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Two theorems on the generation of systems of functions

by

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This paper deals with two basic questions about multiplace functions ("functions of several variables") defined on a finite set $N_m = \{1, \ldots, m\}$. How many functions can k functions generate by composition, and how many functions are needed to generate by composition all p-place functions?

The essential feature of the paper is its algebraic approach to the subject matter in contrast to the traditional treatment of functions in logic [1]. Consider e.g. the functions over N_2. By composition, the two basic logical functions, negation and disjunction, do not generate more than eight functions, namely, the four 1-place functions, four of the sixteen 2-place functions and none of the higherplace functions (see Example 2). All that Sheffer's stroke (herein denoted by a frontal A) generates are four of the 2-place functions. The traditional statement that $A(x, y)$ also generates e.g. the 1-place negation $n(x)$ is based on the fact that $n(x) = A(x, x)$. But in so saying one substitutes x for y; and similarly one substitutes $A(y, z)$ for y in saying that $A(x, y)$ generates $A(x, A(y, z))$. Substitution of an expression for a variable, however, is not the composition of functions. Nor is it possible to obtain any 1-place of 3-place function from A by compositions.

From our strictly algebraic point of view, we prove that the maximum number of functions that k functions can generate depends upon k but (except for trivial limitations) is independent of the place-numbers of the functions (Corollary 2 of Theorem I). At least p functions are necessary (Corollary 3 of Theorem I), and p properly chosen functions are sufficient (Theorem II), to generate all p-place functions for $p > 1$ with one important exception: the 2-place functions over N_2. Thus while three functions are needed to generate all the 2-place func-

* Theorem I and its Corollaries are due to the first author, Theorem II is the work of the second.

[1] Another algebraic approach to the study of multiplace functions is the Marczewski abstract algebra which, however, stresses the domains of the functions rather than their composition.

tions of the 2-valued logic, only two functions are necessary for all other finite-valued logics; and three functions are sufficient to generate all 3-place functions of the 2-valued as well as of all other finite-valued logics.

Let S be a set, and p a natural number. By a p-*place function over* S we mean (²) a mapping of S^p (the set of all ordered p-tuples of elements of S) into S. In other words, a p-place function over S is a set of pairs (T, s) containing, for each ordered p-tuple T of elements of S exactly one pair where s is an element of S. If the p-place function is denoted by F, and (T, s) belongs to F, then we write, as is customary, $s = F(T)$. The set of all elements of S that are second members of pairs (T, s) belonging to F is called the *range* of F—briefly, ran$F$.

If F is a p-place function over S, and $F_1, \ldots, F_p$ are q-place functions over S for some natural number q, then $F(F_1, \ldots, F_p)$ will denote the q-place function consisting of the pairs $\big(U, F(F_1(U), \ldots, F_p(U))\big)$ for all elements U of S^q. This q-place function is said to be the result of the *composition* of F with $F_1, \ldots, F_p$ or of the *application* of F to $F_1, \ldots, F_p$ or of the *substitution* of $F_1, \ldots, F_p$ into F. (Even in the case $p = 1$, substitution of a function F_1 *into* F has of course nothing whatever to do with the substitution of F_1 *for* F in the sense of replacing F by F_1 or y by x). Clearly,

$$\mathrm{ran} F(F_1, \ldots, F_p) \subseteq \mathrm{ran} F.$$

The main property of the operation just defined is what we have called *superassociativity* (³):

$$\big(F(F_1, \ldots, F_p)\big)(G_1, \ldots, G_q) = F\big(F_1(G_1, \ldots, G_q), \ldots, F_p(G_1, \ldots, G_q)\big)$$

for any p-place function F, any q-place functions $F_1, \ldots, F_p$, and any functions $G_1, \ldots, G_q$ of one and the same place-number—all of them over S.

If $\boldsymbol{G}$ is a set of functions over S (not necessarily of the same place-number), then the smallest set of functions containing $\boldsymbol{G}$ (as a subset) that is closed under substitution will be called the set *generated* by $\boldsymbol{G}$ and denoted by $\mathfrak{S}\boldsymbol{G}$. Thus $\mathfrak{S}\boldsymbol{G}$ is the smallest set including 1) $\boldsymbol{G}$ as a subset, and 2) the function $F(F_1, \ldots, F_p)$ if the p-place function F

(²) Cf. H. I. Whitlock [7]. It will be noted that we herein adhere to the typographical convention introduced by the senior author of this paper: All references to functions are in *italic* type (e.g., *F, A, I, O, n, deg, Max*); all references to numbers and to elements of the domains and ranges of functions are in lower case roman type (e.g., p, s, x, y); sets of numbers or subsets of domains and ranges are denoted by capital letters in roman type (e.g., S, T); sets of functions in bold face (e.g., **G**, **F**).

