жеством. Так как пространство X метрическое, то можно построить σ-локально конечную систему множеств, такую что в каждой точке отображение f не зависит от индекса α, если этот индекс α не соответствует некоторому элементу системы, содержащему точку x. Далее строим отображение g пространства X на некоторое пространство X^* так же как и в предложении 3.4 [6]. Легко видеть, что пространство X^* можно так отобразить на пространство Y, что $f = h \circ g$, где h — последнее отображение.

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Diagonal algebras

by

J. Płonka (Wrocław)

Introduction. The class of n-dimensional diagonal algebras defined here (section I) is a generalization of a certain class of semigroups considered by Liapin, and the representation theorem for diagonal algebras (section II) is a generalization of the representation theorem for those semigroups. Besides, the paper contains theorems characterizing diagonal algebras in a different manner, and also theorems concerning independence (in the sense of Marczewski, see [4]), which in diagonal algebras presents itself particularly clearly (section III). The results of this paper were announced in [6].

Diagonal algebras turn to be extremal in a problem of estimation of the number of independent elements in algebras with a noncommutative binary operation and appear also in an investigation of sets of algebraic operations (cf. [7], [9] and [10]).

In diagonal algebras the condition of exchange of independent sets formulated by Marczewski is also fulfilled (see, e.g. [8]).

I. Definitions and the simplest properties. Let us consider an algebra $\mathfrak{D} = (x; d)$ with a unique fundamental operation $d(x_1, \ldots, x_n)$ satisfying the following postulates:

I. $d(x, \ldots, x) = x$,

II. $d\big(d(x_1^1, \ldots, x_n^1), d(x_1^2, \ldots, x_n^2), \ldots, d(x_1^n, \ldots, x_n^n)\big) = d(x_1^1, x_2^2, \ldots, x_n^n)$.

This algebra will be called an *n-dimensional diagonal algebra*.

If the operation $d(x_1, \ldots, x_n)$ depends on each variable, then the n-dimensional diagonal algebra will be called *proper*.

The axioms I and II imply the following simple properties:

(i) *Each algebraic operation $f \in A^{(m)}$* (see [6]) *in the n-dimensional diagonal algebra is of the form:*

(1) $\quad f(x_1, \ldots, x_m) \equiv d(x_{i_1}, \ldots, x_{i_n}) \quad (1 \leqslant i_p \leqslant m$ *for* $p = 1, \ldots, n)$.

In fact, from I it follows that

$$e_j^{(m)}(x_1, \ldots, x_m) \equiv d(x_j, \ldots, x_j);$$

hence, from II and the definition of algebraic operations we obtain (i).

(ii) *In an n-dimensional diagonal algebra (containing at least two elements) there are no algebraic constants.*

In fact, if any algebraic operation is constant, then, on account of (1), we have

$$d(x_{i_1}, \ldots, x_{i_n}) \equiv c,$$

and further, on account of I, $x_1 \equiv c$. The algebra would therefore be one-element, contrary to our assumption.

We say that element a of an n-dimensional diagonal algebra is collinear in the pth direction ($1 \leqslant p \leqslant n$) with element b of that diagonal algebra; we shall denote this by

$$a \equiv_p b \quad \text{if} \quad a = d(\underbrace{a, \ldots, a}_{p-1}, b, a, \ldots, a).$$

(iii) *If $a_p \equiv_p b$, then*

$$d(a_1, \ldots, a_n) = d(a_1, \ldots, a_{p-1}, b, a_{p+1}, \ldots, a_n).$$

This property shows that in operation d the pth argument may be replaced by any element collinear with it in the pth direction.

In fact, we have

$$d(a_1, \ldots, a_n) = d\big(a_1, \ldots, a_{p-1}, d(\underbrace{a_p, \ldots, a_p}_{p-1}, b, a_p, \ldots, a_p), a_{p+1}, \ldots, a_n\big)$$
$$= d(a_1, \ldots, a_{p-1}, b, a_{p+1}, \ldots, a_n).$$

We apply postulate II.

