Absolute Borel sets in their Stone-Čech compactifications *

by

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The following theorem is well known (definitions and notation are given in Section 1):

THEOREM 1.1. *For a metrizable space X, the following conditions are equivalent*:

(i) *X is completely metrizable,*

(ii) *X is a G_δ set in some complete metric space,*

(iii) *X is an absolute G_δ,*

(iv) *X is a G_δ set in βX,*

(v) *X is a G_δ set in some compactification, BX.*

Alexandroff proved the equivalence of (i) and (ii), while the equivalence of (i) with (iii) is due of Sierpiński. Both these results appeared in the 1920's. In 1937, Čech ([1], p. 838) proved (i) equivalent to (iv), and (v) is clearly equivalent to (iv) by the mapping property of the Stone-Čech compactification (Theorem 1.4 below).

In this paper, it is our intention to generalize Čech's result as follows:

THEOREM. *For a metrizable space X, the following conditions are equivalent*:

(i) *X is an absolute G_α,*

(ii) *X is a G_α set in βX,*

(iii) *X is a G_α set in some compactification, BX.*

Since every absolute Borel set X is an absolute G_α for large enough α (depending on X), we derive as a corollary that *every absolute Borel set*

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* I wish to thank Professor A. H. Stone for suggesting the topic treated here. This work was done in partial fulfillment of the requirements for the Ph. D. degree at the University of Rochester.

** The author was supported while doing this research by a National Science Foundation Cooperative Graduate Fellowship.

is a Borel set in its Stone-Čech compactification. Our theorem also gives a start toward the reverse implication, completion of which is related to two problems:

1) Can a result similar to the theorem above be obtained for F_α sets?

2) Is there a convenient classification of the possible Borel types of a metrizable space X in βX?

1. Preliminaries. Throughout this paper, we reserve the symbols X to mean a metrizable topological space, Y to mean a Hausdorff topological space, and $\alpha, \beta, \ldots$ to mean ordinals $<$ the first uncountable ordinal, ω_1. The interior and closure of $A \subset Y$ will be denoted by $\mathrm{Int}_Y A$ and $\mathrm{Cl}_Y A$, respectively.

A subset A of Y is a G_0-set (F_0-set) in Y iff A is open (closed) in Y and a *G_α-set* (*F_α-set*) in Y iff either $A = \bigcup_{n=1}^{\infty} A_n$ (the additive case) or $A = \bigcap_{n=1}^{\infty} A_n$ (the multiplicative case), where each A_n is a G_{β_n} set (F_{β_n} set) in Y for some $\beta_n < \alpha$. Our notation in this respect follows that in [3], but we will not make the usual distinction between additive and multiplicative sets, except to note that G_α sets (F_α sets) are multiplicative for odd (even) α and additive for even (odd) α, with limit ordinals considered even. We assume standard facts about G_α and F_α sets (see [3]) such as:

1) the complement of a G_α set is an F_α set, and vice versa,

2) every G_α set (F_α set) is a G_β set (F_β set), whenever $\alpha < \beta$,

3) every G_α set (F_α set) in a metric space is *both* a G_β set and an F_β set, whenever $\alpha < \beta$,

4) finite unions or intersections of G_α sets (F_α sets) are G_α sets (F_α sets),

and so on.

The collection of *Borel sets* in a space Y is the smallest collection of sets containing the open sets and closed under complementation and countable union and intersection. The F_α sets and G_α sets, $0 \leqslant \alpha < \omega_1$, are Borel sets and in a metrizable space they are the only Borel sets.

A metrizable space X is an *absolute G_α* (*absolute F_α*) iff X is a G_α set (F_α set) in every metric space in which it is embedded. From the above remarks, every absolute G_α (and every absolute F_α) is both an absolute G_β and an absolute F_β, whenever $\alpha < \beta$. The absolute G_α and F_α spaces are referred to collectively as the *absolute Borel sets*. We will need the following well-known result:

THEOREM 1.2. *A metrizable space X is an absolute G_α for $\alpha \geqslant 1$ (absolute F_α for $\alpha \geqslant 2$) iff X is a G_α set (F_α set) in some complete metric space.*

Proof. See [3], p. 339.

A metrizable space X is *completely metrizable* iff there is a complete metric on X which gives the topology on X. Theorem 1.1 gives the relationship between a completely metrizable space and its absolute Borel properties.