(³) Cf. K. Menger [3] and the bibliography in that paper.

and the functions $F_1, \ldots, F_p$ having one and the same place-number belong to the set. Clearly, each function in $\mathfrak{S}\boldsymbol{G}$ must have the same place-number as one of the functions in $\boldsymbol{G}$. A set of functions will be called *homogeneous* if all its elements have the same place-number. If $\boldsymbol{G}$ is homogeneous, then so is $\mathfrak{S}\boldsymbol{G}$.

EXAMPLE 1. If $S = \{0, 1\}$, consider the homogeneous set $\boldsymbol{G} = \{A, I\}$ where I is the first 2-place *selector* assuming the value $I(x, y) = x$ for every pair (x, y) in S^2, and A is *incompatibility*, for which $A(0,0) = A(0,1) = A(1,0) = 1$ and $A(1,1) = 0$. The set $\mathfrak{S}\{I, A\}$ consists of the eight functions $I, A, A' = A(A, A)$, $I' = A(I, I)$, $C = A(I, A)$, $C' = A(C, C)$, $1 = A(A, A')$ and $1' = 0 = A(1, 1)$. Here, C is the *implication*, for which $C(0,0) = C(0,1) = C(1,1) = 1$ and $C(1,0) = 0$; 1 is the *constant* 2-place function of value 1, for which $1(x, y) = 1$ for every (x, y) in S^2; and $F'(x, y) = 1 - F(x, y)$ for each (x, y) in S^2, where $F = A, I, C, 1$. The function A', for which $A'(x, y) = Min(x, y)$ is the *conjunction*; and 0 is the other constant function.

EXAMPLE 2. If $S = \{0, 1\}$, consider the nonhomogeneous set $\{n, B\}$, where n is the 1-place function for which $n(x) = 1 - x$ for $x = 0, 1$, and B is the *disjunction*, for which $B(x, y) = Max(x, y)$. Setting $nB = B'$ and $nn = j$ one readily verifies that

$$\mathfrak{S}\{n\} = \{n, j\}, \quad \mathfrak{S}\{B\} = \{B\}, \quad \mathfrak{S}B' = \{B', B, 1, 0\},$$
$$\text{and} \quad \mathfrak{S}\{n, B\} = \{n, j, o, i, B, B', 1, 0\}.$$

Here, $o = B'(n, j) = 0(n, n)$ and $i = B'(o, o) = no$ are the *constant* 1-*place* functions over S, and j is the identity function over {0, 1}, for which $j(x) = x$ for $x = 0, 1$.

REMARK 1. *For every* p-*place function* F *in* $\mathfrak{S}\boldsymbol{G} - \boldsymbol{G}$, *there is, for some natural number* q, *a* q-*place function*, G, *in* $\boldsymbol{G}$, *and* p-*place functions* $F_1, \ldots, F_q$ *in* $\mathfrak{S}\boldsymbol{G}$ *such that* $F = G(F_1, \ldots, F_q)$.

If $\boldsymbol{G}$ is given we first associate, with some elements F of $\mathfrak{S}\boldsymbol{G}$, a natural number, called the *degree* of F relative to $\boldsymbol{G}$. We get $deg(F, \boldsymbol{G}) = 1$ if and only if F belongs to $\boldsymbol{G}$. If the elements of degree $\leqslant n$ are defined, let F be a p-place function such that $deg(F, \boldsymbol{G})$ is not $\leqslant n$. We set $deg(F, \boldsymbol{G}) = n+1$ if there exist 1) a function in $\boldsymbol{G}$, say a q-place function G, and 2) p-place functions $F_1, \ldots, F_q$ of degree $\leqslant n$ such that $F = G(F_1, \ldots, F_q)$.

Parenthetically we remark that $deg(F, \boldsymbol{G})$ expresses a relation between F and the set $\boldsymbol{G}$ (and not a property of F, nor even a relation between F and $\mathfrak{S}\boldsymbol{G}$). Relative to $\boldsymbol{G} = \{I, A\}$ in Example 1, I and A have the degree 1; I', A', and C, the degree 2; C' and 1, the degree 3; and 0 has the degree 4. Relative to $\{I, A, 1\}$, the degree of 1 is 1, and that of 0 is 2. Relative to $\{C, A\}$ the degree of 1 is 2, and that of 0 is 3, since $C(C, C) = 1$ and $A(1, 1) = 0$. Set $\mathfrak{S}\{I, A\} = \mathfrak{S}\{I, A, 1\} = \mathfrak{S}\{A, C\}$.

