \in \mathfrak{A}} A_x$, for an x-ideal B_x in S, $hB_x = \{A_x \in \mathfrak{I};\ B_x \subseteq A_x\}$. Furthermore, put $k\mathfrak{X} = O_x$.

§ 1. Structure spaces and their separation properties.

Let S be a commutative semigroup with an x-system, and let $\mathfrak{I} \subseteq \mathfrak{X}$ be a family of proper x-ideals such that

$$(1.1) \qquad B_x \cap C_x \subseteq A_x \Rightarrow B_x \subseteq A_x \vee C_x \subseteq A_x$$

whenever $A_x \in \mathfrak{I}$ and B_x and C_x are arbitrary intersections of ideals from $\mathfrak{I}$. Under this assumption $\mathfrak{I}$ is said to be a *structure-family* for S.

For $\mathfrak{A} \subseteq \mathfrak{I}$, put $\overline{\mathfrak{A}} = hk\mathfrak{A}$ if $\mathfrak{A} \neq \emptyset$, $\overline{\emptyset} = \emptyset$.

The above definitions are justified by

PROPOSITION 1. $\mathfrak{A} \to \overline{\mathfrak{A}}$ *defines a topology on* $\mathfrak{I}$ *if and only if* $\mathfrak{I}$ *is a structure-family for* S.

Proof. (1.1) is equivalent to $\overline{\mathfrak{A} \cup \mathfrak{B}} \subseteq \overline{\mathfrak{A}} \cup \overline{\mathfrak{B}}$ for $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{I}$. As $\mathfrak{A} \subseteq \overline{\mathfrak{A}}$, $\overline{\overline{\mathfrak{A}}} = \overline{\mathfrak{A}}$ and $\overline{\mathfrak{A}} \cup \overline{\mathfrak{B}} \subseteq \overline{\mathfrak{A} \cup \mathfrak{B}}$ are satisfied, without assuming (1.1), the proposition follows.

This topology is referred to as the *Zaryski topology* and also as the *Stone topology* on $\mathfrak{I}$, the corresponding topological space is called a *structure-space* for S. It will be denoted by $\mathfrak{I}(S)$.

There is an obvious 1-1 correspondence between the subfamilies of a structure-family and the subspaces of the corresponding structure-space, in the sense that every sub-family of a structure-family is a structure-family, and all subspaces are obtained in this way.

$\mathfrak{P}$ is a structure-family for S. If S has an x-identity, $\mathfrak{M} \subseteq \mathfrak{P}$, and consequently $\mathfrak{M}$ is a structure-family for S.

A structure-space is invariant under (y, z)-isomorphisms, and every closed subset of a structure-space for S is homeomorphic to a structure-space for a (y, z)-homomorphic image of S:

PROPOSITION 2. *Let S, T be commutative semigroups, with x-systems, denoted by y, z respectively. Let $\mathfrak{I}$ be a structure-family for S, φ a (y, z)-isomorphism of S onto T, and put $\varphi(\mathfrak{I}) = \{\varphi(A_y);\ A_y \in \mathfrak{I}\}$. Then $\varphi(\mathfrak{I})$ is a structure-family for T, and $\mathfrak{I}(S)$ is homeomorphic to $(\varphi(\mathfrak{I}))(T)$.*

For $A_y \subseteq S$, put $\tilde{\mathfrak{I}} = \{B_y;\ B_y \in \mathfrak{I}$ and $B_y \supseteq A_y\}$.

Then $\tilde{\mathfrak{I}}$ is a closed subset of $\mathfrak{I}(S)$, and is homeomorphic to $(\psi(\tilde{\mathfrak{I}}))(S/A_y)$ where ψ is the canonical homomorphism of S onto S/A_y. Conversely, if $\mathfrak{F} \subseteq \mathfrak{I}(S)$ is closed, $\mathfrak{F} = \tilde{\mathfrak{I}}$ where $A_y = k\mathfrak{F}$.

Proof. The first part of the proposition follows from the fact that structure-families and structure-spaces are defined by properties invariant under (y, z)-isomorphisms. Clearly, $\tilde{\mathfrak{I}}$ is closed in $\mathfrak{I}(S)$. Put $\Phi(B_y) = \psi(B_y)$. We hare

$$(1.2) \qquad B_y \supseteq C_y \Leftrightarrow \psi(B_y) \supseteq \psi(C_y) \quad \text{for every} \quad B_y \in \tilde{\mathfrak{I}},\ C_y \in \tilde{\mathfrak{I}},$$

$$(1.3) \qquad \psi\Big(\bigcap_{k \in K} C_y^{(k)}\Big) = \bigcap_{k \in K} \psi(C_y^{(k)}), \quad \text{where} \quad C_y^{(k)} \in \tilde{\mathfrak{I}} \text{ for every } k \in K .$$

It follows at once that $\psi(\tilde{\mathfrak{I}})$ is a structure-family for S/A_y and that Φ: $\tilde{\mathfrak{I}}(S) \to \psi(\tilde{\mathfrak{I}})(S/A_y)$ is a homeomorphism. The last part of the proposition is obvious.