A *compactification* K of a space Y is a compact Hausdorff space in which Y is densely embedded. If Y is completely regular, it has a *Stone-Čech compactification*, βY: with the property that if $f\colon Y \to K$ is continuous, where K is a compact Hausdorff space, then f has a continuous extension $f^\beta\colon \beta Y \to K$ (f^β is called the *Stone extension* of f). By applying this property in conjunction with Lemma 1.3, which we will also need later, we get the important Theorem 1.4 below.

LEMMA 1.3. *If $h\colon Y \to Y'$ is continuous and $h|X$ is a homeomorphism where X is a dense subset of Y, then we have $h(Y-X) \subset Y - h(X)$.*

Proof. See [2], p. 92.

THEOREM 1.4. *If Y is completely regular and K is any compactification of Y, there is a continuous function $g\colon \beta Y \to K$ such that $g(y) = y$ for each $y \in Y$ and $g(\beta Y - Y) = K - Y$.*

2. Covering uniformities. A covering $\mathcal{U}$ (not necessarily open) of Y is said to *refine* a covering $\mathcal{U}'$ of Y, $\mathcal{U} < \mathcal{U}'$, if for each $U \in \mathcal{U}$, there is a $U' \in \mathcal{U}'$ such that $U \subset U'$. We also say that $\mathcal{U}$ is a *refinement* of $\mathcal{U}'$.

If $U \in \mathcal{U}$, the *star* of U in $\mathcal{U}$, $\mathrm{St}(U, \mathcal{U})$, is the set $\bigcup [V \in \mathcal{U} \mid V \cap U \neq \emptyset]$. The covering $\mathcal{U}$ is said to *star-refine* the covering $\mathcal{U}'$, $\mathcal{U} \mathbin{*<} \mathcal{U}'$, if for each $U \in \mathcal{U}$, there is a $U' \in \mathcal{U}'$ such that $\mathrm{St}(U, \mathcal{U}) \subset U'$. We also say that $\mathcal{U}$ is a *star-refinement* of $\mathcal{U}'$.

A (covering) *uniformity* for Y is a collection μ of open coverings of Y (called the *uniform coverings*) satisfying the conditions:

(A) *If $\mathcal{U} < \mathcal{U}'$, $\mathcal{U} \in \mu$, then $\mathcal{U}' \in \mu$.*

(B) *If $\mathcal{U}, \mathcal{U}' \in \mu$, then there is a $\mathcal{U}'' \in \mu$ such that $\mathcal{U}'' \mathbin{*<} \mathcal{U}$, $\mathcal{U}'' \mathbin{*<} \mathcal{U}'$.*

A *basis* (or base) for the uniformity μ is a subcollection μ_0 of μ with the property that

$$\mu = [\mathcal{U} \mid \text{for some } \mathcal{U}_0 \in \mu_0,\ \mathcal{U}_0 < \mathcal{U}].$$

A collection μ_0 of open coverings of Y is a basis for some uniformity iff it satisfies condition (B) above.

A topology on Y can be obtained from a uniformity as follows. Let μ be a uniformity, or even a base for a uniformity, and if $y \in Y$, let $\mathrm{St}(y, \mathcal{U}) = \bigcup [U \in \mathcal{U} \mid y \in U]$. Then $[\mathrm{St}(y, \mathcal{U}) \mid \mathcal{U} \in \mu]$ is a neighborhood basis at y. The topology which results is smaller (has no more open sets) than the original topology on Y. If, in fact, it is the same topology, then the uniformity μ on Y is said to be *compatible* with the topology on Y.

A uniformity μ on a set X will be called *metrizable* (*pseudometrizable*) if there is a metric (pseudometric) on X such that $[\mathfrak{U}_1, \mathfrak{U}_2, \ldots]$ is a basis for μ, where $\mathfrak{U}_n$ is the set of all spheres of radius $1/3^n$ in X.

THEOREM 2.1. (Wei.) *A uniformity μ is pseudometrizable iff it has a countable basis.*

Proof. See [6], p. 61.

Let X be a metric space and for each $x \in X$ and each integer $n > 0$, let U_{nx} be the $1/3^n$ sphere about x. If we let $\mathfrak{U}_n = [U_{nx} | x \in X]$, then $[\mathfrak{U}_1, \mathfrak{U}_2, \ldots]$ is a basis for a metrizable uniformity on X which is clearly compatible with the metric topology on X. Note that we have $\mathfrak{U}_n *< \mathfrak{U}_{n-1}$ for each $n > 1$. We will now show that there is a basis $[\mathfrak{W}_1, \mathfrak{W}_2, \ldots]$ for this uniformity which also has the property that $\mathfrak{W}_n *< \mathfrak{W}_{n-1}$ for each $n > 1$, and which has the additional property that its elements consist solely of regularly open sets (i.e. sets A such that $\mathrm{Int}_X \mathrm{Cl}_X A = A$).

