Two theorems on the generation of systems of functions

by

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This paper deals with two basic questions about multiplace functions ("functions of several variables") defined on a finite set \( \mathbb{N}_m = \{1, \ldots, m\} \). How many functions can \( k \) functions generate by composition, and how many functions are needed to generate by composition all \( p \)-place functions?

The essential feature of the paper is its algebraic approach to the subject matter in contrast to the traditional treatment of functions in logic (1). Consider e.g. the functions over \( \mathbb{N}_2 \). By composition, the two basic logical functions, negation and disjunction, do not generate more than eight functions, namely, the four 1-place functions, four of the sixteen 2-place functions and none of the higherplace functions (see Example 2). All that Sheffer's stroke (herein denoted by a frontal \( A \)) generates are four of the 2-place functions. The traditional statement that \( A(x, y) \) also generates e.g. the 1-place negation \( n(x) \) is based on the fact that \( n(x) = A(x, x) \). But in so saying one substitutes \( x \) for \( y \); and similarly one substitutes \( A(y, x) \) for \( y \) in saying that \( A(x, y) \) generates \( A(x, A(y, x)) \). Substitution of an expression for a variable, however, is not the composition of functions. Nor is it possible to obtain any 1-place of 3-place function from \( A \) by compositions.

From our strictly algebraic point of view, we prove that the maximum number of functions that \( k \) functions can generate depends upon \( k \) but (except for trivial limitations) is independent of the place-numbers of the functions (Corollary 2 of Theorem I). At least \( p \) functions are necessary (Corollary 3 of Theorem I), and \( p \) properly chosen functions are sufficient (Theorem II), to generate all \( p \)-place functions for \( p > 1 \) with one important exception: the 2-place functions over \( \mathbb{N}_2 \).

* Theorem I and its Corollaries are due to the first author, Theorem II is the work of the second.

Another algebraic approach to the study of multiplace functions is the Marewski abstract algebra which, however, stresses the domains of the functions rather than their composition.
tions of the 2-valued logic, only two functions are necessary for all other finite-valued logics; and three functions are sufficient to generate all 3-place functions of the 2-valued as well as of all other finite-valued logics.

Let S be a set, and p a natural number. By a p-place function over S we mean (1) a mapping of $S^p$ (the set of all ordered p-tuples of elements of S) into S. In other words, a p-place function over S is a set of pairs $(T, s)$ containing, for each ordered p-tuple T of elements of S exactly one pair where s is an element of S. If the p-place function is denoted by F, and $(T, s)$ belongs to F, then we write, as is customary, $s = F(T)$. The set of all elements of S that are second members of pairs $(T, s)$ belonging to F is called the range of F—briefly, ran F.

If F is a p-place function over S, and $F_1, ..., F_p$ are q-place functions over S for some natural number q, then $F(F_1, ..., F_p)$ will denote the q-place function consisting of the pairs $[U, F(F_1(U), ..., F_q(U))]$ for all elements U of S. This q-place function is said to be the result of the composition of F with $F_1, ..., F_p$ or of the application of F to $F_1, ..., F_p$ or of the substitution of $F_1, ..., F_p$ into F. (Even in the case $p=1$, substitution of a function $F_1$ into F has of course nothing whatever to do with the substitution of $F_1$ for $F$ in the sense of replacing $F$ by $F_1$ or y by x). Clearly,

$$\text{ran } F(F_1, ..., F_p) \subseteq \text{ran } F.$$

The main property of the operation just defined which we have called superassociativity (2):

$$(F(F_1, ..., F_p))(G_1, ..., G_q) = F(F_1(G_1, ..., G_q), ..., F_p(G_1, ..., G_q))$$

for any p-place function F, any q-place functions $F_1, ..., F_p$, and any functions $G_1, ..., G_q$ of one and the same place-number—all of them over S.

