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On relations between some algebraic and topological properties of lattices

by

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Dedicated to Professor A. D. Wallace on the occasion of his 60-th birthday

§ 1. Introduction. Let $\Gamma = (L, \cup, \cap, 0, 1)$ denote a distributive lattice. Γ is called *Brouwerian* (see [4]) if there is an operation $a-b$ (called *pseudo-difference*) such that

$$(a-b \subset c) \equiv [a \subset (b \cup c)].$$

We shall consider in this paper the following three algebraic (structural) properties of lattices:

1. The property of being *Wallman*, which means that:

(1) $(a \not\subset b) \Rightarrow$ *there is* d *such that* $(0 \neq d \subset a)(b \cap d = 0)$.

2. The *regularity* of Γ:

(2) $(a \not\subset b) \Rightarrow$ *there are* c *and* d *such that* $(c \cup d = 1)(a \not\subset c)(b \cap d = 0)$.

2. The *normality* of Γ:

(3) $(a \cap b = 0) \Rightarrow$ *there are* c *and* d *such that* $(c \cup d = 1)$ $(a \cap c = 0 = b \cap d)$

Remark. It is easy to see that assuming the lattice to be Brouwerian one can replace the formulas (2) and (3) by the following:

(2′) $(a \not\subset b) \Rightarrow$ *there is* d *such that* $b \cap d = 0$ *and* $a \not\subset 1-d$,

(3′) $(a \cap b = 0) \Rightarrow$ *there is* d *such that* $b \cap d = 0$ *and* $a \cap (1-d) = 0$.

The three above defined properties of Γ have algebraic aspect (they have been defined without introducing any topology in Γ). Nevertheless, they origin is topological. In fact, in order that the lattice 2^X of closed subsets of a topological space X be structurally regular (resp. normal) it is necessary and sufficient that the space X be regular (resp. normal) in the usual topological sense. If X is a $\mathcal{T}_1$-space, then 2^X is structurally Wallman (the converse is not true).

The exponential topology of a lattice that we are going to consider (and which in the case of the space 2^X is its Vietoris topology) is defined as follows (see [2] where an extensive list of references is given).

Denote by $I(a)$ and $J(a)$ the ideals:

(4) $$I(a) = \{x\colon x \subset a\} \quad \text{and} \quad J(a) = \{x\colon x \frown a = 0\}.$$

The exponential topology of L is the coarsest topology in which the ideals $I(a)$ are closed and $J(a)$ are open. In other words: the open base of L is composed of sets of the form:

(5) $$B(a_0, a_1, \ldots, a_n) = J(a_0) - I(a_1) - \ldots - I(a_n) = \{x\colon (x \frown a_0 = 0)(x \not\subset a_1) \ldots (x \not\subset a_n)\}.$$

It is worthy noticing that the following assumption can be made about the sets $B(a_0, a_1, \ldots, a_n)$:

(6) $$a_0 \subset a_i \quad \text{if} \quad 1 \leqslant i \leqslant n.$$

For, it is easy to see that

(7) $$B(a_0, a_1, \ldots, a_n) = B(a_0, a_0 \smile a_1, \ldots, a_0 \smile a_n)$$

(of course, if $n = 0$, we have $B(a_0) = J(a_0)$).

In §§ 3 and 4 we shall establish for Brouwer and Wallman lattices Γ equivalence between structural regularity, respectively structural normality of Γ, and its corresponding topological properties. Among others we shall show that the topological regularity of these lattices is equivalent to their complete regularity (this theorem and a number of other theorems here considered have been proved by Michael for the case $L = 2^X$, X being a topological space; see [5]).