Let $V_{nx} = \mathrm{Int}_X \mathrm{Cl}_X U_{nx}$, for each $x \in X$ and $n > 0$, and let $\mathfrak{V}_n = [V_{nx} | x \in X]$ for each $n > 0$.

LEMMA 2.2. *V_{nx} is regularly open for each $n > 0$ and $x \in X$.*

Proof. Certainly $V_{nx} \subset \mathrm{Int}_X \mathrm{Cl}_X V_{nx}$. On the other hand, $\mathrm{Int}_X \mathrm{Cl}_X U_{nx} \subset \mathrm{Cl}_X U_{nx}$, and hence $\mathrm{Int}_X \mathrm{Cl}_X(\mathrm{Int}_X \mathrm{Cl}_X U_{nx}) \subset \mathrm{Int}_X \mathrm{Cl}_X U_{nx}$. That is $\mathrm{Int}_X \mathrm{Cl}_X V_{nx} \subset V_{nx}$. Thus V_{nx} is regularly open.

THEOREM 2.3. *$[\mathfrak{V}_1, \mathfrak{V}_3, \ldots]$ is a basis for the same uniformity as $[\mathfrak{U}_1, \mathfrak{U}_2, \ldots]$, and $\mathfrak{V}_{2n+1} *< \mathfrak{V}_{2n-1}$, for $n > 0$.*

Proof. Clearly $\mathfrak{U}_n < \mathfrak{V}_n$. On the other hand, if $V_{nx} = \mathrm{Int}_X \mathrm{Cl}_X U_{nx} \in \mathfrak{V}_n$, then $V_{nx} \subset \mathrm{St}(U_{nx}, \mathfrak{U}_n) \subset U_{n-1y}$ for some y, and hence $\mathfrak{V}_n < \mathfrak{U}_{n-1}$. Thus we have the following relationship:

$$\ldots < \mathfrak{V}_{n+1} < \mathfrak{U}_n < \mathfrak{V}_n < \mathfrak{U}_{n-1} < \ldots$$

It follows immediately that

(i) $[\mathfrak{V}_1, \mathfrak{V}_3, \ldots]$ is a basis for the same uniformity as $[\mathfrak{U}_1, \mathfrak{U}_2, \ldots]$, and

(ii) $\mathfrak{V}_{2n+1} *< \mathfrak{V}_{2n-1}$, for all $n > 0$.

This proves the theorem.

Thus the statement made prior to Lemma 2.2 is true; for take $\mathfrak{W}_n = \mathfrak{V}_{2n-1}$.

3. Extension of uniformities. Let X be a metrizable space, BX an arbitrary compactification of X, and A an open subset of BX. From the previous section, there is a uniformity on X, compatible with the metrizable topology on X, having a base $\mu_0 = [\mathfrak{U}_1, \mathfrak{U}_2, \ldots]$ with the following properties ($\mathfrak{U}_n$ here is the $\mathfrak{W}_n$ of Section 2):

(i) $\mathfrak{U}_n *< \mathfrak{U}_{n-1}$,

(ii) The sets of $\mathfrak{U}_n$ are indexed by the points of X, for each $n > 0$ (this is just a convenience),

(iii) Each $U_{nx} \in \mathfrak{U}_n$ is regularly open.

Define $W_{nx} \subset BX$ for each $n > 0$ and $x \in X$ as follows:

$$W_{nx} = \begin{cases} (\mathrm{Int}_{BX} \mathrm{Cl}_{BX} U_{nx}) \cap A & \text{if } U_{nx} \subset A, \\ (\mathrm{Int}_{BX} \mathrm{Cl}_{BX} U_{nx}) & \text{otherwise}. \end{cases}$$

Let $G = \bigcap_{n=1}^{\infty} (\bigcup_{x \in X} W_{nx})$. Then G is a G_δ in BX and $X \subset G$. Now define $V_{nx} \subset G$ for each $n > 0$ and $x \in X$ by: $V_{nx} = W_{nx} \cap G$. Then for each $n > 0$, the collection $\mathfrak{V}_n = [V_{nx} | x \in X]$ is an open covering of G and we have:

LEMMA 3.1. *For each $n > 0$ and $x \in X$, $U_{nx} = V_{nx} \cap X$.*

Proof. Since U_{nx} is regularly open in X and X is dense in BX, we have $U_{nx} = (\mathrm{Int}_{\beta X} \mathrm{Cl}_{\beta X} U_{nx}) \cap X$. The lemma follows immediately from the definitions of W_{nx} and V_{nx}.