COROLLARY. *Assume that $\mathfrak{M}$ is a structure-family for S, and let $\mathfrak{F}$ be a closed subset of $\mathfrak{M}(S)$. Then the family $\mathfrak{M}'$, of maximal ideals in $S/k\mathfrak{F}$, is a structure-family for $S/k\mathfrak{F}$, and $\mathfrak{M}'(S/k\mathfrak{F})$ is homeomorphic to $\mathfrak{F}$.*

Proof. This follows from Proposition 2 with $\mathfrak{M}' = \psi(\mathfrak{F})$ and ψ: $S \to S/k\mathfrak{F}$ (the canonical homomorphism).

$$(1.4) \qquad \text{For } a \in S, \text{ put} \quad \mathfrak{F}(a) = h\{a\}, \quad \mathfrak{U}(a) = \complement\mathfrak{F}(a) .$$

PROPOSITION 3. $\{\mathfrak{U}(a)\}_{a \in S}$ *constitutes a basis for the topology on* $\mathfrak{I}(S)$.

Proof. Since $\mathfrak{F}(a)$ is closed, $\mathfrak{U}(a)$ is open. For $\mathfrak{U}$ open in $\mathfrak{I}(S)$ and $A_x \in \mathfrak{U}$, we may find $a \in S$ such that $a \notin A_x$, $a \in k\complement\mathfrak{U}$. Then $A_x \in \mathfrak{U}(a) \subseteq \mathfrak{U}$.

For the study of separation properties of a structure-space $\mathfrak{I}(S)$, we introduce, for $A_x \in \mathfrak{I}(S)$,

$$(1.5) \qquad N(A_x) = \bigcup^x_{\{b|b \notin A_x\}} \Big[\bigcap_{\{B_x \in \mathfrak{I}|b \notin B_x\}} B_x\Big] .$$

By Lemma 1 below this definition coincides with Definition 2.1 of [4] if S is a ring with the usual ideal system.

LEMMA 1. *Let $\mathfrak{B}$ be a basis for the neighbourhood system of A_x. Then*

$$N(A_x) = \bigcup_{\mathfrak{U} \in \mathfrak{B}}{}_x k\mathfrak{U}. \tag{1.6}$$

If the x-system is of finite character, we have

$$N(A_x) = \bigcup_{\mathfrak{U} \in \mathfrak{B}} k\mathfrak{U}. \tag{1.7}$$

Proof. For two bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$ we get $\bigcup_{\mathfrak{U} \in \mathfrak{B}_1} k\mathfrak{U} = \bigcup_{\mathfrak{U} \in \mathfrak{B}_2} k\mathfrak{U}$ and the first part follows from Proposition 3. To prove the second half of the lemma, we first observe that $N(A_x) = \emptyset$ implies $\bigcup_{\mathfrak{U} \in \mathfrak{B}} k\mathfrak{U} = \emptyset$. For $a \in \bigcup_{\mathfrak{U} \in \mathfrak{B}}{}_x k\mathfrak{U}$ we find, by the finite character property, a finite set $\{a_1, \ldots, a_n\} \subseteq \bigcup_{\mathfrak{U} \in \mathfrak{B}} k\mathfrak{U}$ such that $a \in (a_1, \ldots, a_n)_x$. Let $a_i \in k\mathfrak{U}_i$, and determine $\mathfrak{U} \in \mathfrak{B}$ such that $\mathfrak{U} \subseteq \bigcap_{i=1}^{n} \mathfrak{U}_i$. Then

$$a \in \bigcup_{i=1}^{n}{}_x k\mathfrak{U}_i \subseteq (k \bigcap_{i=1}^{n} \mathfrak{U}_i)_x = k \bigcap_{i=1}^{n} \mathfrak{U}_i \subseteq k\mathfrak{U}.$$

We thus have $\bigcup_{\mathfrak{U} \in \mathfrak{B}}{}_x k\mathfrak{U} \subseteq \bigcup_{\mathfrak{U} \in \mathfrak{B}} k\mathfrak{U}$, and the lemma is proved.