The subsequent proof of Remark 1 pertains to one and the same set $\boldsymbol{G}$ of functions over the same S, whence $deg(F, \boldsymbol{G})$ will be abbreviated to $degF$. The set $\boldsymbol{F}$, of all functions in $\mathfrak{S}\boldsymbol{G}$ that have a finite degree relative to $\boldsymbol{G}$ is a subset of $\mathfrak{S}\boldsymbol{G}$ which 1) contains the subset $\boldsymbol{G}$, and 2) is closed under substitution. We prove that, more precisely,

$$degF(F_1, \ldots, F_p) \leqslant degF + Max(degF_1, \ldots, degF_p)$$

for any p-place function F and any p-tuple of functions $F_1, \ldots, F_p$ having one and the same place-number. Indeed, this inequality clearly holds if $degF = 1$. Assume its validity for all F of a degree $\leqslant$n, and suppose that $degK = n+1$. By the definition of degree, there exist a function in $\boldsymbol{G}$, say a q-place function G, and functions $H_1, \ldots, H_q$ in $\mathfrak{S}\boldsymbol{G}$ whose degrees are $\leqslant$n such that $K = G(H_1, \ldots, H_q)$. By the superassociative law,

$$\begin{aligned} K(F_1, \ldots, F_p) &= \big(G(H_1, \ldots, H_q)\big)(F_1, \ldots, F_p) \\ &= G\big(H_1(F_1, \ldots, F_p), \ldots, H_q(F_1, \ldots, F_p)\big)\,. \end{aligned}$$

Since, by the inductive assumption, the inequality holds for each of the functions $H_i(F_1, \ldots, F_p)$, it holds for $K(F_1, \ldots, F_p)$.

By definition, $\mathfrak{S}\boldsymbol{G}$ is the smallest set with properties 1) and 2). Hence $\boldsymbol{F} = \mathfrak{S}\boldsymbol{G}$. In other words, each function $\mathfrak{S}\boldsymbol{G}$ has a finite degree relative to $\boldsymbol{G}$. This clearly entails Remark 1.

COROLLARY. *If F belongs to $\mathfrak{S}\boldsymbol{G}$, then* $\operatorname{ran}F \subseteq \operatorname{ran}G$ *for some function G in $\boldsymbol{G}$.* (This function G need not have the same place-number as F.)

REMARK 2. *If* T *and* T′ *are two elements of* S^p *such that* $G(T) = G(T')$ *for each* p*-place function G belonging to the set $\boldsymbol{G}$, then* $F(T) = F(T')$ *for each* p*-place function F belonging to $\mathfrak{S}\boldsymbol{G}$.*

In view of Remark 1, the proof by induction is straight forward.

If $\boldsymbol{G}$ is a homogeneous set of, say p-place, functions, then by a substitutive base—briefly, a *base*—of $\boldsymbol{G}$ we mean a subset B of S^p with the following properties: 1) for each T in S^p, there exists an element T′ in B such that $G(T) = G(T')$ for all G in $\boldsymbol{G}$; 2) if T′ and T″ are two elements of B, then $G(T') \neq G(T'')$ for at least one function G in $\boldsymbol{G}$. In other words, a base of $\boldsymbol{G}$ is a minimal subset of S^p with property 1). By Remark 2, a base of a homogeneous set G is also a base of $\mathfrak{S}\boldsymbol{G}$.

Thus *the set of all* p*-place functions with one and the same base* B *is closed with respect to substitution into* (i.e. left-side composition with) *functions as well as with respect to the application to* (i.e. right-side composition with) *functions having the base* B.

If F is a *constant* p-place function over S, then any single element of S^p constitutes a base of F. Now let S be $\{0, 1\}$. The base of each non-constant p-place function F consists of exactly two elements of S^p: one for which F assumes the value 0, and for which F assumes the value 1. Any base of the function A in Example 1 necessarily contains $(1, 1)$ and any one of the three other pairs. A base of $\{A, F\}$ consists of $(1, 1)$ and one or two of the other pairs, for any F. Any one other pair in conjunction with $(1, 1)$ constitutes a base of $\{A, I\}$. The bases of $\{A, C\}$ are $\{(1, 1), (1, 0), (0, 0)\}$ and $\{(1, 1), (1, 0), (0, 1)\}$.

One furthermore readily proves

REMARK 3. *Let $\boldsymbol{E}$ be the set $\{I, I', J, J', E, E'\}$ where J is the second* 2*-place selector, for which* $J(x, y) = y$ *and* $E(x, y) = 1$ *or* 0 *according as* $x = y$ *or* $x \neq y$. *A pair of* 2*-place functions* $\{F, G\}$ *over* $\{0, 1\}$ *has a base including all four pairs if and only if F and G belong to $\boldsymbol{E}$ without constituting one of the pairs* $\{I, I'\}$, $\{J, J'\}$, $\{E, E'\}$.

If $(G_1, \ldots, G_k)$ is an ordered k-tuple of p-place functions, then by the *range* of the k-tuple—briefly, ran $(G_1, \ldots, G_k)$—we mean the subset of S^k consisting of all k-tuples $(s_1, \ldots, s_k)$ for which there exists an element T of S^p such that $s_i = G_i(T)$ $(i = 1, \ldots, k)$. Clearly, $\operatorname{ran}(G_1, \ldots, G_k)$ is a subset of the Cartesian product, $\operatorname{ran}G_1 \times \ldots \times \operatorname{ran}G_k$. For example,

$$\operatorname{ran}(A, C) = \{(0, 1), (1, 0), (1, 1)\},$$

$$\operatorname{ran}(A, C, I) = \{(0, 1, 1), (1, 0, 1), (1, 1, 1)\},$$

$$\operatorname{ran}(A, C, I) = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}.$$

REMARK 4. *Any base of a homogeneous ordered set $\boldsymbol{G}$, is in a one-to-one correspondence with* $\operatorname{ran}\boldsymbol{G}$. *The power of any such set* $\boldsymbol{G} = \{G_1, \ldots, G_k\}$ *does not exceed the product of the powers of* $\operatorname{ran}G_1, \ldots, \operatorname{ran}G_k$.