LEMMA 3.2. $\mathrm{St}(V_{nx}, \mathfrak{V}_n) = \bigcup [V_{ny} | \ U_{ny} \cap U_{nx} \neq \emptyset]$.

Proof. If $V_{nx} \cap V_{ny} \neq \emptyset$, then $V_{ny} \cap V_{nx}$ is an open set in G and X is dense in G, so $(V_{ny} \cap V_{nx}) \cap X \neq \emptyset$. Thus

$$U_{ny} \cap U_{nx} = (V_{ny} \cap X) \cap (V_{nx} \cap X) = (V_{ny} \cap V_{nx}) \cap X \neq \emptyset,$$

by the previous lemma. On the other hand, clearly $U_{ny} \cap U_{nx} \neq \emptyset \Rightarrow V_{ny} \cap V_{nx} \neq \emptyset$.

LEMMA 3.3. *$\mathfrak{V}_n *< \mathfrak{V}_{n-1}$, for each $n > 1$.*

Proof. Let $V_{nx} \in \mathfrak{V}_n$, for $n > 1$. By 3.2, $\mathrm{St}(V_{nx}, \mathfrak{V}_n) = \bigcup [V_{ny} | U_{ny} \cap U_{nx} \neq \emptyset]$. Since $\mathfrak{U}_n *< \mathfrak{U}_{n-1}$, there is a set $U_{n-1,z} \in \mathfrak{U}_{n-1}$ such that $\mathrm{St}(U_{nx}, \mathfrak{U}_n) \subset U_{n-1,z}$. We claim that $\mathrm{St}(V_{nx}, \mathfrak{V}_n) \subset V_{n-1,z}$. Suppose $x' \in \mathrm{St}(V_{nx}, \mathfrak{V}_n)$; say $x' \in V_{ny}$, where $V_{ny} \cap V_{nx} \neq \emptyset$. Then by 3.2, $U_{ny} \subset \mathrm{St}(U_{nx}, \mathfrak{U}_n) \subset U_{n-1,z}$. But from the definition of the V_{nx}, $U_{ny} \subset U_{n-1,z}$ implies $V_{ny} \subset V_{n-1,z}$. Thus $x' \in V_{n-1,z}$. That is, $\mathrm{St}(V_{nx}, \mathfrak{V}_n) \subset V_{n-1,z}$, and we have shown that $\mathfrak{V}_n *< \mathfrak{V}_{n-1}$. This completes the proof of the lemma.

Thus $\mathfrak{V}_1, \mathfrak{V}_2, \ldots$ is a sequence of coverings of G such that $\ldots *< \mathfrak{V}_n *< \mathfrak{V}_{n-1} *< \ldots *< \mathfrak{V}_1$. Hence $[\mathfrak{V}_1, \mathfrak{V}_2, \ldots]$ is a basis for some pseudometrizable uniformity on G. We will call the set G with the pseudometric topology (G, ϱ) or G_ϱ denoting by (G, BX) or G_{BX} the set G with the BX-induced topology. The relationship between (G, ϱ) and (G, BX) is given by the following lemma.

LEMMA 3.4. *The topology on (G, ϱ) is smaller (has no more open sets) than the topology on (G, BX).*

Proof. It suffices to note that sets like $\mathrm{St}(x, \mathfrak{V}_n)$, which are elements of a neighborhood basis at x in (G, ϱ), are open in (G, BX), since each $V \in \mathfrak{V}_n$ is open in (G, BX). This proves Lemma 3.4.

LEMMA 3.5. *X is topologically embedded in (G, ϱ).*

Proof. It is sufficient to note that the restriction of our pseudometrizable uniformity on (G, ϱ) is just the original uniformity on X, with base $[\mathfrak{U}_1, \mathfrak{U}_2, \ldots]$, since $U_{nx} = X \cap V_{nx}$, for each $n > 0$ and $x \in X$. Thus Lemma 3.5 is established.

We note in passing that, in the case $BX = \beta X$, unless $G = X$ (hence, unless X is a G_δ in βX and thus an absolute G_δ) the topology on (G, ϱ) must be *strictly* smaller than that on (G, BX). Otherwise, (G, BX) would be metrizable and hence each of its points would be a G_δ in G and thus in βX. But no point of $\beta X - X$ is a G_δ in βX (see [2] p. 132).