(iv) *If $d(a_1, \ldots, a_n) = d(b_1, \ldots, b_n)$, then*

$$a_p \equiv_p b_p \quad \textit{for} \quad p = 1, \ldots, n.$$

Indeed, by I, II we have

$$a_p = d(a_p, \ldots, a_p) = d\big(\underbrace{a_p, \ldots, a_p}_{p-1}, d(a_1, \ldots, a_n), a_p, \ldots, a_p\big)$$
$$= d\big(\underbrace{a_p, \ldots, a_p}_{p-1}, d(b_1, \ldots, b_n), a_p, \ldots, a_p\big)$$
$$= d(a_p, \ldots, a_p, b_p, a_p, \ldots, a_p).$$

(v) *If $a = d(a_1, \ldots, a_n)$, then*

$$a \equiv_p a_p \quad \textit{for} \quad p = 1, \ldots, n.$$

To prove it let us remark that

$$a = d(a, \ldots, a) = d(a_1, \ldots, a_n),$$

and apply (iv).

(vi) *In an n-dimensional proper diagonal algebra each algebraic operation of form* (1) *depends on each of the variables on the right side of* (1).

In fact, as the operation $d(x_1, \ldots, x_n)$ depends on each variable, there exist elements $a_1, \ldots, a_n, b$ such that

$$d(a_1, \ldots, a_n) \neq d(a_1, \ldots, a_{p-1}, b, a_{p+1}, \ldots, a_n). \tag{2}$$

If the operation $d(x_{i_1}, \ldots, x_{i_n})$ does not depend on x_{i_p}, then in place of that variable we can put first element a_p, and then—element b, leaving without change the remaining arguments. Then we obtain the equation

$$d(x_{i_1}, \ldots, x_{i_{p-1}}, a_p, x_{i_{p+1}}, \ldots, x_{i_n}) = d(x_{i_1}, \ldots, x_{i_{p-1}}, b, x_{i_{p+1}}, \ldots, x_{i_n})$$

from which in view of (iv) we have

$$a_p \equiv_p b.$$

Further, by (iii) we can write

$$d(a_1, \ldots, a_n) = d(a_1, \ldots, a_{p-1}, b, a_{p+1}, \ldots, a_n),$$

which contradicts (2).

(vii) *A subset J of an n-dimensional proper diagonal algebra is dependent if and only if there exist two different elements a and b in J collinear in some direction.*

Indeed, for a one-dimensional diagonal algebra the condition is fulfilled, because then the diagonal algebra is trivial, i.e. the operation d is trivial, each subset J is independent, and obviously $a = d(b)$ cannot hold for $a \neq b$. Let us assume that $n > 1$. Then the proof of sufficiency follows from the definition of collinearity and (vi).

In fact, if

$$a = d(\underbrace{a, \ldots, a}_{p-1}, b, a, \ldots, a),$$

and $a, b \in J$ ($a \neq b$), then J is depend since

$$x \neq d(\underbrace{x, \ldots, x}_{p-1}, y, x, \ldots, x).$$

The necessity of the conditions follows from (i) and (iv).

Property (vii) implies immediately

(viii) *A subset I of an n-dimensional proper diagonal algebra is independent if and only if each pair of its elements constitutes an independent set.*

(ix) *If $\mathfrak{D}$ is proper and*

$$(i_1, \ldots, i_n) \neq (1, \ldots, n) \quad (1 \leqslant i_j \leqslant n \ \textit{for} \ j = 1, 2, \ldots, n),$$

then

$$d(x_1, \ldots, x_n) \not\equiv d(x_{i_1}, \ldots, x_{i_n}).$$

In fact, from the assumption it follows that there exist p_0 for which $p_0 \neq i_{p_0}$. If

$$d(x_1, \ldots, x_n) \equiv d(x_{i_1}, \ldots, x_{i_n})$$

then, on account of (iv), we always have

$$x_{p_0} \equiv_{p_0} x_{i_{p_0}}.$$

For example, let $i_{p_0} < p_0$. On account of (iii) we have

$$d(x_1, \ldots, x_n) \equiv d(x_1, \ldots, x_{i_{p_0}-1}, x_{i_{p_0}}, x_{i_{p_0}+1}, \ldots, x_{p_0-1}, x_{i_{p_0}}, x_{p_0+1}, \ldots, x_n),$$

but the last form contradicts the assumption that the operation d depends on each variable, because it does not depend on variable x_{p_0}.