What is the maximum number of functions that a set $\boldsymbol{G}$ of k functions over one and the same set S can generate? What is the minimum number of p-place functions generating all p-place functions over S?

First consider the case where $\boldsymbol{G} = \{F\}$. Let B be a base of F and let r be the number of elements in $\operatorname{ran}F$ and the number of elements in B. The values of any function $\boldsymbol{H}$ generated by F determine $\boldsymbol{H}$, whence $\mathfrak{S}F$ includes at most r^r functions, regardless of the place-number of F.

Next consider the case where $\boldsymbol{G} = \{F_1, F_2\}$ and F_1 and F_2 have the same place-number, say p. Let r_1 and r_2 be the numbers of elements in their ranges. According to the Corollary of Remark 1, the range of each function in $\mathfrak{S}\{F_1, F_2\}$ is a subset of either $\operatorname{ran}F_1$ or $\operatorname{ran}F_2$. According to Remark 2, a base of $\{F_1, F_2\}$ is also a base of $\mathfrak{S}\{F_1, F_2\}$. If b is the number of elements in such a base, then at most r_1^b and r_2^b functions are generated whose ranges are $\subseteq \operatorname{ran}F_1$ and $\subseteq \operatorname{ran}F_2$, respectively. If r_{12} is the number of elements in the intersection of $\operatorname{ran}F_1$ and $\operatorname{ran}F_2$, then the maximum number of functions in $\mathfrak{S}\{F_1, F_2\}$ is $r_1^b + r_2^b - r_{12}^b$, regardless of the place number p. By induction, one readily sees that, if $r_{i_1,\ldots,i_h}$ is the number of elements in the intersection of $\operatorname{ran}F_{i_1},\ldots,\operatorname{ran}F_{i_h}$, then the maximum number of functions in $\mathfrak{S}\{F_1,\ldots,F_k\}$ is

$$\sum_{h=1}^{k}\sum_{i_1\ldots i_h}(-1)^{h+1}r^{b}_{i_1,\ldots,i_h}.$$

Suppose now that $F_1,\ldots,F_k$ are all the p-place functions in G and that G also contains functions $G_{k+1},\ldots,G_t$ having the range numbers $r_{k+1},\ldots,r_t$ but place-numbers $\neq p$. The range of a function in $\mathfrak{S}G$ is a subset of one of the t functions in G. According to Remark 2, all p-place functions in $\mathfrak{S}G$ have the same base as $\{F_1,\ldots,F_k\}$. The total number of functions with those b base elements whose ranges are subsets of one of the t ranges of the functions in G is a sum like the one above, the only difference being that the summation of h ranges from 1 to t instead of from 1 to k. We can express this result in

THEOREM I. *If G is a set of t functions, $F_1,\ldots,F_t$, having s different place-numbers, $p_1,\ldots,p_k$, let b_k be the number of base elements of the set of all p_k-place functions in G; and, for any h such that $1\leqslant h\leqslant t$ and any set of h functions $F_{i_1},\ldots,F_{i_h}$, let $r_{i_1,\ldots,i_h}$ denote the number of elements in the intersection of $\operatorname{ran}F_{i_1},\ldots,\operatorname{ran}F_{i_h}$. Then the number of p_k-place functions in $\mathfrak{S}G$ does not exceed*

$$a_k=\sum_{h=1}^{t}\sum_{i_1,\ldots,i_h}(-1)^{h+1}r^{b_k}_{i_1,\ldots,i_h},$$

where one summation extends over all sets $\{i_1,\ldots,i_h\}$ of he of the numbers $1,\ldots,t$, and the other summation extends over the numbers h from 1 to t. The number of functions in $\mathfrak{S}G$ does not exceed $\sum_{k=1}^{s}a_k$.

It should be noted that *the place-numbers $p_1,\ldots,p_s$ themselves do not enter into the upper bounds given above for the numbers of functions generated.* An obvious limitation for the number of p-place functions that can be generated is of course *the number of all p-place functions.* This number is less than the given upper bound when there are too many generators.

In Example 2, we have $t=s=2$; $p_1=1$, $b_1=2$; $p_2=2$, $b_2=2$; $r_1=r_2=r_{12}=2$. Hence, according to Theorem I, the number of functions in $\mathfrak{S}\{n,B\}$ is at most $(2^2+2^2-2^2)+(2^2+2^2-2^2)$. It actually is 8.