Now let us define an open subset B of $A \cap G$ as follows:

$$B = \mathrm{Int}_{G\varrho}(A \cap G).$$

Then we have:

LEMMA 3.6. *$B \cap X = A \cap X$.*

Proof. Since $B \subset A$, we have $B \cap X \subset A \cap X$. On the other hand, suppose $x \in A \cap X$. Then there is an $n > 0$ such that $\mathrm{St}(x, \mathfrak{U}_n) \subset A \cap X$, since $A \cap X$ is open in X. But clearly $\mathrm{St}(x, \mathfrak{V}_n) = \bigcup [V_{ny} |\ x \in U_{ny}]$ and since $U_{ny} \subset A \Rightarrow V_{ny} \subset A$, we have $\mathrm{St}(x, \mathfrak{V}_n) \subset A$. Thus $x \in B$, so that $A \cap X \subset B \cap X$, establishing the claim.

Thus given an open set A in βX, we have found a pseudometric space (G, ϱ) and an open set B in (G, ϱ) such that:

1) G is a G_δ set in βX containing X,

2) The pseudometric topology on G is smaller than the βX-induced topology,

3) $B \subset A$ and $B \cap X = A \cap X$.

4. Some facts about F_a and G_a sets. We gather together in this section several results about F_a and G_a sets which will be referred to often in the ensuing development. Any proofs which are omitted are easy transfinite induction arguments.

LEMMA 4.1. *If H is a G_a set (F_a set) in Y' and $h\colon Y \to Y'$ is a continuous map, then $h^{-1}(H)$ is a G_a set (F_a set) in Y.*

COROLLARY 4.2. *If τ and τ' are two topologies on Y with τ smaller than τ', then a G_a set (F_a set) in (Y, τ) is also a G_a set (F_a set) in (Y, τ').*

LEMMA 4.3. *If H is a G_a set (F_a set) in a pseudometric space (M, ϱ) and $x \in H$, then $\varrho(x, y) = 0 \Rightarrow y \in H$.*

LEMMA 4.4. *If $Y' \subset Y$, then a subset A' of Y' is a G_a set (F_a set) iff there is a G_a set (F_a set) A in Y such that $A' = A \cap Y'$.*

Our final result in this section requires the introduction of some terminology. Let Y be a topological space, $\mathcal{E}$ a family of subsets of Y. Define $\mathcal{E}_0 = \mathcal{E}$, and for each ordinal $a > 0$, define $\mathcal{E}_a = [E \subset Y |\ E = \bigcup_{n=1}^{\infty} E_n$ or $E = \bigcap_{n=1}^{\infty} E_n$, where each $E_n \in \mathcal{E}_{\beta_n}$ for some $\beta_n < a]$.

LEMMA 4.5. *If $\mathcal{E} \subset \mathcal{F}$, then $\mathcal{E}_a \subset \mathcal{F}_a$, for each $a > 0$.*

THEOREM 4.6. *If $\mathcal{E}$ is any family of subsets of Y, then $E \in \mathcal{E}_a \Leftrightarrow$ there is a countable collection $\mathcal{C} \subset \mathcal{E}$ such that $E \in \mathcal{C}_a$.*

Proof. Sufficiency is proved by Lemma 4.5. Suppose that $\mathcal{E}$ is a family of subsets of Y and $E \in \mathcal{E}_1$. Then certainly $E \in \mathcal{C}_0$ for a countable subcollection $\mathcal{C}$ of $\mathcal{E}$, so the theorem is true for $a = 0$. Suppose it true for all $\beta < a$ and let $E \in \mathcal{E}_a$. Then $E = \bigcup_{n=1}^{\infty} E_n$ or $E = \bigcap_{n=1}^{\infty} E_n$, where each $E_n \in \mathcal{E}_{\beta_n}$ for some $\beta_n < a$. Now by the inductive hypothesis, for each n there is a countable collection $\mathcal{C}^n \subset \mathcal{E}$ such that $E_n \in \mathcal{C}^n_{\beta_n}$. Letting $\mathcal{C} = \bigcup \mathcal{C}^n$ we have $E_n \in \mathcal{C}_{\beta_n}$, by Lemma 4.5, for each n. Hence $E \in \mathcal{C}_a$. Since $\mathcal{C}$ is a countable subcollection of $\mathcal{E}$, we are done.

If $\mathcal{E}$ is taken to be the collection of all open (closed) sets in Y, then $E \in \mathcal{E}_a \Leftrightarrow E$ is a G_a set (F_a set) in Y and Theorem 4.6 can be rephrased as follows.