II. Representation Theorem and corollaries.

Lemma 1. *Each of the relations $\equiv_p$ is a congruence.*

Proof. Reflectivity follows from formula I. Further, if $a \equiv_p b$, then

$$a = d(a, \ldots, a, b, a, \ldots, a),$$

i.e. on account of I

$$d(\underbrace{a, \ldots, a}_{p-1}, b, a, \ldots, a) = d(a, \ldots, a).$$

Applying (iv) to the last formulae we obtain $b \equiv_p a$, i.e. relation $\equiv_p$ is symmetrical. The transitivity of relation $\equiv_p$ easily follows from II. Further, if $a_j \equiv_p b_j$ $(j = 1, \ldots, n)$, then

$$d(a_1, \ldots, a_n) = d(a_1, \ldots, a_{p-1}, b_p, a_{p+1}, \ldots, a_n)$$
$$= d\big(a_1, \ldots, a_{p-1}, d(b_1, \ldots, b_n), a_{p+1}, \ldots, a_n\big)$$

by (iii) and II; hence, $d(a_1, \ldots, a_n) \equiv_p d(b_1, \ldots, b_n)$, q.e.d.

Lemma 1 permits the introduction of the concept of a coset in the n-dimensional diagonal algebra, just as in a group, viz.:

Set W_a^p of all elements collinear with a in the pth direction $(p = 1, \ldots, n)$ will be called the *coset* in the pth direction determined by element a.

The set of all cosets in the pth direction will be denoted by W^p.

Obviously, in view of Lemma 1, two cosets in the pth direction are either disjoint or identical.

Let $A_1, \ldots, A_n$ be arbitrary non-empty sets. We define the algebra

$$\mathfrak{P}_{A_1, \ldots, A_n} = \big(A_1 \times \ldots \times A_n;\ d^*(x_1, \ldots, x_n)\big),$$

the unique fundamental operation $d^*(x_1, \ldots, x_n)$ of this algebra being defined as follows:

$$d^*(\langle a_1^1, \ldots, a_n^1\rangle, \langle a_1^2, \ldots, a_n^2\rangle, \ldots, \langle a_1^n, \ldots, a_n^n\rangle) = \langle a_1^1, \ldots, a_n^n\rangle. \tag{3}$$

It is easy to see that the algebra $\mathfrak{P}_{A_1, \ldots, A_n}$ is an n-dimensional diagonal algebra, i.e. it satisfies the axioms I and II.

Let $\mathfrak{D}$ be an arbitrary n-dimensional diagonal algebra, and W^p the set of its cosets in the pth direction $(p = 1, \ldots, n)$.

Theorem 1. *The n-dimensional diagonal algebra $\mathfrak{C}$ is isomorphic to the algebra $\mathfrak{P}_{W^1, \ldots, W^n}$.*

The proof of this theorem will be preceded by one lemma.

Lemma 2. *Every intersection of the form*

$$W_{a_1}^1 \cap \ldots \cap W_{a_n}^n \tag{4}$$

contains exactly one element.

Proof. Indeed, this intersection contains $a = d(a_1, \ldots, a_n)$ because, as follows from (v),

$$a \equiv_p a_p \quad (p = 1, \ldots, n).$$

However, if the elements a and b belong to intersection (4), then $a \equiv_p b$ for each p, and so in view of axiom I and (iii)

$$a = d(a, \ldots, a) = d(b, \ldots, b) = b.$$

Proof of Theorem 1. Now by Lemma 2 the map $\varphi(a) = \langle W_a^1, \ldots, W_a^n\rangle$ is an isomorphism of $\mathfrak{D}$ onto $\mathfrak{P}_{W^1, \ldots, W^n}$ since by (v) we have

$$\varphi\big(d(a_1, \ldots, a_n)\big) = \langle W^1_{d(a_1, \ldots, a_n)}, \ldots, W^n_{d(a_1, \ldots, a_n)}\rangle$$
$$= \langle W_{a_1}^1, \ldots, W_{a_n}^n\rangle = d^*\big(\varphi(a_1), \ldots, \varphi(a_n)\big).$$

Remark. Theorem 1 may be proved also with the aid of a theorem of Birkhoff (see [1] chapter VI, Theorem 4, p. 87).