Obviously $k\,\mathfrak{J} \subseteq N(A_x) \subseteq A_x$. The equalities are reflected in $\mathfrak{J}(S)$ as follows ((1.9) generalizes Theorem 2.7 of [4]).

THEOREM 1.

(1.8) $N(A_x) = k\mathfrak{J} \Leftrightarrow \overline{\mathfrak{U}} = \mathfrak{J}(S)$ *for every neighbourhood* $\mathfrak{U}$ *of* A_x.

If the x-system is of finite character, we get

(1.9) $N(A_x) = A_x$ *for every* $A_x \in \mathfrak{J}(S) \Leftrightarrow \mathfrak{F}(a)$ *is open for every* $a \in S$.

Proof. Clearly, $N(A_x) = k\mathfrak{J} \Leftrightarrow k\mathfrak{U} = k\mathfrak{J}$ for every neighbourhood $\mathfrak{U}$ of A_x, and (1.8) follows. Assume that the x-system of S is of finite character, and that $N(A_x) = A_x$ for every $A_x \in \mathfrak{J}(S)$. Then, for $A_x \in \mathfrak{F}(a)$, $a \in k\mathfrak{U}_0$ for some neighbourhood $\mathfrak{U}_0$ of A_x (Lemma 1). Thus $A_x \in \mathfrak{U}_0 \subseteq \mathfrak{F}(a)$ and $\mathfrak{F}(a)$ is open. Consverely, assume that $\mathfrak{F}(a)$ is open for every $a \in S$, and let $A_x \in \mathfrak{J}(S)$. For $a \in A_x$, we get $a \in k\mathfrak{F}(a) \subseteq N(A_x)$ and $A_x \subseteq N(A_x)$. Obviously $N(A_x) \subseteq A_x$ and (1.9) is proved.

COROLLARY. *Let S be a commutative semigroup with an x-system of finite character, satisfying* (0.7). *Assume also that $k\mathfrak{X} = 0_x \neq \emptyset$. Then the structure-space of the minimal prime ideals, $\mathfrak{N}(S)$, is a totally disconnected Hausdorff space.*

Proof. Let $P_x \in \mathfrak{N}(S)$. Then for every $a \in P_x$ there exists by Proposition 0.2, $b \in P_x$ such that $|a|b \in O_x$. For every $Q_x \in \mathfrak{N}$, $b \notin Q_x \Rightarrow |a| \in Q_x \Rightarrow a \in Q_x$, and $a \in k\mathfrak{U}(b) \subseteq N(P_x)$. We conclude $P_x = N(P_x)$. As $\mathfrak{N}(S)$ is T_1, the proof is complete.

The connection between the Hausdorff-property of $\mathfrak{J}(S)$ and algebraic properties of S is of the same kind as in the case where S is a ring with the d-system. The following theorem is a generalization of Theorem 3.1 of [4].

THEOREM 2. (A), (B), (C) *and* (D) *are equivalent.*

(A) $\mathfrak{J}(S)$ *is a Hausdorff-space.*

(B) *For A_x, B_x different elements of $\mathfrak{J}(S)$, there exists $a \notin A_x$, $b \notin B_x$ such that for every $C_x \in \mathfrak{J}(S)$, $a \in C_x \vee b \in C_x$.*

(C) *For A_x, B_x different elements of $\mathfrak{J}(S)$, $N(A_x) \nsubseteq B_x$.*

(D) *For every $A_x \in \mathfrak{J}(S)$, $N(A_x)$ is contained in exactly one ideal from $\mathfrak{J}$.*

If $\mathfrak{J} \subseteq \mathfrak{M}$ and the x-system is of finite character, (A) *is also equivalent to*

(E) *For A_x, B_x different ideals from $\mathfrak{J}$, and $a \in S$, there exists $b \in N(A_x)$ such that $a \equiv b(B_x)$.*

If every proper ideal in S is contained in an ideal from $\mathfrak{J}$, (A) *is also equivalent to*

(F) *For A_x, B_x different ideals of $\mathfrak{J}$, $N(A_x) \cup_x N(B_x) = S$.*