If G is a set of functions over S (not necessarily of the same place-number), then the smallest set of functions containing G (as a subset) that is closed under substitution will be called the set generated by G and denoted by $\mathfrak{G}$. Thus $\mathfrak{G} = \mathfrak{G}$ is the smallest set including 1) G as a subset, and 2) the function $F(F_1, ..., F_p)$ if the p-place function F

(1) Cf. H. J. Whitlock [7]. It will be noted that we herein adhere to the type-graphical convention introduced by the senior author of this paper: All references to functions are in stoic type (e.g., $F, A, I, G, n, \text{deg, Max}$); all references to numbers and to elements of the domains and ranges of functions are in lower case roman type (e.g., p, n, x, y); sets of numbers or subsets of domains and ranges are denoted by capital letters in roman type (e.g., $S, T$); sets of functions in bold face (e.g., $F, G$).

(2) Cf. K. Menger [3] and the bibliography in that paper.

and the functions $F_1, ..., F_p$ having one and the same place-number belong to the set. Clearly, each function in $\mathfrak{G}$ must have the same place-number as one of the functions in G. A set of functions will be called homogeneous if all its elements have the same place-number.

If G is homogeneous, then so is $\mathfrak{G}$.

Example 1. If $S = \{0, 1\}$, consider the homogeneous set $G = \{A, I\}$ where I is the first 2-place selector assuming the value $I(x, y) = x$ for every pair $(x, y)$ in $S^2$, and A is incompatibility, for which $A(0, 0) = A(0, 1) = A(1, 0) = 1$ and $A(1, 1) = 0$. The set $\mathfrak{G} = \{A, I\}$ consists of the eight functions $I, A, A', A'' = (A, A)$, $I' = (I, I)$, $C = (A, A)$, $A = (A, C)$, $I = A(A, A')$, and $I = 0 = A(I, I)$. Here, C is the implication, for which $C(0, 0) = C(0, 1) = C(1, 1) = 1$, and $C(1, 0) = 0$; I is the constant 2-place function of value 1, for which $I(x, y) = 1$ for every $(x, y)$ in $S^2$; and $F'(x, y) = 1 - F(x, y)$ for each $(x, y)$ in $S^2$, where $F = A, I, C, I$. The function $A'$, for which $A'(x, y) = \text{Max}(x, y)$ is the conjunction; and $0$ is the other constant function.

Example 2. If $S = \{0, 1\}$, consider the nonhomogeneous set $(n, B)$, where n is the 1-place function for which $n(x) = 1 - x$ for $x = 0, 1$, and B is the disjunction, for which $B(x, y) = \text{Max}(x, y)$. Setting $n B = B'$ and $n o = o$ one readily verifies that

$\mathfrak{G} = \{n, I\}, \quad \mathfrak{G} = \{B, B'\}, \quad \mathfrak{G} = \{B', B, 1, 0\},$

and $\mathfrak{G} = \{n, o, o, i, o, B, B', 1, 0\}$.

Here, $o = B'(o, o) = n o, o, o = o$ are the constant 1-place functions over S, and j is the identity function over $(0, 1)$, for which $j(x) = x$ for $x = 0, 1$.

Remark 1. For every p-place function F in $\mathfrak{G} = G$, there is, for some natural number q, a q-place function G, in G, and p-place functions $F_1, ..., F_q$ in $\mathfrak{G}$ such that $G = F(F_1, ..., F_q)$.

If G is given we first associate, with some elements F of $\mathfrak{G}$, a natural number, called the degree of F relative to G. We get $\text{deg}(F, G) = 1$ if and only if G belongs to G. If the elements of degree <n are defined, let F be a p-place function such that $\text{deg}(F, G) = n$. We set $\text{deg}(G, F) = n + 1$ if there exist 1) a function in G, say a q-place function G, and 2) p-place functions $F_1, ..., F_q$ of degree <n such that $F = G(F_1, ..., F_q)$.