COROLLARY 1. *If G is a homogeneous set of k functions with b base elements, and the ranges of the functions in G, which contain $r_1,\ldots,r_k$ elements, are disjoint, then $\mathfrak{S}G$ includes at most $r_1^b+\ldots+r_k^b$ functions.*

We now come to the most important special cases of Theorem I. They concern the sets $S=N_m=\{1,\ldots,m\}$ for some natural number m.

The range of each function contains at most m elements. The base of any homogeneous set of k functions contains at most m^k elements. Hence

COROLLARY 2. *A homogeneous set of k functions over N_m generates at most m^{m^k} functions, regardless of their place-number. If G is a set of functions over N_m having s different place-numbers, $p_1,\ldots,p_s$, and if k_i is the number of p_i-place functions in G, then $\mathfrak{S}G$ includes at most $\sum_{i=1}^{s}m^{m^{k_i}}$ functions.*

COROLLARY 3. *A set G generating all p-place functions over N_m includes at least p functions.*

We now turn to the questions whether there actually exist sets of k functions over N_m that generate m^{m^k} functions, and whether p functions are sufficient to generate all p-place functions over N_m.

Obviously, the lower bound, p, stipulated in Corollary 3 is unsharp in three simple cases:

a) $m=1$ and $p>1$. There is only one single p-place function over N_m for each p.

b) $m>1$ and $p=1$. No single function generates the full semigroup of 1-place functions over N_m. Two or three functions are needed (cf. Piccard [4]) to generate those m^m functions according as $m=2$ or $m>2$.

c) $p=m=2$. Three functions are needed (cf. Menger [2]) to generate all 2-place functions over N_2. If one considers $S=\{0,1\}$ instead of N_2, then from Remark 3 one readily concludes: Unless the pair of 2-place functions $\{F,G\}$ is a subset of the set E, it has a base of at most three elements and, therefore, cannot generate more than eight functions. If $\{F,G\}$ is a subset of E, then $\mathfrak{S}\{F,G\}\subset\mathfrak{S}E$, and $\mathfrak{S}E$ is easily seen to consist of eight elements: the six functions in E and the constant functions *1* and *0*. All sixteen 2-place functions over S are indeed generated by some triples of functions, e.g., by $\{A,I,J\}$.

Except for these cases, however, the lower bound, p, stipulated in Corollary 3 will now be proved to be sharp.

In the proof, we shall make extensive use of the *selectors*. For any two natural numbers, m and p, there are p such p-place functions over N_m. Where m is kept fixed, the k-th p-place selector over N_m will be denoted by $I_k^{(p)}$ and is defined for $1\leqslant k\leqslant p$ by

$$I_k^{(p)}(x_1,\ldots,x_p)=x_k \quad \text{for any} \quad x_1,\ldots,x_p \text{ in } N_m.$$

In some cases, we shall continue to write I and J for $I_1^{(2)}$ and $I_2^{(2)}$ over N_m, respectively. The single 1-place selector $I_1^{(1)}$ is the identity function j over N_m. It has the fundamental property that any function F (of any number of places) remains unchanged upon substitution into j; that is to say, $jF = F$ for any F.

If $(F_1, \dots, F_k)$ is an ordered k-tuple of k-place funcions over N_m, then, according to Remark 4, $\operatorname{ran}(F_1, \dots, F_k)$ and any base of the set $\boldsymbol{F} = \{F_1, \dots, F_k\}$ consist of equally many elements of N_m^k. If the (unique) base of $\boldsymbol{F}$ includes all k^m elements of N_m^k, then the set $\boldsymbol{F}$ will be called *perfect*. If $\boldsymbol{F}$ is perfect, then $\operatorname{ran}(F_1, \dots, F_k)$ (as well as the range of $\boldsymbol{F}$ in any order) is a permutation of the base of $\boldsymbol{F}$. For any k-place function H, we set $H(F_1, \dots, F_k) = H(F_1, \dots, F_k)^1$ and define

$$(F_1, \dots, F_k)^{r+1} = \big(F_1(F_1, \dots, F_k)^r, \dots, F_k(F_1, \dots, F_k)^r\big).$$

Clearly, k^m iterations of the permutation of the k^m elements yield the identical permutation; that is to say,

$$(F_1, \dots, F_k)^{k^m} = (I_1^{(k)}, \dots, I_k^{(k)}).$$

Since each component of each $(F_1, \dots, F_k)^r$ belongs to $\mathfrak{S}\boldsymbol{F}$ we thus have

LEMMA 1. *If $\boldsymbol{F}$ is a perfect set of* k*-place functions over* N_m, *then* $\mathfrak{S}\boldsymbol{F}$ *includes all* k*-place selectors*, $I_i^{(k)}$ $(1 \leqslant i \leqslant k)$.

