Diagonal algebras
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Introduction. The class of n-dimensional diagonal algebras defined here (section I) is a generalization of a certain class of semigroups considered by Liapin, and the representation theorem for diagonal algebras (section II) is a generalization of the representation theorem for those semigroups. Besides, the paper contains theorems characterizing diagonal algebras in a different manner, and also theorems concerning independence (in the sense of Marczewski, see [4]), which in diagonal algebras presents itself particularly clearly (section III). The results of this paper were announced in [6].

Diagonal algebras turn to be extremal in a problem of estimation of the number of independent elements in algebras with a noncommutative binary operation and appear also in an investigation of sets of algebraic operations (cf. [7], [9] and [10]).

In diagonal algebras the condition of exchange of independent sets formulated by Marczewski is also fulfilled (see, e.g. [8]).

I. Definitions and the simplest properties. Let us consider an algebra \( D = (a, d) \) with a unique fundamental operation \( d(a_1, \ldots, a_n) \) satisfying the following postulates:

I. \( d(a, \ldots, a) = a \),

II. \( d(a_{i_1}, \ldots, a_{j_1}), d(a_{i_2}, \ldots, a_{j_2}), \ldots, d(a_{i_n}, \ldots, a_{j_n}) = d(a_{i_1}, a_{j_1}, \ldots, a_{j_n}) \).

This algebra will be called an \( n \)-dimensional diagonal algebra.

If the operation \( d(a_1, \ldots, a_n) \) depends only on each variable, then the \( n \)-dimensional diagonal algebra will be called proper.

The axioms I and II imply the following simple properties:

(i) Each algebraic operation \( f \in A^{m,n} \) (see [6]) in the \( n \)-dimensional diagonal algebra is of the form:

\[ f(a_{i_1}, \ldots, a_{j_n}) = d(a_{i_1}, \ldots, a_{j_n}), \quad (1 \leq i_p \leq m \text{ for } p = 1, \ldots, n). \]

In fact, from I it follows that

\[ d(a_{i_1}, \ldots, a_{i_m}) = d(a_{j_1}, \ldots, a_{j_n}); \]

hence, from II and the definition of algebraic operations we obtain (i).
(ii) In an \(n\)-dimensional diagonal algebra \(A\) (containing at least two elements) there are no algebraic constants.

In fact, if any algebraic operation is constant, then, on account of (i), we have
\[
\bar{d}(x_1, \ldots, x_n) = c,
\]
and further, on account of (i), \(x_1 = c\). The algebra would therefore be one-element, contrary to our assumption.

We say that element \(a\) of an \(n\)-dimensional diagonal algebra is collinear in the \(p\)th direction \(1 \leq p \leq n\) with element \(b\) of that diagonal algebra; we shall denote this by
\[
a \equiv_p b
\]
if
\[
a = \bar{d}(a_1, \ldots, a_{p-1}, a, b, a_{p+1}, \ldots, a_n).
\]

(iii) If \(a_p \equiv_p b\), then
\[
\bar{d}(a_1, \ldots, a_n) = \bar{d}(a_1, \ldots, a_{p-1}, a, b, a_{p+1}, \ldots, a_n).
\]

This property shows that in operation \(\bar{d}\) the \(p\)th argument may be replaced by any element collinear with it in the \(p\)th direction.

In fact, we have
\[
\bar{d}(a_1, \ldots, a_n) = \bar{d}(a_1, \ldots, a_{p-1}, \bar{d}(a_p, \ldots, a_{p-1}, b, a_{p+1}, \ldots, a_n))
\]
\[
= \bar{d}(a_1, \ldots, a_{p-1}, b, a_{p+1}, \ldots, a_n).
\]

We apply postulate II.

(iv) If \(\bar{d}(a_1, \ldots, a_n) = \bar{d}(b_1, \ldots, b_n)\), then
\[
a_p \equiv_p b_p
\]
for \(p = 1, \ldots, n\).

Indeed, by I, II we have
\[
a_p = \bar{d}(a_p, \ldots, a_n) = \bar{d}(a_p, \ldots, a_n, a_p, \ldots, a_p)
\]
\[
\quad = \bar{d}(a_p, \ldots, a_p, \bar{d}(a_1, \ldots, a_n), a_p, \ldots, a_p)
\]
\[
\quad = \bar{d}(a_p, \ldots, a_p, b, a_{p+1}, \ldots, a_p).
\]

(v) If \(a = \bar{d}(a_1, \ldots, a_n)\), then
\[
a \equiv_p a_p
\]
for \(p = 1, \ldots, n\).