Parenthetically we remark that $\text{deg}(F, G)$ expresses a relation between F and the set G (and not a property of F, nor even a relation between G and $\mathfrak{G}$). Relative to $G = (I, A)$ in Example 1, I and 'A have the degree 1; I', A', and C, the degree 2; 'C and I, the degree 3; and 0 has the degree 4. Relative to $(I, A, I)$, the degree of I is 1, and that of 0 is 2. Relative to $(G, A)$ the degree of I is 2, and that of 0 is 3, since $C(G, C) = I$ and $A(I, I) = 0$. Set $\mathfrak{G} = \{I, A\}$, $\mathfrak{G} = \{I, A, I\} = \mathfrak{G} = \{A, C\}$. 

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The subsequent proof of Remark 1 pertains to one and the same set \( G \) of functions over the same \( S \), whence \( \deg F, G \) will be abbreviated to \( \deg F \). The set \( F \) of all functions in \( S \) that have a finite degree relative to \( G \) is a subset of \( \mathcal{G} \) which 1) contains the subset \( G \), and 2) is closed under substitution. We prove that, more precisely,
\[
\deg F(F_1, \ldots, F_p) \leq \deg F + \max \{\deg F_1, \ldots, \deg F_p\}
\]
for any \( p \)-place function \( F \) and any \( p \)-tuple of functions \( F_1, \ldots, F_p \) having one and the same place-number. Indeed, this inequality clearly holds if \( \deg F = 1 \). Assume its validity for all \( F \) of a degree \( \leq n \), and suppose that \( \deg F = n + 1 \). By the definition of degree, there exist a function in \( G \), say a \( q \)-place function \( G \), and functions \( H_1, \ldots, H_q \) in \( \mathcal{G} \) whose degrees are \( \leq n \) such that \( K = G(H_1, \ldots, H_q) \). By the superassociative law,
\[
K(F_1, \ldots, F_p) = G(H_1(F_1, \ldots, F_p), \ldots, H_q(F_1, \ldots, F_p))
\]
for all \( p \)-place functions \( K \) and \( p \)-tuples of functions \( F_1, \ldots, F_p \).

Since, by the inductive assumption, the inequality holds for each of the functions \( H_i(F_1, \ldots, F_p) \), it holds for \( K(F_1, \ldots, F_p) \).

Remark 3. Let \( B \) be the set of \( \{(1, 1), (0, 1), (1, 0), (0, 0)\} \). If \( F \) is the second 2-place selector, for which \( F(x, y) = y \) and \( E(x, y) = 1 \) or \( 0 \) according as \( x = y \) or \( x \neq y \). A pair of \( 2 \)-place functions \( \{F, G\} \) over \( \{0, 1\} \) has a base including all pairs if and only if \( F \) and \( G \) belong to \( B \) without constituting one of the pairs \( \{(1, 1), (0, 1), (1, 0), (0, 0)\} \).

Remark 4. Any base of a homogeneous ordered set \( G \) is in a one-to-one correspondence with \( \mathcal{G} \). The power of any such set \( G \) does not exceed the product of the powers of \( \mathcal{G}_1, \ldots, \mathcal{G}_k \).

What is the maximum number of functions of a group \( G \) of functions over one and the same set \( S \) that can be generated? What is the minimum number of \( p \)-place functions generating all \( \mathcal{S} \)-place functions over \( S \)?

First consider the case where \( G = \{F_1, F_2\} \). Let \( B \) be a base of \( F \) and let \( r \) be the number of elements in \( \mathcal{F} \) and the number of elements in \( B \). The values of any function \( F \) generated by \( F \) determine \( F \), whence \( \mathcal{F} \) includes at most \( r \) functions, regardless of the place-number of \( F \).
The range of each function contains at most m elements. The base of any homogeneous set of k functions contains at most m^2 elements. Hence

**Corollary 2.** A homogeneous set of k functions over \( N_m \) generates at most \( m^{m^2} \) functions, regardless of their place-number. If \( G \) is a set of \( m \) functions over \( N_m \) having different place-numbers, \( \alpha_1, \ldots, \alpha_m \), and \( x_k \) is the number of \( x \)-place functions in \( G \), then \( \Sigma G \) includes at most \( \sum_{i=1}^m m^{x_i} \) functions.