EXAMPLE 3. Consider the triple (F, I_1, I_2) of 3-place functions over N_2, where

$$F(1,1,1) = F(1,2,2) = F(2,1,1) = F(2,2,1) = 2,$$

$$F(2,2,2) = F(1,2,1) = F(2,1,2) = F(1,1,2) = 1;$$

$$I_1(x, y, z) = x, \quad I_2(x, y, z) = y \quad \text{for each} \quad (x, y, z) \text{ in } N_2^3.$$

The set $\{F, I_1, I_2\}$ is easily seen to be perfect. (F, I_1, I_2) produces a cyclical permutation of the triples

$$(1,1,1),\ (2,1,1),\ (2,2,1),\ (2,2,2),\ (1,2,2),\ (2,1,2),\ (1,2,1),\ (1,1,2).$$

Hence $(F, I_1, I_2)^8 = (I_1, I_2, I_3)$, where $I_3(x, y, z) = z$. It follows that I_3 belongs to $\mathfrak{S}\boldsymbol{F}$. Indeed, $I_3 = I_2(F, I_1, I_2)^7 = I_1(F, I_1, I_2)^6 = F(F, I_1, I_2)^5$.

A classical theorem in Boolean algebra asserts that each "function of p variables $x_1, \dots, x_p$" over $\{0, 1\}$ can be represented in two (so-called *normal*) forms: as a sum of products and as a product of sums. The first half states, more precisely, that each $F(x_1, \dots, x_p)$, except the function assuming only the value 0, is, for some number k, where $1 \leqslant k \leqslant 2^p$, the sum of k products of the form $y_1 \cdot \ldots \cdot y_p$, where, for each $i = 1, \dots, p$, one has $y_i = x_i$ or $y_i = 1 - x_i$. This theorem can be expressed in terms of the functions n, A', and B, mentioned in Examples 1 and 2, and the p-place selectors $I_i^{(p)}$, which we shall denote, briefly, by $I_1, \dots, I_p$. It is convenient to set $n^1 = n$, $n^2 = nn = j$. Hence $n^k F = nF$ or $= F$ according as $k = 1$ or 2. If $(i_1, \dots, i_p)$ is an ordered p-tuple of numbers belonging to $\{1, 2\}$, then we define a p-place function

$$P_{i_1, \dots, i_p} = A'\Big(\dots\big(A'(A'(n^{i_1}I_1, n^{i_2}I_2), n^{i_3}I_3), \dots, n^{i_p}I_p\big)\Big).$$

Each such function corresponds to one of the products $y_1 \cdot \ldots \cdot y_p$ and obviously belongs to $\mathfrak{S}\{n, A', I_1, \dots, I_p\}$.

The classical theorem asserts that, for any p-place function F over $\{0, 1\}$ there exist k ordered p-tuples $(i_{h1}, \dots, i_{hp})$ where $1 \leqslant h \leqslant k$, for some k $(1 \leqslant k \leqslant 2^p)$ such that

$$F = B\Big(\dots\big(B(B(P_{i_{11}, \dots, i_{1p}}, P_{i_{21}, \dots, i_{2p}}), P_{i_{31}, \dots, i_{3p}}), \dots, P_{i_{k1}, \dots, i_{kp}}\big)\Big).$$

Hence F belongs to $\mathfrak{S}\{n, A', B, I_1, \dots, I_p\}$.

Post [5] generalized the theorem just mentioned to the set $\mathfrak{S}_m^p$ of all p-place functions over N_m. We assume N_m to be ordered according to $1 < 2 < \dots < m$, and define

$$A'(x, y) = \mathit{Min}(x, y), \quad B(x, y) = \mathit{Max}(x, y), \quad n(x) = x + 1$$
$$\text{for} \quad 1 \leqslant x \leqslant m-1 \text{ and } n(m) = 1.$$

We set $n^{k+1} = nn^k$ for $1 \leqslant k \leqslant m-1$. Clearly, $n^m = j$, where $jF = F$ for each F over N_m. The functions $P_{i_1, \dots, i_p}$ are defined as in the classical theorem, but for all ordered p-tuples $(i_1, \dots, i_p)$ of numbers $1, \dots, m$. Any p-place function F over N_m, except the constant p-place function of value 1, can be expressed, just as in the classical case, in terms of B and k functions $P_{i_{h1}, \dots, i_{hp}}$ $(1 \leqslant i \leqslant k)$ for some k such that $1 \leqslant k \leqslant m^p$.