To prove it let us remark that
\[
a = \bar{d}(a, \ldots, a) = \bar{d}(a_1, \ldots, a_n),
\]
and apply (iv).

(vi) In an \(n\)-dimensional proper diagonal algebra each algebraic operation \(\bar{d}\) depends on each of the variables on the right side of (1).

In fact, as the operation \(\bar{d}(x_1, \ldots, x_n)\) depends on each variable, there exist elements \(a_1, \ldots, a_n, b\) such that
\[
\bar{d}(a_1, \ldots, a_n) \neq \bar{d}(a_1, \ldots, a_{p-1}, b, a_{p+1}, \ldots, a_n).
\]

If the operation \(\bar{d}(x_1, \ldots, x_n)\) does not depend on \(x_p\), then in place of that variable we can put first element \(a_p\), and then—element \(b\), leaving without change the remaining arguments. Then we obtain the equation
\[
\bar{d}(a_1, \ldots, a_{p-1}, a, \bar{d}(a_p, \ldots, a_n), a) = \bar{d}(a_1, \ldots, a_{p-1}, b, a_{p+1}, \ldots, a_n)
\]
from which in view of (iv) we have
\[
a_p \equiv_p b.
\]

Further, by (iii) we can write
\[
\bar{d}(a_1, \ldots, a_n) = \bar{d}(a_1, \ldots, a_{p-1}, b, a_{p+1}, \ldots, a_n),
\]
which contradicts (2).

(vii) A subset \(J\) of an \(n\)-dimensional proper diagonal algebra is dependent if and only if there exist two different elements \(a\) and \(b\) in \(J\) collinear in some direction.

Indeed, for a one-dimensional diagonal algebra the condition is fulfilled, because then the diagonal algebra is trivial, i.e. the operation \(\bar{d}\) is trivial, each subset \(J\) is independent, and obviously \(a = \bar{d}(b)\) cannot hold for \(a \neq b\). Let us assume that \(n > 1\). Then the proof of sufficiency follows from the definition of collinearity and (vii).

In fact, if
\[
a = \bar{d}(a_1, \ldots, a_{p-1}, b, a_{p+1}, \ldots, a_n),
\]
and \(a, b \in J\) \(a \neq b\), then \(J\) is dependent since
\[
\bar{d}(a, \ldots, a, \bar{d}(a_1, \ldots, a_n), a) = \bar{d}(a, \ldots, a, \bar{d}(a_1, \ldots, a_n), a).
\]

(viii) A subset \(J\) of an \(n\)-dimensional proper diagonal algebra is independent if and only if each pair of its elements constitutes an independent set.

(ix) If \(\mathcal{D}\) is proper and
\[
(a_1, \ldots, a_n) \neq (1, \ldots, n) \quad (1 \leq i_j \leq n \text{ for } j = 1, 2, \ldots, n),
\]
then
\[
\bar{d}(a_1, \ldots, a_n) = \bar{d}(a_{i_1}, \ldots, a_{i_n}).
\]
In fact, from the assumption it follows that there exist \( p_{a} \) for which \( p_{a} \neq i_{a} \). If
\[
d(a_{1}, \ldots, a_{n}) = d(a_{1}, \ldots, a_{n})
\]
then, on account of (iv), we always have
\[
x_{a} = y_{a} \Rightarrow x_{a} = y_{a}.
\]
For example, let \( i_{a} < p_{a} \). On account of (iii) we have
\[
d(x_{1}, \ldots, x_{n}) = d(x_{1}, \ldots, x_{n+1}, y_{a}, x_{n+1}, \ldots, y_{a}, x_{n+1}, \ldots, x_{n}),
\]
but the last form contradicts the assumption that the operation \( d \) depends on each variable, because it does not depend on variable \( x_{a} \).