**Corollary 3.** A set \( G \) generating all \( p \)-place functions over \( N_m \) includes at least \( p \) functions.

We now turn to the questions whether there actually exist sets \( k \) functions over \( N_m \) that generate \( m^m \) functions, and whether \( p \) functions are sufficient to generate all \( p \)-place functions over \( N_m \).

Obviously, the lower bound, \( p \), stipulated in Corollary 3 is unsharp in three simple cases:

a) \( m = 1 \) and \( p > 1 \). There is only one single \( p \)-place function over \( N_m \) for each \( p \).

b) \( m > 1 \) and \( p = 1 \). No single function generates the full semigroup of \( 1 \)-place functions over \( N_m \). Two or more functions are needed (cf. Piccard [4]) to generate those \( m^m \) functions according as \( m = 2 \) or \( m > 2 \).

c) \( p = m = 2 \). Three functions are needed (cf. Menger [2]) to generate all \( 2 \)-place functions over \( N_1 \). If one considers \( S = \{0, 1\} \) instead of \( N_1 \), then from Remark 3 one readily concludes: Unless the pair of \( 2 \)-place functions \( (F, G) \) is a subset of the set \( E \), it has a base of at least three elements and, therefore, cannot generate more than eight functions. If \( (F, G) \) is a subset of \( E \), then \( \Sigma (F, G) \) is a subset of \( E \), and \( \Sigma E \) is easily seen to consist of eight elements: the six functions in \( E \) and the constant functions \( I_0 \) and \( I_1 \). All sixteen \( 2 \)-place functions over \( S \) are indeed generated by some triples of functions, e.g., by \((A, I_1, I_2)\). Except for these cases, however, the lower bound, \( p \), stipulated in Corollary 3 will now be proved to be sharp.

To the proof, we shall make extensive use of the selectors. For any two natural numbers, \( m \) and \( p \), there are \( p \) such \( p \)-place functions over \( N_m \). Where \( m \) is kept fixed, the \( k \)-th \( p \)-place selector over \( N_m \) will be denoted by \( S^k_p \) and is defined for \( 1 \leq k \leq p \) by

\[
S^k_p(x_1, \ldots, x_k) = x_k \quad \text{for any} \quad x_1, \ldots, x_k \in N_m.
\]

In some cases, we shall continue to write \( I \) and \( j \) for \( S^1_p \) and \( S^2_p \) over \( N_m \), respectively. The single \( 1 \)-place selector \( S^1_p \) is the identity function \( j \) over \( N_m \). It has the fundamental property that any function \( F \) of any
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The number of places) remains unchanged upon substitution into \( j \); that is to say, \( j^P = F \) for any \( P \).

If \( (F_1, \ldots, F_k) \) is an ordered \( k \)-tuple of \( k \)-place functions over \( N_m \), then, according to Remark 4, \( \text{ran}(F_1, \ldots, F_k) \) and any base of the set \( F = (F_1, \ldots, F_k) \) consist of equally many elements of \( N_m^k \). If the (unique) base of \( F \) includes all \( k^m \) elements of \( N_m^k \), then the set \( F \) will be called perfect. If \( F \) is perfect, then \( \text{ran}(F_1, \ldots, F_k) \) (as well as the range of \( F \) in any order) is a permutation of the base of \( F \). For any \( k \)-place function \( H \), we set \( H(F_1, \ldots, F_k) = H(F_1, \ldots, F_k)^j \) and define

\[
(F_1, \ldots, F_k)^{kn} = (F_1^{k_1}, \ldots, F_k^{k_m}).
\]

Clearly, \( k^m \) iterations of the permutation of the \( k^m \) elements yield the identical permutation; that is to say,

\[
(F_1, \ldots, F_k)^{kn} = (F_1^{k_1}, \ldots, F_k^{k_m}).
\]

Since each component of each \( (F_1, \ldots, F_k)^j \) belongs to \( \mathbb{S} \), we thus have

**Lemma 1.** If \( F \) is a perfect set of \( k \)-place functions over \( N_m \), then \( \mathbb{S} \) contains all \( k \)-place selectors, \( I_i^k (1 \leq i \leq k) \).