§ 2. Basic properties of $I(a)$ and $J(a)$ in Brouwerian Wallman lattices. Let us start with the obvious statement: if $a \smile b = 1$, then $J(a) \subset I(b)$. As $a \smile (1-a) = 1$, it follows that

(i) $$J(a) \subset I(1-a),$$

(ii) $$J(1-a) \subset I(a).$$

We shall show that the formulas (i) and (ii) can be strengthened as follows:

(iii) $$I(1-a) = \overline{J(a)},$$

(iv) $$J(1-a) = \operatorname{Int} I(a).$$

In fact we shall establish the more general theorem:

Theorem. *Let Γ be a Brouwerian Wallman lattice. Suppose that condition* (6) *of* § 1 *is satisfied. Then we have*

(0) $$\overline{J(a_0) - I(a_1) - \ldots - I(a_n)} = I(1-a_0) - J(1-a_1) - \ldots - J(1-a_n).$$

Proof. For the sake of brevity, denote by $\bar{B}$ the first term of the above identity (B being defined by formula (5) of § 1) and by C its second term.

According to formulas (i), (ii), and to the fact that the ideals I are closed and J open, we have $\bar{B} \subset C$. We shall show that $C \subset \bar{B}$.

Let $p \in C \frown G$ and G open. We have to define an element q of L such that

(1) $$q \in G \frown B.$$

We may suppose of course that G belongs to an open base of L. Hence we may put (comp. § 1 (5)):

(2) $$G = J(b_0) - I(b_1) - \ldots - I(b_m),$$

(2') $$b_0 \subset b.$$

As $p \in C$, we have

(3) $$p \subset 1 - a_0,$$

(4) $$p \frown (1 - a_i) \neq 0 \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

and as $p \in G$, it follows that

(5) $$p \frown b_0 = 0,$$

(6) $$p \not\subset b_j \quad \text{for} \quad 1 \leqslant j \leqslant m.$$

Formulas (3) and (6) give

(7) $1 - a_0 \not\subset b_j$, hence (8) $a_0 \smile b_j \neq 1$,

since $x \smile y = 1 \Rightarrow 1 - y \subset x$.

(4) and (5) imply

(9) $1 - a_i \not\subset b_0$, hence (10) $a_i \smile b_0 \neq 1$.

The lattice Γ being Wallman, it follows from (8) and (10) that there are c_j and d_i such that

(11) $c_j \neq 0$, (12) $c_j \frown a_0 = 0$, (13) $c_j \frown b_j = 0$,

(14) $d_i \neq 0$, (15) $d_i \frown a_i = 0$, (16) $d_i \frown b_0 = 0$.

Put $q = (c_1 \smile \ldots \smile c_m) \smile (d_1 \smile \ldots \smile d_n)$. Formula (1) is fulfilled, i.e.,

(17) $q \frown b_0 = 0$, (18) $q \not\subset b_j$, (19) $q \frown a_0 = 0$, (20) $q \not\subset a_i$.

Indeed: (13), (2'), (16) $\Rightarrow$ (17); (13), (11) $\Rightarrow c_j \not\subset b_j \Rightarrow$ (18); (12), (15) $\Rightarrow$ (19) by virtue of § 1 (6); (14), (15) $\Rightarrow d_i \not\subset a_i \Rightarrow$ (20).

Remark 1. In order to derive (iii) from the preceding theorem, we put $n = 0$. (iv) is obtained by putting $n = 1$ and $a_0 = 0$. This implies the following

COROLLARY. *Under the assumptions of the theorem we have*

$$\overline{J(a_0)-I(a_1)-\ldots-I(a_n)} = \overline{J(a_0)} \cap \overline{-I(a_1)} \cap \ldots \cap \overline{-I(a_n)}.$$

Remark 2. Without assuming condition (6) of § 1, we have

$$\overline{J(a_0)-I(a_1)-\ldots-I(a_n)} = \overline{J(a_0)} \cap \overline{-I(a_0 \cup a_1)} \cap \ldots \cap \overline{-I(a_0 \cup a_n)}$$
$$= I(1-a_0)-J[1-(a_0 \cup a_1)]-\ldots-J[1-(a_0 \cup a_n)].$$

§ 3. Relations between structural regularity and the Hausdorff topology of a lattice.