Corollary. *The isomorphism type of an n-dimensional diagonal algebra is determined by an n-tuple $\alpha_1, \ldots, \alpha_n$ of cardinal numbers, where α_p is the power of the set of cosets in the p-th direction. The n-dimensional diagonal algebra is proper if and only if $\alpha_p > 1$ for $p = 1, \ldots, n$.*

The first part of the corollary follows immediately from Theorem 1. If in an n-dimensional proper diagonal algebra we had $\alpha_p = 1$ for some p, there would exist in that diagonal algebra only one coset in the pth direction. Hence, every two elements would be collinear in the pth direction. In the operation $d(x_1, \ldots, x_n)$ in view of (iii) it would be possible to put any element in place of variable x_p without changing the value of the operation, and so the function $d(x_1, \ldots, x_n)$ would not be dependent on variable x_p, in contradiction to the assumption. The converse implication is obvious.

To denote what is the isomorphism type of the diagonal algebra, a corresponding n-tuple will be written as a lower index of the letter $\mathfrak{D}$.

For example $\mathfrak{D}_{2,3,4}$ is a 3-dimensional diagonal algebra having two cosets in the first direction, three cosets in the second direction and four cosets in the third direction.

THEOREM 2. *Every two maximal independent sets of a given n-dimensional proper diagonal algebra have the same power. Every two minimal sets of generators of an n-dimensional diagonal algebra have the same power.*

Proof. Let I be a maximal independent set of an n-dimensional proper diagonal algebra. In view of (vii) the set I cannot contain two different elements of the same coset in the pth direction. So

$$|I| \leqslant \min(a_1, \ldots, a_n).$$

As the set I is maximal, it must contain one element from each coset on p_0 direction for some p_0. Hence

$$|I| \geqslant \min(a_1, \ldots, a_n).$$

The last two formulae give

$$|I| = \min(a_1, \ldots, a_n). \tag{5}$$

The first part of the theorem is thus proved.

Let G be a minimal set of generators of an n-dimensional diagonal algebra. As it is a set of generators, it must contain at least one element of each coset in each direction. Hence,

$$|G| \geqslant \max(a_1, \ldots, a_n).$$

As G is a minimal set of generators, it contains at most one element of each coset. Hence,

$$|G| \leqslant \max(a_1, \ldots, a_n).$$

The last two formulae give

$$|G| = \max(a_1, \ldots, a_n). \tag{6}$$

Formula (6) proves the second part of the theorem.

Let $\mathfrak{A}$ be an arbitrary algebra. We shall denote by $a(\mathfrak{A})$, $\iota(\mathfrak{A})$ and $\gamma(\mathfrak{A})$, respectively, the cardinal number of $\mathfrak{A}$, the cardinal number of a maximal independent subset of $\mathfrak{A}$ and the cardinal number of a minimal set of generators of $\mathfrak{A}$.

From Theorem 1 (and from formulae (5) and (6)) we immediately obtain the following theorem:

THEOREM 3.

$$a(\mathfrak{D}_{a_1,\ldots,a_n}) = a_1 \cdot \ldots \cdot a_n, \tag{7}$$

$$\iota(\mathfrak{D}_{a_1,\ldots,a_n}) = \min(a_1, \ldots, a_n), \tag{8}$$

$$\gamma(\mathfrak{D}_{a_1,\ldots,a_n}) = \max(a_1, \ldots, a_n), \tag{9}$$

where formula (8) *refers to the n-dimensional proper diagonal algebra.*

THEOREM 4. *An n-dimensional diagonal algebra $\mathfrak{D}_{a,\ldots,a}$, where $a > 1$, has a basis.*

Proof. In view of Theorem 1 it is enough to prove it for the algebra $\mathfrak{P}_{A,\ldots,A}$, where $|A| = a$. Obviously, the basis of this algebra is the set $\{\langle a, \ldots, a\rangle \colon a \in A\}$, q.e.d.