COROLLARY 4.7. *A subset E of Y is a G_a set (F_a set) in $Y \Leftrightarrow E \in \mathcal{C}_a$ for some countable collection $\mathcal{C}$ of open (closed) sets in Y.*

5. The pseudometric construction. The material of the next two sections is directed specifically toward proving the most difficult part of our theorem; namely that if X is a G_a set in βX then X is an absolute G_a. Hence, for the next two sections we assume that X is a G_a set in X.

From section 4, X belongs to $\mathcal{C}_a$, where $\mathcal{C}$ is some countable collection of open sets in βX. From section 3, if $\mathcal{C} = [C_1, C_2, \ldots]$, then for each n there is a G_δ set G_n in βX, a pseudometric ϱ_n on G_n, and an open set B_n in (G_n, ϱ_n) such that $B_n \subset C_n$ and $B_n \cap X = C_n \cap X$. We may assume that each ϱ_n is bounded by 1.

Let $G = \bigcap G_n$ and define $\varrho(x, y) = \sum_{i=1}^{\infty} \dfrac{\varrho_i(x, y)}{2^i}$ on G. Then (G, ϱ) is a pseudometric space and a G_δ set in βX and we have the following important result:

LEMMA 5.1. (i) *If H is open in (G_n, ϱ_n), then $H \cap G$ is open in (G, ϱ) (in particular, $B_n \cap G$ is open in G, for each $n > 0$).*

(ii) *If J is open in (G, ϱ), then J is open in G with the βX-induced subspace topology.*

Proof: (i) If H is open in (G_n, ϱ_n) and $x \in H \cap G$, then since $H \cap G$ is open in (G, ϱ_n), there is an $\varepsilon > 0$ such that $[y \in G| \ \varrho_n(x, y) < \varepsilon] \subset H \cap G$. But then clearly we have $[y \in G| \ \varrho(x, y) < \varepsilon/2^n] \subset [y \in G| \ \varrho_n(x, y) < \varepsilon] \subset H \cap G$, so that $H \cap G$ is open in (G, ϱ).

(ii) Let J be open in (G, ϱ) and $x \in J$. Pick $\varepsilon > 0$ so that $[y \in G| \ \varrho(x, y) < \varepsilon] \subset J$. Pick N large enough that $\sum_{i=1}^{\infty} 1/2^i < \varepsilon/2$. Now for $k = 1, \ldots, N$ the set $U_k = [z \in G| \ \varrho_k(x, z) < 2^k\varepsilon/2N]$ is open in (G, ϱ_k) and hence in $(G, \beta X)$. But if $y \in \bigcup_{k=1}^{N} U_k = U$, then

$$\varrho(x, y) \leqslant \sum_{k=1}^{N} [\varrho_k(x, y)/2^k] + \sum_{k=N+1}^{\infty} 1/2^k < \sum^{N} (\varepsilon/2N) + \varepsilon/2 = \varepsilon$$

and consequently $y \in J$. Hence U is a neighborhood of y in $(G, \beta X)$ which is contained in J, and we have proved the claim.

LEMMA 5.2. *X is topologically embedded in (G, ϱ); that is, ϱ is a metric on X and is equivalent to the original metric σ.*

Proof. Part (ii) of Lemma 5.1 shows that every open set in (X, ϱ) is open in (X, σ). Conversely, if H is open in (X, σ), then H is open in (X, ϱ_1), since by Lemma 3.5, X is embedded in (G_1, ϱ_1). But by Lemma 5.1, part (i), every open set in (X, ϱ_1) is open in (X, ϱ). Hence H is open in (X, ϱ) and the lemma is proved.

THEOREM 5.3. *$\mathcal{B} = [B_1 \cap G, B_2 \cap G, \ldots]$ is a collection of open sets in (G, ϱ) and $X \in \mathcal{B}_a$.*

Proof. (The sets B_i were introduced at the start of this section.) $B_i \cap G$ is open in (G, ϱ) by part (i) of Lemma 5.1. To prove that $X \in \mathcal{B}_a$, we will establish the following statement by transfinite induction: If $Y \in \mathcal{C}_a$ in βX, there is a subset Y' of G, $Y' \in \mathcal{B}_a$, such that $Y' \subset Y$ and $Y' \cap X = Y \cap X$.