Proof. Throughout the proof, let A_x, B_x denote ideals from $\mathfrak{J}$. We first verify (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (D) $\Rightarrow$ (A). By Proposition 3, (A) $\Rightarrow$ (B). Assume (B). Let $A_x \neq B_x$, and let a, b satisfy (B). Then $b \in k\mathfrak{U}(a) \subseteq N(A_x)$. As $b \in B_x$, (C) follows. Obviously (C) $\Rightarrow$ (D). Assume (D). Then, for $A_x \neq B_x$, Lemma 1, (1.6) gives that $\bigcup_{\mathfrak{U} \in \mathfrak{B}} k\mathfrak{U} \nsubseteq B_x$ where $\mathfrak{B}$ is the neighbourhood-system of A_x. Choose $a \in \bigcup_{\mathfrak{U} \in \mathfrak{B}} k\mathfrak{U}$, $a \notin B_x$. Then $a \in k\mathfrak{U}_0$ for some $\mathfrak{U}_0 \in \mathfrak{B}$. Now $B_x \in \mathfrak{U}(a)$ and clearly $\mathfrak{U}_0 \cap \mathfrak{U}(a) = \emptyset$. Thus (A) follows.

Now, assume that the x-system is of finite character, and let $\mathfrak{J} \subseteq \mathfrak{M}$. If (A) is satisfied, and $A_x \neq B_x$, then for $a \in S$ we get two possibilities: for $a \in B_x$, choose $b \in N(A_x) \cap B_x$. This is possible, for by Theorem 1, $N(A_x) \neq \emptyset$. Then $(a, B_x)_x = (b, B_x)_x = B_x$ and $a \equiv b(B_x)$. On the other hand, if $a \notin B_x$, (C) implies the existence of b such that $b \in N(A_x)$, $b \notin B_x$. Here $(a, B_x)_x = (b, B_x)_x = S$, and $a \equiv b(B_x)$. Assume (E). Let $A_x \neq B_x$ and choose $a \notin B_x$. For b given by (E), $B_x \in \mathfrak{U}(b)$ and $b \in N(A_x)$. By Lemma 1 (1.7), $b \in k\mathfrak{U}_0$ for some neighbourhood $\mathfrak{U}_0$ of A_x. As $\mathfrak{U}_0 \cap \mathfrak{U}(b) = \emptyset$, (A) follows.

To prove the last part of the theorem, assume that every proper ideal in S is contained in an ideal from $\mathfrak{J}$. If $\mathfrak{J}(S)$ is a Hausdorff-space and $A_x \neq B_x$, (D) implies that $N(A_x) \cup_x N(B_x) = S$. If (F) is satisfied, and for some $A_x \neq B_x$ $N(A_x) \subseteq B_x$, we find $B_x = S$, in contradiction to the definition of a structure-family.

LEMMA 2. *Assume that the x-system is of finite character and that $\mathfrak{J} \subseteq \mathfrak{P}$. Then $N(A_x)$ is half prime for every $A_x \in \mathfrak{J}$.*

Proof. Let $a \in \operatorname{rad} N(A_x)$, i.e., $a^n \in N(A_x)$. For some neighbourhood $\mathfrak{U}_0$ of A_x, $a^n \in k\mathfrak{U}_0$ by Lemma 1. For every $B_x \in \mathfrak{U}_0$, $a^n \in B_x$, and since $B_x \in \mathfrak{P}$, we have $a \in B_x$. Thus $a \in k\mathfrak{U}_0 \subseteq N(A_x)$.

The following is a generalization of Corollary 3.8 of [4].

THEOREM 3. *Assume that the x-system is of finite character. Then the following statements are equivalent:*

(A) $\mathfrak{P}(S)$ *is a Hausdorff-space.*

(B) $\mathfrak{P}(S)$ *is totally disconnected.*

(C) *For every* $A_x \in \mathfrak{P}$, $N(A_x) = A_x$.

Proof. We verify (C) ⇒ (B) ⇒ (A) ⇒ (C). (C) ⇒ (B) follows from Theorem 1 (B), (B) ⇒ (A) is obvious. Assume (A). By the Krull-Stone theorem, Lemma 2 gives

$$N(A_x) = \bigcap_{\{P_x \in \mathfrak{P} \mid P_x \supseteq N(A_x)\}} P_x .$$

If $N(A_x) \neq A_x$, then for some $P_x \neq A_x$, $P_x \supseteq N(A_x)$, contradicting (A).

§ 2. Compactness. We now turn to the compactness of structure-spaces, and make the following observation:

PROPOSITION 4. *$\{\mathfrak{U}(a)\}_{a \in R}$ is an open covering of $\mathfrak{J}(S)$ if and only if $R \not\subseteq A_x$ for every $A_x \in \mathfrak{J}$.*

THEOREM 4. *$\mathfrak{J}(S)$ is compact if and only if every subset R of S with the property $R \not\subseteq A_x$ for every $A_x \in \mathfrak{J}$, contains a finite subset N with the property $N \not\subseteq A_x$ for every $A_x \in \mathfrak{J}$.*

Proof. The theorem follows from Proposition 3 and Proposition 4.