**Example 3.** Consider the triple \( (F_1, I_1, I_2) \) of 3-place functions over \( N_m \), where

\[
F(1, 1, 1) = F(1, 2, 2) = F(2, 1, 1) = F(2, 2, 2) = F(2, 1, 2) = 1,
\]

\[
I_1(x, y, z) = x, \quad I_2(x, y, z) = x \quad \text{for each} \quad (x, y, z) \in I_1^3.
\]

The set \( (F_1, I_1, I_2) \) is easily seen to be perfect. \( (F_1, I_1, I_2) \) produces a cyclical permutation of the triples

\[
(1, 1, 1), (2, 1, 1), (2, 2, 1), (2, 2, 2), (1, 2, 1), (1, 2, 2), (1, 1, 2), (1, 1, 2).
\]

Hence \( (F_1, I_1, I_2)^j = (I_1, I_2, I_1) \), where \( I_2(x, y, z) = x \). It follows that \( I_2 \) belongs to \( \mathbb{S} \). Indeed, \( I_1 = I_2(F_1, I_1, I_2)^j = I_2(F_1, I_1, I_2)^j = F(F_1, I_1, I_2)^j \).

A classical theorem in Boolean algebra asserts that each \( \text{"function of \( p \) variables} \ x_1, \ldots, x_p \text{" over} \ (0, 1) \text{" can be represented in two (so-called normal) forms: as a sum of products and as a product of sums.} \) The first half states, more precisely, that each \( F(x_1, \ldots, x_p) \), except the function assuming only the value 0, is, for some number \( k \), where \( 1 \leq k \leq 2^p \), the sum of \( k \) products of the form \( y_1 \cdots y_p \), where, for each \( i = 1, \ldots, p \), one has \( y_i = x_i \) or \( y_i = 1 - x_i \), for all \( i \). This theorem can be expressed in terms of the functions \( n, n', A, \) and \( B \), mentioned in Examples 1 and 2, and the \( p \)-place selectors \( I_i^p \), which we shall denote, briefly, by \( I_{1, \ldots, p} \). It is convenient to set \( n^p = n, n'^p = n \). Hence \( nF = nF \) or \( =F \) according

\[
\begin{align*}
F_{n_1, \ldots, n_p} &= A'(\cdots), \\
B_{n_1, \ldots, n_p} &= B'((\cdots)).
\end{align*}
\]

Each such function corresponds to one of the products \( x_1 \cdots x_p \) and obviously belongs to \( \mathbb{S} \). Hence \( \mathbb{S} \) contains all \( p \)-place functions \( F \).

The classical theorem asserts that, for any \( p \)-place function \( F \) over \( \{0, 1\} \) there exist \( k \) ordered \( p \)-tuples \((i_1, \ldots, i_p)\) where \( 1 \leq i \leq k \), for some \( k \), such that

\[
F = B((\cdots)).
\]

Hence \( F \) belongs to \( \mathbb{S} \).

Post [5] generalized the theorem just mentioned to the set \( \mathbb{S}^p \) of all \( p \)-place functions. We assume \( N_m \) to be ordered according to \( 1 < 2 < \ldots < m \), and define

\[
A'(x, y) = \min(x, y), \quad B(x, y) = \max(x, y), \quad n(x) = x + 1
\]

for \( 1 < x < m \) and \( n(m) = 1 \).

We set \( n^{k+1} = n^{k+1} \) for \( 1 \leq k \leq m-1 \). Clearly, \( n^m = j \), where \( j = F \) for each \( F \) over \( N_m \). The functions \( F_{n_1, \ldots, n_p} \) are defined as in the classical theorem, but for all ordered \( p \)-tuples \((i_1, \ldots, i_p)\) of numbers \( 1, \ldots, m \). Any \( p \)-place function \( F \) over \( N_m \), except the constant \( p \)-place function of value 1, can be expressed, just as in the classical case, in terms of \( B \) and \( k \) functions \( F_{n_1, \ldots, n_p} \) for some \( k \) such that \( 1 < k < m^p \).