II. Representation Theorem and corollaries.

**Lemma 1.** Each of the relations \( \equiv \) is a congruence.

**Proof.** Reflectivity follows from formula I. Further, if \( a \equiv b \), then
\[
a = d(a, \ldots, a, b, a, \ldots, a),
\]
that is, on account of I
\[
d(a_{1}, \ldots, a, b, a, \ldots, a) = d(a_{1}, \ldots, a).
\]
Applying (iv) to the last formulae we obtain \( b = a \), i.e. relation \( \equiv \) is symmetrical. The transitivity of relation \( \equiv \) easily follows from II.

Further, if \( a \equiv b \), then
\[
d(a_{1}, \ldots, a) = d(a_{1}, \ldots, a_{i+1}, b_{i}, a_{i+1}, \ldots, a_{i}),
\]
by (iii) and II; hence, \( d(a_{1}, \ldots, a_{i}) = d(b_{1}, \ldots, b_{i}) \), q.e.d.

Lemma 1 permits the introduction of the concept of a coset in the \( n \)-dimensional diagonal algebra, just as in a group, viz.:

Set \( W^{a}_{n} \) of all elements collinear with \( a \) in the \( p \)th direction (\( p = 1, \ldots, n \)) will be called the coset in the \( p \)th direction determined by element \( a \).

The set of all cosets in the \( p \)th direction will be denoted by \( W^{n} \).

Obviously, in view of Lemma 1, two cosets in the \( p \)th direction are either disjoint or identical.

Let \( A_{1}, \ldots, A_{n} \) be arbitrary non-empty sets. We define the algebra
\[
\Psi_{A_{1}, \ldots, A_{n}} = (A_{1} \times \ldots \times A_{n}) \setminus d^{*}(x_{1}, \ldots, x_{n}),
\]
the unique fundamental operation \( d^{*}(x_{1}, \ldots, x_{n}) \) of this algebra being defined as follows:

\[
(3) \quad d^{*}(a_{1}, \ldots, a_{n}, a_{n+1}, a_{n+1}, \ldots, a_{n}) = (a_{1}, \ldots, a_{n}).
\]

It is easy to see that the algebra \( \Psi_{A_{1}, \ldots, A_{n}} \) is an \( n \)-dimensional diagonal algebra, i.e. it satisfies the axioms I and II.

Let \( D \) be an arbitrary \( n \)-dimensional diagonal algebra, and \( W^{n} \) the set of its cosets in the \( p \)th direction (\( p = 1, \ldots, n \)).

**Theorem 1.** The \( n \)-dimensional diagonal algebra \( \Psi \) is isomorphic to the algebra \( W^{n} \).

The proof of this theorem will be preceded by one lemma.

**Lemma 2.** Every intersection of the form
\[
W^{1}_{a} \cap \ldots \cap W^{n}_{a}
\]
contains exactly one element.

**Proof.** Indeed, this intersection contains \( a = d(a_{1}, \ldots, a_{n}) \) because, as follows from \((v)\),
\[
a = \equiv a_{p} \equiv a_{p} \equiv \ldots \equiv a_{p} \equiv a_{p} \equiv a_{p} \equiv a_{p} \equiv a_{p}.
\]
However, if the elements \( a \) and \( b \) belong to intersection \((4)\), then
\[
a = \equiv a_{p} \equiv a_{p} \equiv b \equiv d(b_{1}, \ldots, b_{n}) = b.
\]

**Proof of Theorem 1.** Now by Lemma 2 the map \( \varphi(a) = \langle W^{1}_{a_{1}}, \ldots, W^{n}_{a_{n}} \rangle \) is an isomorphism of \( D \) onto \( \Psi_{A_{1}, \ldots, A_{n}} \), since by \((v)\) we have
\[
\varphi(d(a_{1}, \ldots, a_{n})) = \langle W^{1}_{a_{1}}, \ldots, W^{n}_{a_{n}} \rangle = \langle W^{1}_{b_{1}}, \ldots, W^{n}_{b_{n}} \rangle = d^{*}(\varphi(a_{1}), \ldots, \varphi(a_{n})).
\]

**Remark.** Theorem 1 may be proved also with the aid of a theorem of Birkhoff (see [1] chapter VI, Theorem 4, p. 87).

**Corollary.** The isomorphism type of an \( n \)-dimensional diagonal algebra is determined by an \( n \)-tuple \( a_{1}, \ldots, a_{n} \) of cardinal numbers, where \( a_{p} \) is the power of the set of cosets in the \( p \)th direction. The \( n \)-dimensional diagonal algebra is proper if and only if \( a_{p} > 1 \) for \( p = 1, \ldots, n \).