THEOREM 1. *If Γ is structurally regular, then L is topologically Hausdorff.*

Proof. Let $a \neq b$, e.g. $a \not\subset b$. In virtue of the regularity of Γ, there are c and d such that $c \cup d = 1$, $a \not\subset c$ and $b \cap d = 0$. Put $U = -I(c)$ and $V = J(d)$. Hence U and V are open and contain a and b, respectively. Furthermore U and V are disjoint, since for every x we have $x = (x \cap c) \cup (x \cap d)$ and if $x \in V$, then $x \cap d = 0$, hence $x = x \cap c$, i.e. $x \subset c$, which means that $x \notin U$.

THEOREM 2. *Let Γ be a Wallman and Brouwer lattice and let the space L be Hausdorff. Then Γ is structurally regular.*

Proof. Let $a \not\subset b$. According to § 1 (2') we have to define c such that

(1) $$b \cap c = 0 \quad \text{and} \quad a \not\subset 1-c.$$

Γ being a Wallman structure, there is p such that

(2) $$p \cap b = 0 \quad \text{and} \quad 0 \neq p \subset a, \quad \text{hence} \quad b \neq b \cup p.$$

L being Hausdorff, the last inequality implies the existence of two disjoint open sets G and H such that $b \in G$ and $(b \cup p) \in H$. Obviously, one can assume that G and H belong to a base of L. In other terms, there are two systems $a_0, \ldots, a_m$ and $b_0, \ldots, b_n$ such that:

(3) $$b \cap a_0 = 0, \quad b \not\subset a_i \quad \text{for} \quad i = 1, \ldots, m,$$

(4) $$(b \cup p) \cap b_0 = 0, \quad (b \cup p) \not\subset b_j \quad \text{for} \quad j = 1, \ldots, n,$$

and there exists no x satisfying simultaneously the conditions:

(5) $$x \cap a_0 = 0, \quad x \not\subset a_i, \quad x \cap b_0 = 0, \quad x \not\subset b_j \quad (i = 1, \ldots, m;\ j = 1, \ldots, n).$$

Put $c = a_0$. In order to prove (1), it remains to show that $a \not\subset 1-c$. Suppose the contrary is true, i.e. $a \subset 1-a_0$, hence by (2)

(6) $$p \subset 1-a_0.$$

We shall define p_i, for $i = 1, \ldots, m$, and q_j, for $j = 1, \ldots, n$, so that:

(7) $$p_i \cap (b_0 \cup a_0) = 0, \quad p_i \not\subset a_i,$$

(8) $$q_j \cap (b_0 \cup a_0) = 0, \quad q_j \not\subset b_j;$$

and then we shall put $x = p_1 \cup \ldots \cup p_m \cup q_1 \cup \ldots \cup q_n$. One sees easily that x satisfies formulas (5), which means a contradiction. Thus the proof will be completed.

Now, we define p_i according to the Wallman condition applied to the formula $b \not\subset a_i \cup a_0$ (which is a consequence of (3)); this yields:

(9) $$p_i \cap (a_i \cup a_0) = 0 \quad \text{and} \quad 0 \neq p_i \subset b.$$

But by (4), $b \cap b_0 = 0$, hence the last inclusion gives $p_i \cap b_0 = 0$. Thus according to (9) p_i satisfies (7).

According to the second part of (4), there are for each j two possibilities:

1. either $b \not\subset b_j$, which yields $b \not\subset b_j \cup a_0$ (by (3)),

2. or $p \not\subset b_j$, therefore $p \not\subset b_j \cup b_0$ (by (4)) and then by (6) $1-a_0 \not\subset b_j \cup b_0$, which implies that $a_0 \cup b_j \cup b_0 \neq 1$.

By the Wallman condition there is $q_j \neq 0$ which satisfies either formulas

(10) $$q_j \cap (b_j \cup a_0) = 0 \quad \text{and} \quad q_j \subset b,$$

or

(11) $$q_j \cap (a_0 \cup b_j \cup b_0) = 0.$$

Formula (10) implies that $q_j \cap b_0 = 0$ since $b \cap b_0 = 0$ (by (4)). Thus in both cases, part one of (8) is satisfied. Part two is satisfied too since $q_j \cap b_j = 0$ and $q_j \neq 0$.