THEOREM 5. *The subalgebra generated by a subset A of an n-dimensional diagonal algebra is the set of all elements a, where*

$$a \in W^1_{a_1} \cap \ldots \cap W^n_{a_n}; \quad a_1, \ldots, a_n \in A.$$

It follows immediately from Theorem 1.

THEOREM 6. *Transformation φ of an n-dimensional diagonal algebra $\mathfrak{D}$ into an n-dimensional diagonal algebra $\mathfrak{D}'$ is a homomorphism if and only if it transforms each coset in the p-th direction of the first algebra into a coset in p-th direction for $p = 1, \ldots, n$ of the second algebra.*

Proof. In fact, if a and b are the elements of the same coset in the pth direction of the diagonal algebra $\mathfrak{D}$ and φ is a homomorphism, we have

$$a = d(\underbrace{a, \ldots, a}_{p-1}, b, a, \ldots, a),$$

$$\varphi(a) = \varphi\big(d(a, \ldots, a, b, a, \ldots, a)\big)$$
$$= d\big(\varphi(a), \varphi(a), \ldots, \varphi(a), \varphi(b), \varphi(a), \ldots, \varphi(a)\big), \quad \text{i.e.} \quad \varphi(a) \equiv_p \varphi(b).$$

Conversely, if φ transforms cosets in the same direction into cosets in the same direction, then on account of (v) we have

$$d(a_1, \ldots, a_n) \equiv_p a_p,$$

$$\varphi\big(d(a_1, \ldots, a_n)\big) \equiv \varphi(a_p)$$

for each p. Since also

$$d\big(\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n)\big) \equiv_p \varphi(a_p),$$

we must have in view of Lemma 2

$$d\big(\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n)\big) = \varphi\big(d(a_1, \ldots, a_n)\big),$$

i.e. φ is a homomorphism, q.e.d.

THEOREM 7. *The direct product of the diagonal algebra $\mathfrak{D}_{a_1,\ldots,a_n}$ and of the diagonal algebra $\mathfrak{D}_{\beta_1,\ldots,\beta_n}$ is a diagonal algebra $\mathfrak{D}_{a_1\cdot\beta_1,\ldots,\beta_n\cdot a_n}$.*

To prove it, let us first consider the diagonal algebra $\mathfrak{D}_{a_1,\ldots,a_n}$. We pick out exactly one element from each coset in the pth direction. The set of those elements is denoted by R_1. We do the same in the algebra $\mathfrak{D}_{\beta_1,\ldots,\beta_n}$ and we obtain the set R_2. An algebra which is the direct product of these two diagonal algebras is of course itself a diagonal algebra and contains as many cosets in the pth direction as there are pairs in the direct product of the sets R_1 and R_2, i.e. $a_p \cdot \beta_p$. In fact, each pair $\langle a, b\rangle$, where $a \in R_1$, $b \in R_2$, determines some coset in the pth direction of the direct product of the two diagonal algebras. It is easy to see that different pairs determine different cosets and thus all the cosets are obtained.

Remark. Theorem 7 was obtained in collaboration with J. Mycielski. The next two theorems were found by J. Mycielski.

THEOREM 8. *The free product in the class of diagonal algebras of* $\mathfrak{P}_{A_1,\ldots,A_n}$ *and* $\mathfrak{P}_{B_1,\ldots,B_n}$, *where* $A_i \cap B_i = 0$ for $i = 1, \ldots, n$, *is isomorphic to* $\mathfrak{P}_{A_1\cup B_1,\ldots,A_n\cup B_n}$.

Proof. This immediately follows from Theorem 6. A topological algebra is a general algebra in the set of which a Hausdorff topology is given such that the fundamental operations are continuous.