The first part of the corollary follows immediately from Theorem 1. If in an \( n \)-dimensional proper diagonal algebra we had \( a_{p} = 1 \) for some \( p \), there would exist in that diagonal algebra only one coset in the \( p \)th direction. Hence, every two elements would be collinear in the \( p \)th direction. In the operation \( d(a_{1}, \ldots, a_{n}) \) in view of (iii) it would be possible to put any element in place of variable \( x_{p} \) without changing the value of the operation, and so the function \( d(a_{1}, \ldots, a_{n}) \) would not be dependent on variable \( x_{p} \), in contradiction to the assumption. The converse implication is obvious.

To denote what is the isomorphism type of the diagonal algebra, a corresponding \( n \)-tuple will be written as a lower index of the letter \( D \).
For example $D_{4,4}$ is a 3-dimensional diagonal algebra having two cosets in the first direction, three cosets in the second direction and four cosets in the third direction.

**Theorem 2.** Every two maximal independent sets of a given $n$-dimensional proper diagonal algebra have the same power. Every two minimal sets of generators of an $n$-dimensional diagonal algebra have the same power.

**Proof.** Let $I$ be a maximal independent set of an $n$-dimensional proper diagonal algebra. In view of (vii) the set $I$ cannot contain two different elements of the same coset in the $p$th direction. So

$$|I| = \min(a_1, \ldots, a_n).$$

As the set $I$ is maximal, it must contain one element from each coset on $p$ direction for some $p$. Hence

$$|I| > \max(a_1, \ldots, a_n).$$

The last two formulae give

$$|I| = \min(a_1, \ldots, a_n).$$

The first part of the theorem is thus proved.

Let $G$ be a maximal independent set of an $n$-dimensional diagonal algebra. As it is a set of generators, it must contain at least one element of each coset in each direction. Hence,

$$|G| > \max(a_1, \ldots, a_n).$$

As $G$ is a maximal independent set, it contains at most one element of each coset. Hence,

$$|G| = \max(a_1, \ldots, a_n).$$

The last two formulae give

$$|G| = \max(a_1, \ldots, a_n).$$

Formula (6) proves the second part of the theorem.

Let $\mathcal{A}$ be an arbitrary algebra. We shall denote by $a(\mathcal{A})$, $i(\mathcal{A})$ and $\gamma(\mathcal{A})$, respectively, the cardinal number of $\mathcal{A}$, the cardinal number of a maximal independent subset of $\mathcal{A}$ and the cardinal number of a minimal set of generators of $\mathcal{A}$.

From Theorem 1 (and from formulae (5) and (6)) we immediately obtain the following theorem:

**Theorem 3.**

$$a(D_{m,n}) = a_1 \cdot \ldots \cdot a_n,$$

$$i(D_{m,n}) = \min(a_1, \ldots, a_n),$$

$$\gamma(D_{m,n}) = \max(a_1, \ldots, a_n),$$

where formula (8) refers to the $n$-dimensional proper diagonal algebra.

**Theorem 4.** An $n$-dimensional diagonal algebra $D_{m,n}$, where $a > 1$, has a basis.

**Proof.** In view of Theorem 1 it is enough to prove for the algebra $D_{m,n}$, where $|A| = a$. Obviously, the basis of this algebra is in the set

$$\{a \in A : a \in A, a \notin A\},$$

**Theorem 5.** The subalgebra generated by a subset $A$ of an $n$-dimensional diagonal algebra is the set of all elements $a$, where

$$a \in W_{a_1} \cap \ldots \cap W_{a_n}, a \in A.$$

It follows immediately from Theorem 1.

**Theorem 6.** Transformation $\varphi$ of an $n$-dimensional diagonal algebra $D$ into an $n$-dimensional diagonal algebra $D'$ is a homomorphism if and only if it transforms each coset in the $p$-th direction of the first algebra into a coset in $p$-th direction for $p = 1, \ldots, n$ of the second algebra.