THEOREM 5. *If the x-system is of finite character, $\mathfrak{J}(S)$ is compact if and only if every ideal R_x which satisfies $R_x \not\subseteq A_x$ for every $A_x \in \mathfrak{J}$, contains a finitely generated ideal N_x such that $N_x \not\subseteq A_x$ for every $A_x \in \mathfrak{J}$.*

Proof. The necessity of the condition follows from Theorem 4. Assume the condition. Let $R \subseteq S$ satisfy $R \not\subseteq A_x$ for every $A_x \in \mathfrak{J}$. Then also $R_x \not\subseteq A_x$ for every $A_x \in \mathfrak{J}$ and by the condition we find a finitely generated ideal $N_x \subseteq R_x$ such that $N_x \not\subseteq A_x$ for every $A_x \in \mathfrak{J}$. By the finite character property we find for every $a \in N$ an $F_a \subseteq R$ such that $a \in (F_a)_x$, F_a finite for every $a \in N$. As N is finite, $F = \bigcup_{a \in N} F_a$ is finite, and $N_x \subseteq F_x$. Clearly $F_x \not\subseteq A_x$ for every $A_x \in \mathfrak{J}$, thus also $F \not\subseteq A_x$ for every $A_x \in \mathfrak{J}$, and as $F \subseteq R$, $\mathfrak{J}(S)$ is compact by Theorem 4.

COROLLARY. *If the x-system is of finite character, and if every proper ideal in S is contained in an ideal from $\mathfrak{J}$, then $\mathfrak{J}(S)$ is compact if and only if S is finitely generated.*

Proof. Under the assumptions of the corollary, $N_x \not\subseteq A_x$ for every $A_x \in \mathfrak{J}$, if and only if $N_x = S$.

The compactness of $\mathfrak{J}(S)$ thus amounts to a finiteness condition on S. On the other hand, we may take a somewhat different point of view:

A proper subset V of S is said to be an *f-set for $\mathfrak{J}$* if for every finite subset N of V there exists an $A_x \in \mathfrak{J}$ such that $N \subseteq A_x$ (see [5]). By Zorn's Lemma we find that every f-set for $\mathfrak{J}$ in S is contained in a maximal f-set for $\mathfrak{J}$ in S.

THEOREM 6. *$\mathfrak{J}(S)$ is compact if and only if every maximal f-set for $\mathfrak{J}$ is a member of $\mathfrak{J}$.*

Proof. Assume that $\mathfrak{J}(S)$ is compact, and let W be a maximal f-set. We contend that $W \in \mathfrak{J}$. As every $A_x \in \mathfrak{J}$ is an f-set, it is sufficient to find $A_x \in \mathfrak{J}$ with $A_x \supseteq W$. Assume that $A_x \not\supseteq W$ for every $A_x \in \mathfrak{J}$. By Theorem 4 we find a finite subset N of W with $N \not\subseteq A_x$ for every $A_x \in \mathfrak{J}$, a contradiction. Conversely, assume that every maximal f-set is an element of $\mathfrak{J}$. If $\mathfrak{J}(S)$ were not compact, we could find an f-set R for $\mathfrak{J}$ with $R \not\subseteq A_x$, in contradiction to the assumption.

§ 3. Disconnected $\mathfrak{M}(S)$. Let R be a commutative ring with identity. If R is semisimple, then $\mathfrak{M}(R)$ is disconnected if and only if R is the direct sum of two of its proper ideals R_1 and R_2. If $\mathfrak{M}(R) = \mathfrak{F}_1 \cup \mathfrak{F}_2$ is a partition of $\mathfrak{M}(R)$ into disjoint open-closed proper subsets, we may determine R_1 and R_2 so that $\mathfrak{M}(R_i)$ is homeomorphic to $\mathfrak{F}_i$, $i = 1, 2$. (See for instance [6].)

We observe that if R is the direct sum of the ideals R_1 and R_2, then $R = R_1 \cup_d R_2$, $R_1 \cap R_2 = \{0\}$ and every d-ideal A of R_i is a d-ideal of R. In view of this the following theorem generalizes the above-mentioned theorem for rings.