COROLLARY. *Let Γ be a Brouwer and Wallman lattice. Then the following conditions are equivalent:*

1. *Γ is structurally regular,*

2. *Γ is topologically Hausdorff.*

Let us recall that these conditions are equivalent to each of the following (see [3] and [1], p. 723):

3. *the set $\{(x, y): x-y = 0\}$ is closed,*

4. *the mapping $x-y$: $L \times L \to L$ is lower semi-continuous* [1].

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[1] The mapping f: $X \to L$ is *lower semi-continuous* if the set $\{x: f(x) \subset a\}$ is closed for each $a \in L$.

§ 4. Relations between structural normality and topological regularity of a lattice.

LEMMA. *Let Γ be a Brouwerian and structurally normal lattice. Let $a \frown (1-c) = 0$. Then there exists a mapping $f\colon R \to L$ (where R denotes the set of rational numbers of the form $k/2^n$, $k = 0, 1, \ldots, 2^n$) such that*

(1) $$f(0) = a,$$

(2) $$f(1) = c,$$

(3) $$r_0 < r_1 \Rightarrow f(r_0) \frown [1 - f(r_1)] = 0,$$

hence

(3′) $$r_0 < r_1 \Rightarrow f(r_0) \subset f(r_1), \quad \text{i.e.} \quad f \textit{ is isotonic.}$$

Proof. We proceed by induction. For $n = 0$, we define $f(0)$ and $f(1)$ according to (1) and (2). Then (3) is obviously satisfied. Let $n > 0$ and k odd. We may suppose that (3) is fulfilled for $n-1$. Hence

$$f\left(\frac{k-1}{2^n}\right) \frown \left[1 - f\left(\frac{k+1}{2^n}\right)\right] = 0.$$

By formula (3′) of § 1, there is an element of L—let us denote it by $f\left(\frac{k}{2^n}\right)$—such that

(4) $$f\left(\frac{k-1}{2^n}\right) \frown \left[1 - f\left(\frac{k}{2^n}\right)\right] = 0 = f\left(\frac{k}{2^n}\right) \frown \left[1 - f\left(\frac{k+1}{2^n}\right)\right].$$

By assumption f is an isotonic mapping for r's having 2^{n-1} as denominator. Hence (4) implies (3) for the denominator 2^n.

AUXILIARY THEOREM. *Let $f\colon R \to L$. Put $C(x) = f^{-1}[-J(x)] = \{r\colon f(r) \frown x \neq 0\}$ and $F(x) = \overline{C(x)}$ (= closure respectively to $\mathfrak{J}$).* [2] *Suppose that f satisfies* (1), (2) *and* (3′); *then the mapping $F\colon L \to 2^{\mathfrak{J}}$ is upper semi-continuous.*

Furthermore, *if f satisfies* (3), *F is continuous.*

Proof. 1. We have to show that, under the assumptions (1), (2) and (3′), if $\emptyset \neq A = \bar{A} \subset \mathfrak{J}$, then the set

$$F^{-1}[J(A)] = \{x\colon F(x) \frown A = \emptyset\}$$

is open. Put

(5) $$a = \sup A.$$

[2] $\mathfrak{J}$ denotes the closed interval (01). A mapping $F\colon X \to 2^Y$ is called *upper semi-continuous* if the set $\{x\colon F(x) \frown A = \emptyset\}$ is open for each A closed in Y.

Let us note that if $a = 1$ and $x \frown c \neq 0$, then $a \in F(x) \frown A$, i.e. $1 \in F(x)$ (since $f(1) \frown x \neq 0$ by (2)). If $x \frown c = 0$, then $C(x) = 0$. Thus (if $a = 1$):

$$[F(x) \frown A = \emptyset] \equiv (x \frown c = 0), \quad \text{i.e.} \quad F^{-1}[J(A)] = J(c)$$

and the latter set is open by definition.