We now prove

LEMMA 2. *Let* p, r, *and* m *be natural numbers* >1. *Then the set* $\mathfrak{S}_m^p$ *of all* p*-place functions over* N_m *is a subset of*

$$\mathfrak{S}^* = \mathfrak{S}\{N_r^*, A_r^*, B_r^*, I_1^{(p)}, \dots, I_p^{(p)}\}$$

where N_r^*, A_r^*, *and* B_r^* *are* r*-place functions defined as follows*:

$$N_r^* = nI_1^{(r)}, \quad A_r^* = A'(I_1^{(r)}, I_2^{(r)}), \quad B^* = B(I_1^{(r)}, I_2^{(r)}).$$

Let F be a function belonging to $\mathfrak{S}_m^p$. According to Post's Theorem, F belongs to $\mathfrak{S} = \mathfrak{S}\{n, A', B, I_1, \dots, I_p\}$, where I_i is an abbreviation for $I_i^{(p)}$. We prove that F belongs to $\mathfrak{S}^*$ by induction on the degree of F relative to the set $\mathfrak{S}$. If $\mathit{deg}F = 1$ then, being a p-place function, F is one of the selectors I_i and therefore belongs to $\mathfrak{S}^*$. Only if $p = 2$, also $\mathit{deg}A' = \mathit{deg}B = 1$; but in this case $A' = A_r^*(I_1, I_2, I_2, \dots, I_2)$ and $B = B_r^*(I_1, I_2, I_2, \dots, I_2)$. For $p > 2$, assume that all p-place functions of a degree $\leqslant n$ relative to $\mathfrak{S}$ belong to $\mathfrak{S}^*$, and let F be a p-place function of degree $n+1$. Clearly, F is either nK or $A'(K,L)$ or $B(K,L)$ for two functions K and L of a degree $\leqslant n$. But

$$nK = N_r^*(K, K, \ldots, K), \quad A'(K,L) = A_r^*(K, L, \ldots, L),$$
$$B(K,L) = B_r^*(K, L, \ldots, L).$$

In any case, F thus belongs to $\mathfrak{S}^*$.

An immediate consequence is

Lemma 3. *For any two natural numbers,* p *and* r, *if* $\mathfrak{S}\{F_1, \ldots, F_k\} = \mathfrak{S}_m^r$, *then* $\mathfrak{S}_m^p \subseteq \mathfrak{S}\{F_1, \ldots, F_k, I_1^{(p)}, \ldots, I_p^{(p)}\}$.

The set $\mathfrak{S}\{F, I_1, I_2\}$ in Example 3 contains, as has been shown, all three 3-place selectors. As one readily verifies, $N_3^* = F(I_1, I_1, I_1)$ and, if one sets $K = F(I_1, I_2, I_2)$ and $L = N_3^*(I_1, I_2, I_2)$, $M = N_3^*(I_2, I_1, I_1)$, then $A_3 = N_3^*(K, K, K)$ and $B_3 = F(L, M, M)$. Thus also N_3^*, A_3^*, and B_3^* belong to $\mathfrak{S}\{F, I_1, I_2\}$. From Lemma 2 it follows that this set is $\mathfrak{S}_2^3$. We thus have established the case $p = 3$ of

Lemma 4. *For each* $p > 2$, *there exist* p *functions generating* $\mathfrak{S}_2^p$.

Assume $p > 3$, and consider

$$\boldsymbol{G} = \{F(I_1, I_2, I_3), I_1, I_2, I_4, \ldots, I_p\},$$

where I_i is an abbreviation of $I_i^{(p)}$. As one easily verifies, $\boldsymbol{G}$ is perfect. Hence, by Lemma 1, $\mathfrak{S}\boldsymbol{G}$ includes all p-place selectors (also I_3). We further show that $\mathfrak{S}\boldsymbol{G}$ includes N_p^*, A_p^*, and B_p^*. Setting $F^* = F(I_1, I_2, I_3)$ one can verify that $N_p = F^*(I_1, I_1, \ldots, I_1)$. Setting $K = F^*(I_1, I_2, \ldots, I_2)$ $L = N^*(I_1, I_2, \ldots, I_2)$ and $M = N_p^*(I_2, I_1, \ldots, I_1)$, one furthermore verifies that

$$A_p^* = N_p^*(K, K, K, \ldots, K) \quad \text{and} \quad B_p^* = F^*(L, M, \ldots, M).$$

By Lemma 2, $\mathfrak{S}_2^p \subset \mathfrak{S}\boldsymbol{G}$, which completes the proof of Lemma 4.

Example 4. Consider the 2-place function F over N_3 defined by

$$F(1,1) = F(1,2) = F(1,3) = 2;$$
$$F(2,2) = F(3,1) = F(3,2) = 3;$$
$$F(3,3) = F(2,1) = F(2,3) = 1.$$

Martin [1] has proved that this function F in conjunction with I and J generates all 3^{3^3} 2-place functions over N_3. If we define G by $G(x,y) = F(y,x)$ for each (x,y) in N_3^2, then the set $\{F, G\}$ is easily seen to be perfect and, therefore, by Lemma 1, includes I and J. It follows that $\mathfrak{S}\{F, G\} = \mathfrak{S}_3^2$.