We now prove

**Lemma 2.** Let \( p, \rho \), and \( m \) be natural numbers \( \geq 1 \). Then the set \( \mathbb{S}^p \) of all \( p \)-place functions over \( N_m \) is a subset of

\[
\mathbb{S}^* = \mathbb{S} \left( N^*_p, A^*_p, B^*_p, I^{(p)}_1, \ldots, I^{(p)}_p \right),
\]

where \( N^*_p, A^*_p, \) and \( B^*_p \) are \( p \)-place functions defined as follows:

\[
N^*_p = n^{(p)}, \quad A^*_p = A(I^{(p)}_1, I^{(p)}_2, \ldots, I^{(p)}_p); \quad B^*_p = B(I^{(p)}_1, I^{(p)}_2, \ldots, I^{(p)}_p).
\]

Let \( F \) be a function belonging to \( \mathbb{S}^p \). According to Post’s Theorem, \( F \) belongs to \( \mathbb{S} \), where \( \mathbb{S} \) is the set of all \( p \)-place functions. Therefore \( \mathbb{S} \rightarrow \mathbb{S}^p \). Let \( \mathbb{S} \rightarrow \mathbb{S} \) and \( \mathbb{S} \rightarrow \mathbb{S}^p \).

We prove that \( \mathbb{S} \rightarrow \mathbb{S} \) by induction on the degree of \( F \) relative to the set \( \mathbb{S} \). If \( deg F = 1 \), then, being a \( p \)-place function, \( F \) is one of the selectors \( I_{1, \ldots, p} \) and therefore belongs to \( \mathbb{S} \). Only if \( p = 2 \), also \( deg A' = deg B = 1 \); but in this case \( A' = A(I_{1, \ldots, p}) \) and \( B = B(I_{1, \ldots, p}) \). For \( p > 2 \), assume that all \( p \)-place functions
of a degree \(\leq n\) relative to \(S\) belong to \(S^*\), and let \(F^p\) be a \(p\)-place function of degree \(n+1\). Clearly, \(F^p\) is either \(nK\) or \(A^p(K, L)\) or \(B^p(K, L)\) for two functions \(K\) and \(L\) of a degree \(\leq n\). But
\[
\begin{align*}
  nK &= N^n(K, K, \ldots, K), \\
  A^p(K, L) &= A^p(K, L, \ldots, L), \\
  B^p(K, L) &= B^p(K, L, \ldots, L).
\end{align*}
\]
In any case, \(F\) thus belongs to \(S^*\).

An immediate consequence is

**Lemma 5.** For any two natural numbers, \(p\) and \(r\), if \(S \subseteq \{F_1, \ldots, F_k\} \subseteq S^m\), then \(S^p \subseteq S \subseteq \{F_1, \ldots, F_k, I_2^{(1)}, \ldots, I_2^{(p)}\}\).

The set \(S(F, I_1, I_2)\) in Example 3 contains, as has been shown, all three \(3\)-place selectors. As one readily verifies, \(N^n = F(I_1, I_1, I_2)\) and, if one sets \(K = F(I_1, I_1, I_2)\), then \(L = N^n(I_1, I_1, I_2)\). Therefore, \(N^n = F(I, I, I, I, I)\) and \(B^p = F(L, M, M)\). Thus also \(N^n, A^p, I_2^{(1)}\), and \(B^p\) belong to \(S(F, I_1, I_2)\). From Lemma 2 it follows that this set is \(S^*\).