THEOREM 7. *Let S be a commutative semigroup with an x-system of finite character and x-identity. Then $\mathfrak{M}(S)$ is disconnected if and only if there exist ideals A_x, B_x in S, different from S and $k\mathfrak{M}$, such that $A_x \cup_x B_x = S$, $A_x \cap B_x = k\mathfrak{M}$.*

If the x-system is additive, and if $k\mathfrak{M} = k\mathfrak{X} = 0_x$, then for every partition of $\mathfrak{M}(S)$ into disjoint open-closed proper subsets $\mathfrak{A}$ and $\mathfrak{B}$, we may determine A_x and B_x such that $\mathfrak{M}(A_x)$ is homeomorphic to $\mathfrak{A}$ and $\mathfrak{M}(B_x)$ homeomorphic to $\mathfrak{B}$, where A_x and B_x are equipped with the x-systems induced from S.

Proof. Let $\mathfrak{M}(S) = \mathfrak{A} \cup \mathfrak{B}$ be a partition of $\mathfrak{M}(S)$ into proper, disjoint open-closed subsets and put $A_x = k\mathfrak{B}$, $B_x = k\mathfrak{A}$. Then $A_x \cap B_x = k\mathfrak{M}$. Furthermore, $A_x \cup_x B_x = S$, for if $A_x \cup_x B_x$ were a proper ideal, it would be contained in some maximal ideal M_x. (This is proved in the usual way by the existence of an x-identity.) Then $M_x \in \mathfrak{A} \cap \mathfrak{B}$, a contradiction. Clearly A_x and B_x are different from S and $k\mathfrak{M}$.

Conversely, assume the existence of A_x and B_x satisfying the condition of the theorem. Put $\mathfrak{A} = hB_x$, $\mathfrak{B} = hA_x$. Clearly, $\mathfrak{A}$ and $\mathfrak{B}$ are closed. Since $\mathfrak{M} \subseteq \mathfrak{P}$ and $A_x \cap B_x = k\mathfrak{M}$, we have $\mathfrak{A} \cup \mathfrak{B} = \mathfrak{M}(S)$. Furthermore $A_x \cup_x B_x = S$ implies $\mathfrak{A} \cap \mathfrak{B} = 0$, and the first part of the theorem is proved.

Finally assume the condition of the last part of the theorem. Define A_x and B_x as above, $A_x = k\mathfrak{B}$, $B_x = k\mathfrak{A}$, where $\mathfrak{A} \cup \mathfrak{B}$ is a partition of $\mathfrak{M}(S)$ into proper, disjoint open-closed subsets. By symmetry it is sufficient to prove that $\mathfrak{M}(A_x)$ is homeomorphic to $\mathfrak{A}$. We denote the x-system induced on A_x from S by x_1, and prove first that A_x/O_x is isomorphic to S/B_x. Denote the canonical mapping of A_x onto A_x/O_x by φ, of S onto S/B_x by ψ; and denote the canonical x-system of S/B_x by $\bar{x}$, of A_x/O_x by $\bar{x}_1$. Now define

$$\bar{\psi}\colon A_x/O_x \to S/B_x$$

by $\bar{\psi}(\varphi(s)) = \psi(s)$ for $s \in A_x$. By definition $A_x/O_x = A_x/A_x \cap B_x$, $S/B_x = A_x \cup_x B_x/B_x$, and $\bar{\psi}$ is bijective by Theorem 0.2. Clearly $\bar{\psi}$ is multiplicative, and every $\bar{x}_1$-ideal in A_x/O_x may be written in the form $\varphi(C_{x_1})$, where C_{x_1} is some x_1-ideal in A_x, i.e., an x-ideal contained in A_x. Now $\bar{\psi}(\varphi(C_{x_1})) = \psi(C_{x_1} \cup_x A_x)$, showing that the image by $\bar{\psi}$ of every $\bar{x}_1$-ideal in A_x/O_x is an $\bar{x}$-ideal in S/B_x. Conversely, every $\bar{x}$-ideal in S/B_x may be written as $\psi(D_x)$, where $D_x \supseteq B_x$. Now

$$\bar{\psi}^{-1}(\psi(D_x)) = \{\varphi(s);\ s \in A_x \text{ and } \psi(s) \in \psi(D_x)\}$$
$$= \{\varphi(s);\ s \in A_x \text{ and } s \in D_x\} = \varphi(A_x \cap D_x)$$

showing that $\bar{\psi}$ is an isomorphism. This shows in particular that A_x/O_x has an x-identity, so that the maximal ideals is a structure-family of A_x/O_x. Furthermore,

(3.1) $$\mathfrak{M}(A_x/O_x) \text{ is homeomorphic to } \mathfrak{M}(S/B_x)\,.$$

By (1.2) and (1.3) the family of maximal ideals in A_x is a structure-family, and

(3.2) $$\mathfrak{M}(A_x) \text{ is homeomorphic to } \mathfrak{M}(A_x/O_x)\,.$$

Finally, the corollary of Proposition 2 gives

(3.3) $$\mathfrak{M}(S/B_x) \text{ is homeomorphic to } \mathfrak{A}\,.$$

(3.1), (3.2) and (3.3) implies that $\mathfrak{M}(A_x)$ is homeomorphic to $\mathfrak{A}$, and the theorem is proved.