**Proof.** In fact, if $a$ and $b$ are the elements of the same coset in the $p$-th direction of the diagonal algebra $D$ and $\varphi$ is a homomorphism, we have

$$\varphi(a) = \varphi(d(a_1, a_2, a_3, \ldots, a_n)).$$

Conversely, if $\varphi$ transforms cosets in the same direction into cosets in the same direction, then on account of (v) we have

$$\varphi(d(a_1, a_2, a_3, \ldots, a_n)) = \varphi(a_p)$$

for each $p$. Since also

$$\varphi(d(a_1, a_2, a_3, \ldots, a_n)) = \varphi(a_p),$$

we must have in view of Lemma 2

$$\varphi(d(a_1, a_2, a_3, \ldots, a_n)) = \varphi(d(a_1, a_2, a_3, \ldots, a_n)),$$

i.e. $\varphi$ is a homomorphism, q.e.d.

**Theorem 7.** The direct product of the diagonal algebra $D_{m,n}$ and of the diagonal algebra $D_{m',n'}$ is a diagonal algebra $D_{m,n'}$.

To prove it, let us first consider the diagonal algebra $D_{m,n}$. We pick out exactly one element from each coset in the $p$th direction. The set of those elements is denoted by $E_p$. We do the same in the algebra $D_{m,n'}$, and we obtain the set $E$. An algebra which is the direct product of these two diagonal algebras is of course itself a diag-
Then the algebra \( \mathcal{A} \) is an \( n \)-dimensional proper diagonal algebra.

Proof. It follows from the assumption that the operation \( g \) is idempotent, i.e.
\[
g(x, \ldots, x) = x.
\]

In fact, the trivial operation \( e(x) = x \) also satisfies (10), and so (11) must hold. Secondly, in the algebra \( \mathcal{A} \) there are no constants, because any constant \( c \) may be considered as a function of one variable, and thus we would have
\[
c = c(x) = g(x, \ldots, x) = x,
\]
which is possible only in a one-element algebra.

Let us consider the operation
\[
g(g(x_1, \ldots, x_m), \ldots, g(x_1, \ldots, x_m)).
\]

In view of (10)
\[
g(g(x_1, \ldots, x_m), \ldots, g(x_1, \ldots, x_m)) = g(x_1, \ldots, x_m)
\]
(1 \( \leq i_p \leq n \), 1 \( \leq j_p \leq n \)).

We must prove that the operation \( g \) fulfills axiom II, i.e. that in (13) if \( i_p = p \) and \( j_q = q \) for \( p = 1, \ldots, n \). We shall show that in each case formula (13) gives a contradiction of the supposition that the operation \( g \) depends on each argument. Let us write the variables placed on the left side of formula (13) in a square matrix:
\[
\begin{array}{cccc}
  a_1 & a_2 & \cdots & a_n \\
  a_1' & a_2' & \cdots & a_n' \\
  \cdots & \cdots & \cdots & \cdots \\
  a_1'' & a_2'' & \cdots & a_n''
\end{array}
\]

1. If on the right side of formula (13) we had \( i_p = i_q \) and \( j_p = j_q \) for \( p \neq q \), where 1 \( \leq i_q \leq n \), 1 \( \leq j_q \leq n \), i.e. if the variables were not all different, then identifying the variables according to the formula
\[
a_i = a_i \quad (i = 1, \ldots, n, j = 1, \ldots, n),
\]
i.e. identifying variables belonging to the same line of table (1) and applying formula (11), we would obtain
\[
g(a_1, \ldots, a_n) = g(a_1', \ldots, a_n'),
\]
where again \( i_p = i_q \) for \( p \neq q \). This contradicts the assertion that \( g \) depends on each variable.
We have therefore proved that in formula (13) for each \( p \) we must have \( \varphi_p = \varphi = p \), which means that function \( g \) satisfies also axiom II, q.e.d.

We shall now prove for the 2-dimensional diagonal algebra a stronger theorem than Theorem 10, namely

**Theorem 11.** If \( g(x, y) \) is an algebraic operation of \( M \) depending on two variables, and each algebraic operation \( f \) of \( \mathbb{A}^{<m} \) \((m < 4)\) of this algebra is of the form

\[
f(x_1, \ldots, x_m) = g(x_{i_1}, x_{i_2}) \quad (1 \leq i_1 < i_2; \quad k = 1, 2),
\]

then the operation \( g \) satisfies axioms III-V of the 2-dimensional diagonal algebra (see Remark, Section II).