For $a = 0$, $Y \in \mathcal{C}_0$ means $Y \in \mathcal{C}$, say $Y = C_1$. Then $Y' = B_1 \cap G$ satisfies the requirements. Suppose that the claim is true for all $\beta < a$ and let $Y \in \mathcal{C}_a$. Then $Y = \bigcup Y_n$ or $Y = \bigcap Y_n$, where $Y_n \in \mathcal{C}_{\beta_n}$ for some $\beta_n < a$. By the inductive hypothesis, for each n there is a $Y'_n \in \mathcal{B}_{\beta_n}$ such that $Y'_n \subset Y_n$ and $Y'_n \cap X = Y_n \cap X$. Define $Y' = \bigcup Y'_n$ or $Y' = \bigcap Y'_n$ (respectively, as Y is defined above). Then clearly $Y' \subset Y$ and $Y' \cap X = Y \cap X$, while Y' is an element of $\mathcal{B}_a$. Hence the inductive step is completed and the statement of the first paragraph is established.

Letting $X = Y$, we find that if $X \in \mathcal{C}_a$, then there is a $Y' \in \mathcal{B}_a$ in (G, ϱ) such that $Y' \subset X$ and $Y' \cap X = X$. Then we must have $Y' = X$ and hence $X \in \mathcal{B}_a$ as claimed.

6. The metric identification.

Let G^* denote the metric space obtained from the pseudometric space (G, ϱ) of section 5 is the usual way. That is, the points of G^* are the equivalence classes in (G, ϱ) under the equivalence relation $x \sim y \Leftrightarrow \varrho(x, y) = 0$, and the metric on G^* is defined by $\varrho^*([x], [y]) = \varrho(x, y)$, where $[x]$ denotes the equivalence class containing x.

LEMMA 6.1. *X is topologically embedded in G^*.*

Proof. Let h: $(G, \varrho) \to G^*$ be the identification map, $h(x) = [x]$, for each $x \in (G, \varrho)$. We assert that $h|H$ is a homeomorphism.

Since ϱ is a metric on X, h is clearly 1-1 on X. In proving h and h^{-1} continuous, it is sufficient to note that

$$h[y \in X| \ \varrho(x, y) < \varepsilon] = \big[[y] \in h(X)| \ \varrho^*([x], [y]) < \varepsilon\big],$$
$$h^{-1}\big[[y] \in h(X)| \ \varrho^*([x], [y]) < \varepsilon\big] = [y \in X| \ \varrho(x, y) < \varepsilon]$$

for each $x \in X$. This establishes Lemma 6.1.

THEOREM 6.2. *For the collection $\mathcal{B}$ of open subsets of (G, ϱ), $h(\mathcal{B}) = [h(B)| \ B \in \mathcal{B}]$ is a collection of open subsets of G^*, and if $H \in \mathcal{B}_a$, then $h(H) \in [h(\mathcal{B})]_a$.*

Proof. Obviously each $h(B)$ is open, since h is an open map. The second claim is true for $a = 0$, by definition. Suppose it true for all $\beta < a$ and let $H \in \mathcal{B}_a$. Then, say, $H = \bigcap H_n$, where each $H_n \in \mathcal{B}_{\beta_n}$ for some $\beta_n < a$. It is sufficient, by the inductive hypothesis, to show that $h(H) = \bigcap h(H_n)$. Certainly we have $h(H) \subset \bigcap h(H_n)$. Conversely, suppose $[x] \in \bigcap h(H_n)$. Then, for each n, there is an $x_n \in H_n$ such that $h(x_n) = [x]$. By the nature of the map h, we must then have $\varrho(x_1, x_n) = 0$ for each n. Then by Lemma 4.3, we have $x_1 \in H_n$ for each n, so that $x_1 \in \bigcap H_n = H$. Thus $[x] = h(x_1) \in h(H)$. Thus $\bigcap h(H_n) \subset h(H)$ and hence equality. The case $H = \bigcup H_n$ being obvious since $\bigcup h(H_n) = h(\bigcup H_n)$, the theorem is proved.

COROLLARY 6.3. *X is a G_a set in G^*.*

Proof. $X \in \mathcal{B}_a$ from section 5, so by the previous result, $h(X) \in [h(\mathcal{B})]_a$, and hence $h(X)$ is a G_a set in G^*. Since we may identify X and $h(X)$, this proves 6.3.

7. The main theorem.

THEOREM 7.1. *If X is metrizable, the following conditions are equivalent for $a \geqslant 1$:*

(i) *X is an absolute G_a,*

(ii) *X is a G_a set in βX,*

(iii) *X is a G_a set in some compactification, BX.*

Proof. We will show that (i) ⇒ (iii) ⇒ (ii) ⇒ (i).