§ 4. A representation theorem. The l-system of a distributive lattice L is a half prime x-system of finite character, when L is considered as a semi-group under $\wedge$. This x-system has the property that every finitely generated ideal is a principal ideal. We shall now prove a theorem closely related to a converse of this statement. In fact, let S be a commutative semigroup with a half-prime x-system of finite character, where every finitely generated ideal is a principal ideal. Denote the canonical x-system of S/O_x by $\bar{x}$. Then we have:

THEOREM 8. *There exists a family $\mathfrak{L}$ of open sets in $\mathfrak{P}(S)$ such that $\mathfrak{L}$ is a lattice under $\cap$ and $\cup$, and such that S/O_x is $(\bar{x}, l)$-isomorphic to $\mathfrak{L}$, where $\mathfrak{L}$ is considered as a semigroup under $\cap$.*

Proof. Put $\mathfrak{L} = \{\mathfrak{U}(s);\ s \in S\}$. For $s_1, s_2 \in S$ there exists $s \in S$ such that $(s)_x = (s_1, s_2)_x$. Then

(4.1) $$\mathfrak{U}(s_1) \cup \mathfrak{U}(s_2) = \mathfrak{U}(s)\,,$$

since $\mathfrak{P}(S)$ consists of prime ideals,

(4.2) $$\mathfrak{U}(s_1) \cap \mathfrak{U}(s_2) = \mathfrak{U}(s_1 s_2)$$

and $\mathfrak{L}$ is a lattice under $\cap$ and $\cup$. For $\bar{s} \in S/O_x$, put $\varphi(\bar{s}) = \mathfrak{U}(s)$. Now, $\bar{s}_1 = \bar{s}_2$ if and only if $(s_1, O_x)_x = (s_2, O_x)_x$. The x-system is half prime, so this is equivalent to $k\mathfrak{F}(s_1) = k\mathfrak{F}(s_2)$, which is again equivalent to $\mathfrak{F}(s_1) = \mathfrak{F}(s_2)$. To sum up, $\bar{s}_1 = \bar{s}_2 \Leftrightarrow \mathfrak{U}(s_1) = \mathfrak{U}(s_2)$. This shows that φ is well defined and injective. Clearly φ is surjective; that φ is multiplicative follows by (4.2), and it remains to be shown that φ establishes a 1-1 correspondence between the ideals in S/O_x and $\mathfrak{L}$. To every $\bar{x}$-ideal $A_{\bar{x}}$ in S/O_x there corresponds an x-ideal B_x in S such that $A_{\bar{x}} = \psi(B_x)$, where ψ is the canonical mapping of S onto S/O_x. Now $\varphi(A_{\bar{x}}) = \{\mathfrak{U}(s);\ s \in B_x\}$. For $s \in S$, $s_1 \in B_x$, we find $\mathfrak{U}(s) \cap \mathfrak{U}(s_1) = \mathfrak{U}(s_1 s) \in \varphi(A_{\bar{x}})$. For $s_1, s_2 \in B_x$ we find $s \in S$ such that $(s)_x = (s_1, s_2)_x$, and $\mathfrak{U}(s_1) \cup \mathfrak{U}(s_2) = \mathfrak{U}(s) \in \varphi(A_{\bar{x}})$, since we clearly have $s \in B_x$. Thus $\varphi(A_{\bar{x}})$ is an l-ideal. Conversely, every l-ideal in $\mathfrak{L}$ may be written in the form $\{\mathfrak{U}(a);\ a \in A\}$ for some $A \subseteq S$. As $\varphi^{-1}(\{\mathfrak{U}(a);\ a \in A_x\}) = \psi(A_x)$, the proof is complete if we show that $A_x = A$. It is sufficient to show that for any finite subset N of A, $N_x \subseteq A$. To see this, we observe that if $N_x = (s)_x$, then $\bigcup_{a \in N} \mathfrak{U}(a) = \mathfrak{U}(s)$, and as $\{\mathfrak{U}(a);\ a \in A\}$ is an l-ideal, $\mathfrak{U}(s) \in \{\mathfrak{U}(a);\ a \in A\}$, $s \in A$ and for every $t \in (s)_x$, we have $s \in P_x \Rightarrow t \in P_x$ for every prime ideal P_x. Thus $\mathfrak{U}(t) \subseteq \mathfrak{U}(s)$, and again since $\{\mathfrak{U}(a);\ a \in A\}$ is an l-ideal, we conclude that $t \in A$. To sum up, $N_x \subseteq A$, and the proof is complete.