THEOREM 9. *For any topological spaces* $A_1, \ldots, A_n$ *the diagonal algebra* $\mathfrak{P}_{A_1,\ldots,A_n}$ *with the product topology in its set* $A_1\times\ldots\times A_n$ *is a topological algebra.*

Proof. The continuity of the fundamental operation of $\mathfrak{P}_{A_1,\ldots,A_n}$ is obvious.

COROLLARY. *Any diagonal algebra admits some compact topology and is a compact algebra in the sense of* [5].

Remark. E. S. Liapin (see [3], p. 108) considers a semi-group characterized by the equations

III. $x\cdot x = x$,
IV. $(x\cdot y)\cdot z = x\cdot(y\cdot z)$,
V. $x\cdot y\cdot z = x\cdot z$,

this system of equations being, as is easy to verify, equivalent to Axioms I and II for $n = 2$. This semi-group is then a particular case of an n-dimensional diagonal algebra, namely, the 2-dimensional diagonal algebra.

This operation was also considered by C. C. Chang, B. Jónsson and A. Tarski, who used it for investigating decompositions of relational structures; see [2].

III. Some characterizations of n-dimensional proper diagonal algebras. It appears that the property expressed in (i) is characteristic for an n-dimensional proper diagonal algebra.

The following theorem is true:

THEOREM 10. *If* $g(x_1, \ldots, x_n)$ *is an algebraic operation of algebra* $\mathfrak{A}$, *depending on each variable, and each algebraic operation* $f \in A^{(m)}$ *of this algebra is of the form*

$$f(x_1,\ldots,x_m) \equiv g(x_{i_1},\ldots,x_{i_n}) \quad (1\leqslant i_p\leqslant m \text{ for } p = 1,\ldots,n), \tag{10}$$

then the algebra $\mathfrak{A}$ *is an* n*-dimensional proper diagonal algebra.*

Proof. It follows from the assumption that the operation g is idempotent, i.e.

$$g(x,\ldots,x) \equiv x. \tag{11}$$

In fact, the trivial operation $e(x) \equiv x$ also satisfies (10), and so (11) must hold. Secondly, in the algebra $\mathfrak{A}$ there are no constants, because any constant c may be considered as a function of one variable, and thus we would have

$$c \equiv c(x) \equiv g(x,\ldots,x) \equiv x,$$

which is possible only in a one-element algebra.

Let us consider the operation

$$g\big(g(x_1^1,\ldots,x_n^1),\ldots,g(x_1^n,\ldots,x_n^n)\big). \tag{12}$$

In view of (10)

$$g\big(g(x_1^1,\ldots,x_n^1),\ldots,g(x_1^n,\ldots,x_n^n)\big) \equiv g(x_{i_1}^{j_1},\ldots,x_{i_n}^{j_n}) \qquad (1\leqslant i_p\leqslant n,\ 1\leqslant j_p\leqslant n). \tag{13}$$

We must prove that the operation g fulfils axiom II, i.e. that in (13) $i_p = p$ and $j_p = p$ for $p = 1, \ldots, n$. We shall show that in each case formula (13) gives a contradiction of the supposition that the operation g depends on each argument. Let us write the variables placed on the left side of formula (13) in a square matrix:

$$\begin{matrix} x_1^1 & x_1^2 & \ldots & x_1^n \\ x_2^1 & x_2^2 & \ldots & x_2^n \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ x_n^1 & x_n^2 & \ldots & x_n^n \end{matrix} \tag{I}$$

1. If on the right side of formula (13) we had $i_p = i_q$ and $j_p = j_q$ for $p \neq q$, where $1\leqslant p\leqslant n$, $1\leqslant q\leqslant n$, i.e. if the variables were not all different, then identifying the variables according to the formula

$$x_i^j = x_i \quad (i = 1,\ldots,n,\ j = 1,\ldots,n), \tag{14}$$

i.e. identifying variables belonging to the same line of table (I) and applying formula (11), we would obtain

$$g(x_1,\ldots,x_n) \equiv g(x_{i_1},\ldots,x_{i_n}), \tag{15}$$

where again $i_p = i_q$ for $p \neq q$. This contradicts the assumption that g depends on each variable.