(i) ⇒ (iii). If X is an absolute G_α, then X is a G_α set in some complete space X'. Now by Čech's result, X' is a G_δ set in $\beta X'$, so that X is a G_δ set intersected with a G_α set in $\beta X'$, and hence a G_α set in $\beta X'$. Then X is a G_α set in $\mathrm{Cl}_{\beta X'} X$, which is a compactification of X.

(iii) ⇒ (ii). Suppose that BX is a compactification of X in which X is a G_α set. Let h: $\beta X \to BX$ be the Stone map (see 1.4). Then $X = h^{-1}(X)$ is a G_α set in βX, by 4.1.

(ii) ⇒ (i). Suppose that X is a G_α set in βX. Then X is a G_α set in G^* from section 6 and the mappings i: $(G, \beta X) \to (G, \varrho)$ and h: $(G, \varrho) \to G^*$ are continuous and preserve X topologically. We will identify $h \circ i(X)$ with X when it is convenient. Letting $f = h \circ i$, we have:

$$\begin{array}{ccccc} X & \overset{\alpha}{\subset} & G_{\beta X} & \overset{1}{\subset} & \beta X \\ & & \downarrow & & \\ X & \overset{\alpha}{\subset} & G^* & \overset{f}{\subset} & \beta G^*, \end{array}$$

where "$X \overset{\alpha}{\subset} Y$" means "$X$ is a G_α set in Y".

Now f can be considered as a continuous mapping of $(G, \beta X)$ into βG^* and, as such, it has a Stone extension f^β: $\beta(G, \beta X) \to \beta G^*$. Since $\beta(G, \beta X) = \beta X$ (see [2], p. 89), we have f^β: $\beta X \to \beta G^*$. Furthermore, $f^\beta(\beta X)$ is a compact subset of βG^* containing G^*, so $f^\beta(\beta X) = \beta G^*$.

Since $f^\beta(G) = f(G) = G^*$ and f^β is onto, we must have $f^\beta(\beta X - G) \subset \subset \beta G^* - G^*$. Furthermore, f is a homeomorphism when restricted to the dense subset X of βX, so by 1.4, $f^\beta(\beta X - X) = \beta G^* - X$. Thus $f^\beta(\beta X - G) \subset \beta G^* - X$. Hence we have $\beta G^* - G^* \subset f^\beta(\beta X - G) \subset \beta G^* - X$. Thus, if we define $H = \beta G^* - f^\beta(\beta X - G)$, then $X \subset H \subset G^*$.

But G is a G_δ set in βX, so $\beta X - G$ is σ-compact and hence $f^\beta(\beta X - G) = f^\beta(\bigcup F_n) = \bigcup f^\beta(F_n)$, each F_n compact, so that $f^\beta(\beta X - G)$ is an F_σ-set in βG^*. Thus H is a G_δ set in βG^* and hence in $\mathrm{Cl}_{\beta G^*} H$. Since this last is a compactification of H, H is completely metrizable by Čech's original result.

Finally, X is a G_α set in G^* and hence in H. Thus X is a G_α set in a complete space and therefore an absolute G_α, by 1.2.

This completes the proof of Theorem 7.1.

A careful examination of the proof of the implications (i) ⇒ (iii) ⇒ (ii) above reveals that it contains nothing which cannot be used in a proof of the following:

COROLLARY 7.2. *If X is metrizable, then* (i) ⇒ (iii) ⇒ (ii) *below*: ($\alpha \geqslant 2$)

(i) *X is an absolute F_α,*

(ii) *X is an F_α set intersected with a G_δ set in βX,*

(iii) *X is an F_α set intersected with a G_δ set in some compactification βX.*

To prove that (ii) ⇒ (i) in 7.2 an analogue for closed sets is needed to the construction of the set B in section 3. This seems to be difficult to accomplish. One might better hope to establish a result like: X is Borel in βX iff X is a G_α-set in βX, for some α.

In the case of F_σ sets, A. H. Stone ([4]) has proved that: *a metrizable space X is an absolute F_σ iff X is σ-locally compact.* This, combined with the well-known fact that a subset of a locally compact space is locally compact iff it is the intersection of an open set with a closed set, yields the following

THEOREM 7.3. *A metrizable space X is an absolute F_σ iff X is the union of countably many sets in βX, each of which is the intersection of an open set with a closed set in βX.*

Since every absolute Borel set is, for some α, an absolute G_α, Theorem 7.1 yields:

THEOREM 7.4. *If a metrizable space X is an absolute Borel set, then X is a Borel set in βX.*

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Reçu par la Rédaction le 14. 6. 1965