§ 5. A characteristic property for Boolean algebras. Finally we give an application of the previous theory to a simple algebraic problem. Let L be a distributive lattice with the l-system.

THEOREM 9. *$\mathfrak{P}(L)$ is compact if and only if L has a greatest element.*

Proof. Since the l-system is half-prime of finite character, and every finitely generated ideal is a principal ideal, the theorem follows from the corollary of Theorem 5.

THEOREM 10. *Let L be a distributive lattice with a greatest element I and a least element* 0. *Then L is a Boolean algebra if and only if $\mathfrak{P}(L)$ is Hausdorff.*

Proof. If L is a Boolean algebra, then for any $a \in L$, $\mathfrak{U}(a) = \mathfrak{F}(Ca)$, and $\mathfrak{P}(L)$ is Hausdorff by Theorem 1 and Theorem 3 (C). Conversely, assume that $\mathfrak{P}(L)$ is Hausdorff and let $a \neq 0, I$. Let $P_l \notin \mathfrak{F}(a)$. For every $Q_l \in \mathfrak{F}(a)$ there exists an element d such that $Q_l \in \mathfrak{U}(d)$ and $P_l \notin \mathfrak{U}(d)$. These $\mathfrak{U}(d)$ form an open covering of the compact set $\mathfrak{F}(a)$ (Theorem 9), and we can find $d_1, d_2, \ldots, d_n$ such that

$$\mathfrak{F}(a) \subseteq \mathfrak{U}(d_1) \cup \ldots \cup \mathfrak{U}(d_n) = \mathfrak{U}(\bigvee_{i=1}^{n} d_i). \tag{5.1}$$

Now put $b = \bigvee_{i=1}^{n} d_i$. By Theorem 3 (C) and Theorem 1, $\mathfrak{F}(b)$ is open, and $P_l \in \mathfrak{F}(b)$. On the other hand $\mathfrak{U}(a)$ is closed and therefore compact. Thus the sets $\mathfrak{F}(b)$ form an open covering of the compact set $\mathfrak{U}(a)$, and we can find $b_1, \ldots, b_m$ such that

$$\mathfrak{U}(a) \subseteq \mathfrak{F}(b_1) \cup \mathfrak{F}(b_2) \cup \ldots \cup \mathfrak{F}(b_m) = (\bigwedge_{i=1}^{m} b_i). \tag{5.2}$$

Now, for every $P_l \in \mathfrak{P}$, $a \wedge (\bigwedge_{i=1}^{m} b_i) \in P_l$, and since $k\mathfrak{P} = \{0\}$ we have $a \wedge (\bigwedge_{i=1}^{m} b_i) = 0$. On the other hand, $a \vee (\bigwedge_{i=1}^{m} b_i) = \bigwedge_{i=1}^{m} (a \vee b_i)$ is contained in no proper prime ideal. In fact, if $\bigwedge_{i=1}^{m} (a \vee b_i) \in P_l$, then for some i_0, $a \vee b_{i_0} \in P_l$, consequently $a \in P_l$ and $b_{i_0} \in P_l$, i.e., $P_l \in \mathfrak{F}(a)$ and $P_l \notin \mathfrak{U}(b_{i_0})$, which is impossible. This gives that $a \vee (\bigwedge_{i=1}^{m} b_i) = I$, and $\bigwedge_{i=1}^{m} b_i$ is a complement of a.

COROLLARY (Nachbin). *A distributive lattice with* 0 *and I is a Boolean algebra if and only if every proper prime ideal is a maximal ideal.*

Proof. For a distributive lattice we always have $\mathfrak{M} \subseteq \mathfrak{P}$. If $\mathfrak{P}(L)$ is Hausdorff, we therefore conclude $\mathfrak{M} = \mathfrak{P}$. On the other hand, if $\mathfrak{M} = \mathfrak{P}$, then clearly $\mathfrak{M} = \mathfrak{N} = \mathfrak{P}$, and by the corollary of Theorem 1, $\mathfrak{P}(L)$ is Hausdorff.

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