**Proof.** For shortness we shall write the operation \( g(x, y) \) as a multiplication, i.e. in the form \( x \odot y \), and we shall call it a diagonal multiplication. In view of the supposition we must have \( x \odot x \), and so Axiom III is fulfilled. This theorem will be proved if we show that the formulae

\[
\forall (x \odot y \odot z \equiv x \odot y) \quad (x \odot y \odot z \equiv x \odot y),
\]

are fulfilled because this set is equivalent to the set of Axioms IV and V. Therefore, on account of (19) each of the operations \( x \odot y \odot z \) and \( x \odot (y \odot z) \) must be identically equal to one of the following nine functions: \( x, y, z, x \odot y, y \odot z, y \odot (y \odot z) \). We shall show that the only possible combination are formulae (20). Namely, we shall prove that in all the other cases the diagonal multiplication would not depend on each variable. In the six lemmas given below by multiplication we shall mean an arbitrary idempotent operation of two variables and we shall denote it by a dot. We shall also say that the multiplication is trivial if it is identically equal to a trivial operation.

**Lemma 3.** The multiplication satisfying the identities

\[
(x \cdot y) \cdot z = x \cdot y, \quad x \cdot (y \cdot z) = x \cdot z
\]

is trivial.

Indeed, taking advantage successively of the two identities given in the Lemmas, we have

\[
y = (x \cdot y) \cdot y = (x \cdot y) \cdot (x \cdot y) = x \cdot y.
\]

**Lemma 4.** The multiplication satisfying the identities

\[
(x \cdot y) \cdot z = x \cdot z, \quad x \cdot (y \cdot z) = x \cdot y
\]

is trivial.

We have

\[
x = (x \cdot y) \cdot z = (x \cdot y) \cdot (x \cdot y) = x \cdot y.
\]
Lemma 5. The multiplication satisfying the identities

\[(x \cdot y) \cdot z = y \cdot z, \quad x \cdot (y \cdot z) = x \cdot y\]

is trivial.

We have

\[x \cdot y = (y \cdot x) \cdot y = (y \cdot x) = y \cdot x,\]

i.e. the multiplication is commutative. Hence,

\[x \cdot z = (y \cdot x) \cdot z = (y \cdot x) = y \cdot z,\]

i.e.

\[x \cdot z = y \cdot z.\]

Putting in the last formula \(y = z\) and applying idempotence, we have

\[x \cdot z = z.\]

Lemma 6. The multiplication satisfying one of the identities

\[(x \cdot y) \cdot z = x \cdot y,\]
\[(x \cdot y) \cdot z = y \cdot z,\]
\[x \cdot (y \cdot z) = y \cdot z,\]
\[x \cdot (y \cdot z) = z \cdot y,\]

is trivial.

For a proof, it is enough to put \(y = z\) in the first two formulae, and \(y = x\) in the remaining ones and to apply the idempotence of multiplication.

Lemma 7. The multiplication satisfying one of the identities

\[(x \cdot y) \cdot z = z \cdot x,\]
\[(x \cdot y) \cdot z = z \cdot y,\]
\[x \cdot (y \cdot z) = z \cdot x,\]
\[x \cdot (y \cdot z) = y \cdot z,\]

is trivial.

We shall prove it for the first formula. The proofs for the remaining ones are similar. Let us put \(y = x\) in the first formula of Lemma 8. We have

\[x \cdot z = z \cdot x,\]

i.e. the multiplication is commutative. Hence,

\[(x \cdot y) \cdot z = (y \cdot x) \cdot z,\]

and since the first of the formulae of Lemma 7 is an identity, we have

\[(y \cdot z) \cdot x = z \cdot y,\]

i.e.

\[z \cdot z = z \cdot y.\]

Let us put \(z = y\) in the last form. We get

\[y \cdot y = y \cdot y = y.\]

Lemma 8. The multiplication satisfying one of the identities

\[(x \cdot y) \cdot z = x,\]
\[(x \cdot y) \cdot z = y,\]
\[(x \cdot y) \cdot z = z,\]

is trivial.

Identifying the corresponding variables we shall easily obtain the proof of the Lemma in each case.

In view of Lemmas 3-8 the proof of the Theorem 11 is finished. Probably in Theorem 10 the suppositions may be weakened so as to make Theorem 11 its particular case.

References