2. If two variables from the right side of formula (13) were in the same line of matrix (I), i.e. if $i_p = i_q$ for $p \neq q$, then by applying formula (14) we would obtain formula (15), and again we would have a contradiction.

3. If two variables on the right side of formula (13) appear in the same column of matrix I, i.e. if $j_p = j_q$ for some $p \neq q$, we put

$$x_i^j = x^j \quad (i = 1, \ldots, n,\ j = 1, \ldots, n) \tag{16}$$

and we get

$$g(x^1, \ldots, x^n) \equiv g(x^{j_1}, \ldots, x^{j_n}), \tag{17}$$

where on the right side of the last formula $j_p = j_q$ for $p \neq q$, which contradicts the assumption that g depends on each variable.

4. If for some p on the right side of (13) $i_p \neq p$ then, applying (14) we obtain (15), and in this case (15) would mean that in function g a permutation is possible moving the pth variable. Applying this permutation to the function $g(x_p^1, \ldots, x_p^n)$ in the pth place, where $j = p$ in the brackets on the left side of (13) we get

$$g\big(g(x_1^1, \ldots, x_n^1), \ldots, g(x_1^{p-1}, \ldots, x_n^{p-1}), g(x_{i_1}^p, \ldots, x_{i_n}^p), g(x_1^{p+1}, \ldots, x_n^{p+1}), \ldots, g(x_1^n, \ldots, x_n^n)\big) \equiv g(x_{i_1}^{j_1}, \ldots, x_{i_n}^{j_n}). \tag{18}$$

Let us now identify the variables according to the formula

$$x_i^j = x_p^{i_j} = x^j \quad (i = 1, \ldots, p-1,\ p+1, \ldots, n).$$

Applying formula (11) to the left side of (18), we obtain

$$g(x^1, \ldots, x^n) \equiv g(x^{j_1}, \ldots, x^{j_{p-1}}, x^{i_p}, x^{j_{p+1}}, \ldots, x^{j_n}).$$

This formula contradicts our assumption that g depends on each variable because on the right side we have only $n-1$ variables in view of $i_p \neq p = j_p$, and $j_1, \ldots, j_n$ is a permutation of the sequence $1, \ldots, n$.

Roughly this argumentation may be presented as follows: Knowing that no two variables from the right side of (13) can appear in the same line or in the same column of table (I), we permute the pth line of table (I) so that two such variables can be placed in one column. Next we identify the variables in the same columns bringing the problem to case 3.

5. If on the right side of (13) we have $j_p \neq p$ for some p, then by identifying the variables from table (I) according to (16) and applying (11) we come to the conclusion that in function g some non-trivial permutation is allowed (formula 17). Further the argumentation is as in case 4.

We have therefore proved that in formula (13) for each p we must have $i_p = j_p = p$, which means that function g satisfies also axiom II, q.e.d.

We shall now prove for the 2-dimensional diagonal algebra a stronger theorem than Theorem 10, namely

THEOREM 11. *If $g(x, y)$ is an algebraic operation of $\mathfrak{A}$ depending on two variables, and each algebraic operation $f \in A^{(m)}$ $(m < 4)$ of this algebra is of the form*

$$f(x_1, \ldots, x_m) \equiv g(x_{i_1}, x_{i_2}) \quad (1 \leqslant i_k \leqslant m;\ k = 1, 2), \tag{19}$$

then the operation g satisfies axioms III-V *of the 2-dimensional diagonal algebra* (see Remark, Section II).

Proof. For shortness we shall write the operation $g(x, y)$ as a multiplication, i.e. in the form $x \odot y$, and we shall call it a diagonal multiplication. In view of the supposition we must have $x \odot x$, and so Axiom III is fulfilled. This theorem will be proved if we show that the formulae

$$(x \odot y) \odot z \equiv x \odot z, \quad x \odot (y \odot z) \equiv x \odot z, \tag{20}$$

are fulfilled because this set is equivalent to the set of Axioms IV and V. Therefore, on account of (19) each of the operations $(x \odot y) \odot z$ and $x \odot (x \odot z)$ must be identically equal to one of the following nine functions: $x, y, z, x \odot y, x \odot z, y \odot z, y \odot x, z \odot x, z \odot y$. We shall show that the only possible combination are formulae (20). Namely, we shall prove that in all the other cases the diagonal multiplication would not depend on each variable. In the six lemmas given below by multiplication we shall mean an arbitrary idempotent operation of two variables and we shall denote it by a dot. We shall also say that the multiplication is trivial if it is identically equal to a trivial operation.