Example 5. For any $m > 3$, consider the 2-place function F defined by

$$F(i,i) = i+1 \quad \text{for} \quad 1 \leqslant i \leqslant m-1, \qquad F(m,m) = 1,$$
$$F(1,2) = F(1,4) = 2, \qquad F(2,3) = F(2,4) = 1,$$
$$F(x,y) = x \quad \text{for all other pairs } (x,y) \text{ in } N_m^2.$$

Set $F = F_1$ and $F(F_k, F_k) = F_{k+1}$, and define 1-place functions g_k over N_m for $1 \leqslant k \leqslant m$ by setting $g_k(i) = F_k(i,i)$ for $1 \leqslant i \leqslant m$. Since $g_1(i) \equiv i+1 \pmod{m}$ it is clear that these m functions g_k are the m cyclical permutations of $(1, 2, \ldots, m)$. From the definition of F, one further sees that $t = F(g_m, g_1)$ is the transposition interchanging 1 and 2, and that $h = F(g_m, g_2)$ has the values $h(1) = h(2) = 1$ and $h(k) = k$ for $3 \leqslant k \leqslant m$. It is well known that g_1, t, and h generate all m^m functions in $\mathfrak{S}_m^1$. Since the functions $T = F(F_m, F_1)$ and $H = F(F_m, F_2)$ belong to $\mathfrak{S}\{F\}$ and

$$F(i,i) = g_1(i), \quad T(i,i) = t(i), \quad H(i,i) = h(i) \quad \text{for} \quad 1 \leqslant i \leqslant m,$$

it is clear that, for each function u in $\mathfrak{S}_m^1$, the set $\mathfrak{S}\{F\}$ contains a function U such that $U(j,j) = u$; that is to say, $U(i,i) = u(i)$, for $1 \leqslant i \leqslant m$. Hence $\mathfrak{S}\{F\}$ includes m^m mutually different functions.

We now define G by setting $G(x,x) = F(x,x)$,

$$G(1,2) = 3, \quad G(2,3) = 2, \quad \text{and}$$
$$G(x,y) = y \quad \text{for all other } (x,y) \text{ in } N_m^2.$$

It is easy to verify that the set $\{F, G\}$ is perfect. Hence $\mathfrak{S}\{F, G\}$ includes the 2-place selectors, I and J.

Słupecki [6] proved, for every natural number $m > 2$, an important theorem which may be formulated as follows. If H is any 2-place function over N_m which, for no f in $\mathfrak{S}_m^1$ is equal to either fI or fJ, then

$$\mathfrak{S}_m^2 \subset \mathfrak{S}\{g_1, t, h, H, I, J\}.$$

In Example 5, $\mathfrak{S}\{F, G\}$ includes I and J and U for every u in $\mathfrak{S}_m^1$. By induction on the degree of 2-place functions relative to $\{g_1, t, h, F, I, J\}$ one sees that $\mathfrak{S}\{F, G\} = \mathfrak{S}_m^2$.

We abbreviate $I_i^{(p)}$ to I_i and define, for each $p > 2$,

$$F_p = F(I_1, I_2), \quad G_p = G(I_1, I_2),$$

where F and G are the functions studied in Example 5 if $m > 3$, and the functions in Example 4 if $m = 3$. In any case, the set

$$\boldsymbol{G} = \{F_p, G_p, I_3, \ldots, I_p\}$$

is readily seen to be perfect, whence $\mathfrak{S}\{F_p, G_p, I_2, \ldots, I_p\}$ includes all p-place functions. Now set

$$\boldsymbol{G}^* = \{F, G, I_1, \ldots, I_p\}.$$

By an inductive proof similar to that of Lemma 2, we see that the p-place functions in $\mathfrak{S}G$ and in $\mathfrak{S}G^*$ are the same. Hence $\mathfrak{S}G = \mathfrak{S}_m^p$.

We thus have the first half of

THEOREM II. *If* $m > 1$ *then, except for the case* $m = p = 2$, *the bounds given in Corollaries* 3 *and* 2 *of Theorem I are sharp*; that is to say, *there are* p *functions generating all* p*-place functions over* N_m; *and there exists a homogeneous set of* k *functions generating* m^{m^k} *functions*. More specifically, *there exists a homogeneous set of* p *functions including* $p-2$ *selectors and generating* $\mathfrak{S}_m^p$; *and there exists a homogeneous set of* k *functions including* $k-2$ *selectors and generating* m^{m^k} *functions*.

In order to obtain a homogeneous set of k functions generating the maximum number of functions, for any place-number $p > k$, consider a homogeneous set F of k functions, $F_1, \ldots, F_k$, generating $\mathfrak{S}_m^k$. (F may be so chosen as to include $k-2$ selectors.) For $p > k$, set $F_1^{(p)} = F_1(I_1^{(p)}, I_2^{(p)})$ and

$$G = \{F_1^{(p)}, \ldots, F_k^{(p)}\}.$$

The number of functions in $\mathfrak{S}F$ and in $\mathfrak{S}G$ is the same. Thus the k functions in G generate m^{m^k} functions.

Addition in the proofs. In Remark 1, a function F in G is not necessarily obtainable by substituting functions belonging to $\mathfrak{S}G$ into a function belonging to G. Cf. [2] p. 291 for an example of two functions F and G in the 3-valued logic such that $F(X, Y) = G(X, Y) = G \neq F$ for each X, Y in $\{F, G\}$.

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