We thus have established the case \(p = 0\) of

**Lemma 4.** For each \(p > 0\), there exist \(p\) functions generating \(S^*\).

Assume \(p > 0\), and consider
\[
G = \{F(I_1, I_2, I_3), I_1, I_1, I_1, I_2, I_2, \ldots, I_p\},
\]
where \(I_i\) is an abbreviation of \(I_2^{(i)}\). As one easily verifies, \(G\) is perfect. Hence, by Lemma 1, \(S\) includes all \(p\)-place selectors (also \(I_2\)). We further show that \(S\) includes \(N^n, A^p, I_2^{(1)}\), and \(B^p\). Setting \(F^p = F(I_1, I_1)\) one can verify that \(N^n = F^p(I_1, I_1, \ldots, I_1)\). Setting \(K = F^p(I_1, I_1, \ldots, I_1)\), \(L = N^n(I_1, I_1, \ldots, I_1)\), and \(M = N^n(I_1, I_1, \ldots, I_1)\), one furthermore verifies that
\[
A^p = N^n(K, K, \ldots, K) \quad \text{and} \quad B^p = F^p(L, M, \ldots, M).
\]
By Lemma 2, \(S^* \subseteq S^m\), which completes the proof of Lemma 4.

**Example 4.** Consider the \(2\)-place function \(F^p\) over \(N_2\) defined by
\[
\begin{align*}
  F(1, 1) &= F(1, 2) = F(1, 3) = 2; \\
  F(2, 2) &= F(3, 1) = F(3, 2) = 3; \\
  F(3, 3) &= F(2, 3) = F(3, 2) = 1.
\end{align*}
\]
Martin [1] has proved that this function \(F\) in conjunction with \(I\) and \(J\) generates all \(3^{n^2}\) \(2\)-place functions over \(N_2\). If we define \(G\) by \(G(x, y) = F(y, x)\) for each \((x, y)\) in \(N_2^n\), then the set \(S(F, G)\) is easily seen to be perfect and, therefore, by Lemma 1, includes \(I\) and \(J\). It follows that \(S(F, G) = S^*\).
By an inductive proof similar to that of Lemma 2, we see that the \( p \)-place functions in \( \mathcal{S}_G \) and in \( \mathcal{S}_G^c \) are the same. Hence \( \mathcal{S}_G = \mathcal{S}_G^c \).

We thus have the first half of

**Theorem II.** If \( m > 1 \) then, except for the case \( m = p = 2 \), the bounds given in Corollaries 3 and 2 of Theorem I are sharp; that is to say, there are \( p \) functions generating all \( p \)-place functions over \( \mathbb{N}_m \); and there exists a homogeneous set \( \mathcal{S}_G \) of \( k \) functions generating \( m^k \) functions. More specifically, there exists a homogeneous set of \( p \) functions including \( p - 2 \) selectors and generating \( \mathcal{S}_G^c \); and there exists a homogeneous set of \( k \) functions including \( k - 2 \) selectors and generating \( m^k \) functions.

In order to obtain a homogeneous set of \( k \) functions generating the maximum number of functions, for any number-\( p \) or \( k \), consider a homogeneous set \( \mathcal{F} \) of \( k \) functions, \( F_1, \ldots, F_k \), generating \( \mathcal{S}_G^c \). (\( F \) may be chosen so as to include \( k - 2 \) selectors.) For \( p \) \( k \), set \( F_{(p)} = F(p, F^{(p)}) \) and

\[
G = \{F_{(p)}^{(1)}, \ldots, F_{(p)}^{(k)}\}.
\]

The number of functions in \( \mathcal{S}_G \) and in \( \mathcal{S}_G^c \) is the same. Thus the \( k \) functions in \( G \) generate \( m^k \) functions.

Addition in the proofs. In Remark 1, a function \( F \in \mathcal{G} \) is not necessarily obtainable by substituting functions belonging to \( \mathcal{S}_G \) into a function belonging to \( \mathcal{S}_G \); cf. [1] p. 391 for an example of two functions \( F \) and \( G \) in the \( 3 \)-valued logic such that \( F(X, Y) = G(X, Y) = G \neq F \) for each \( X, Y \) in \( F, G \).

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**Publié par la Réduction le 20, 1, 1965**

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