LEMMA 3. *The multiplication satisfying the identities*

$$(x \cdot y) \cdot z \equiv y \cdot z, \quad x \cdot (y \cdot z) \equiv x \cdot z$$

is trivial.

Indeed, taking advantage successively of the two identities given in the Lemma, we have

$$y \equiv (x \cdot y) \cdot y \equiv (x \cdot y) \cdot (x \cdot y) \equiv x \cdot y.$$

LEMMA 4. *The multiplication satisfying the identities*

$$(x \cdot y) \cdot z \equiv x \cdot z, \quad x \cdot (y \cdot z) \equiv x \cdot y$$

is trivial.

We have

$$x \equiv (x \cdot y) \cdot z \equiv (x \cdot y) \cdot (x \cdot y) \equiv x \cdot y.$$

LEMMA 5. *The multiplication satisfying the identities*

$$(x\cdot y)\cdot z \equiv y\cdot z, \qquad x\cdot(y\cdot z)\equiv x\cdot y$$

is trivial.

We have

$$x\cdot y \equiv (y\cdot x)\cdot y \equiv (y\cdot x)\cdot(y\cdot x) \equiv y\cdot x,$$

i.e. the multiplication is commutative. Hence,

$$x\cdot z \equiv (y\cdot x)\cdot z \equiv (x\cdot y)\cdot z \equiv y\cdot z,$$

i.e.

$$x\cdot z \equiv y\cdot z.$$

Putting in the last formula $y = z$ and applying idempotence, we have

$$x\cdot z \equiv z.$$

LEMMA 6. *The multiplication satisfying one of the identities*

$$(x\cdot y)\cdot z \equiv x\cdot y,$$
$$(x\cdot y)\cdot z \equiv y\cdot x,$$
$$x\cdot(y\cdot z) \equiv y\cdot z,$$
$$x\cdot(y\cdot z) \equiv z\cdot y$$

is trivial.

For a proof, it is enough to put $y = x$ in the first two formulae, and $y = z$ in the remaining ones and to apply the idempotence of multiplication.

LEMMA 7. *The multiplication satisfying one of the identities*

$$(x\cdot y)\cdot z \equiv z\cdot x,$$
$$(x\cdot y)\cdot z \equiv z\cdot y,$$
$$x\cdot(y\cdot z) \equiv z\cdot x,$$
$$x\cdot(y\cdot z) \equiv y\cdot x$$

is trivial.

We shall prove it for the first formula. The proofs for the remaining ones are similar. Let us put $y = x$ in the first formula of Lemma 8. We have

$$x\cdot z \equiv z\cdot x,$$

i.e. the multiplication is commutative. Hence,

$$(x\cdot y)\cdot z \equiv (y\cdot x)\cdot z,$$

and since the first of the formulae of Lemma 7 is an identity, we have

$$(y\cdot x)\cdot z \equiv z\cdot y,$$

i.e.

$$z\cdot x \equiv z\cdot y.$$

Let us put $z = y$ in the last form. We get

$$y\cdot x \equiv y\cdot y \equiv y.$$

LEMMA 8. *The multiplication satisfying one of the identities*

$$(x\cdot y)\cdot z \equiv x,$$
$$(x\cdot y)\cdot z \equiv y,$$
$$(x\cdot y)\cdot z \equiv z,$$
$$x\cdot(y\cdot z) \equiv x,$$
$$x\cdot(y\cdot z) \equiv y,$$
$$x\cdot(y\cdot z) \equiv z$$

is trivial.

Identifying the corresponding variables we shall easily obtain the proof of the Lemma in each case.

In view of Lemmas 3-8 the proof of the Theorem 11 is finished.

Probably in Theorem 10 the suppositions may be weakened so to make Theorem 11 its particular case.

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