Hence we may assume that $a < 1$. We shall show that

(6) $$F^{-1}[J(A)] = \bigcup_{r > a} J[f(r)],$$

what will complete the proof.

First, suppose that $x \in F^{-1}[J(A)]$, i.e. that $F(x) \frown A = \emptyset$. Since $a \in A$, it follows that $a \notin F(x)$. Consequently, there is an $r > a$ such that $r \notin C(x)$, i.e. $f(r) \frown x = 0$, or equivalently $x \in J[f(r)]$.

Next, suppose that for an $r_0 > a$, we have $x \in J[f(r_0)]$, i.e. $f(r_0) \frown x = 0$. It follows by (3′) that if $f(r_1) \frown x \neq 0$ (i.e. $r_1 \in C(x)$), then $r_1 > r_0$. In other terms, $C(x)$ is contained in the closed interval $(r_0, 1)$, and so is $F(x)$. As $r_0 > a$, it follows by (5) that $F(x) \frown A = \emptyset$, i.e. $x \in F^{-1}[J(A)]$.

This completes the proof of (6).

2. Suppose now that condition (3) is fulfilled. We have to show that F is continuous. It remains to show that F is lower semi-continuous, i.e. that the set $F^{-1}[I(A)]$ is closed for each $A = \bar{A} \subset \mathfrak{J}$. We shall prove indeed that

$$F^{-1}[I(A)] = \bigcap_{r \notin A} I[1 - f(r)],$$

what will complete the proof since the sets $I(x)$ are closed.

First, suppose that $x \in F^{-1}[I(A)]$, i.e. $F(x) \subset A$. Hence $C(x) \subset A$, which means that $[f(r) \frown x \neq 0] \Rightarrow r \in A$ for each $r \in R$. Otherwise stated:

$$r \notin A \Rightarrow [f(r) \frown x = 0] \Rightarrow x \subset 1 - f(r) \equiv x \in I[1 - f(r)].$$

Next, suppose that $x \notin F^{-1}[I(A)]$, i.e. $F(x) \not\subset A$. Hence there is $r_0 < 1$ such that $r_0 \in F(x) - A$. As $F(x) = \overline{C(x)}$, we may assume that $r_0 \in C(x)$. As $r_0 \notin A = \bar{A}$, there is $r_1 > r_0$ such that $r_1 \notin A$, and as $r_0 \in C(x)$, i.e. $f(r_0) \frown x \neq 0$, it follows by (3), that $x \not\subset 1 - f(r_1)$. Consequently $x \notin I[1 - f(r_1)]$.

COROLLARY 1. (GENERALIZED URYSOHN LEMMA.) *Let Γ be a Brouwerian and structurally normal lattice. Let $a \frown b = 0$ where $a \neq 0 \neq b$. Then there is a continuous mapping $\varphi\colon L \to \mathfrak{J}$ such that*

(7) $$\varphi(a) = 0 \quad \textit{and, more generally,} \quad a \frown x \neq 0 \Rightarrow \varphi(x) = 0,$$

(8) $$\varphi(b) = 1 \quad \textit{and, more generally,} \quad x \subset b \Rightarrow \varphi(x) = 1.$$

Proof. According to formula (3′) there is c such that

(9) $$a \frown (1 - c) = 0 \quad \text{and} \quad b \frown c = 0.$$

Hence by the lemma there is f satisfying conditions (1)-(3). Define F like in the Auxiliary Theorem and put

$$\varphi(x) = \inf F(x)$$

(assuming that $\inf \emptyset = 1$).

As $\inf: 2^{\mathfrak{J}} \to \mathfrak{J}$ is continuous ($\emptyset$ being isolated in $2^{\mathfrak{J}}$) and as $F: L \to 2^{\mathfrak{J}}$ is continuous by the Auxiliary Theorem, the composed mapping $\varphi: L \to \mathfrak{J}$ is also continuous.

In order to show (7), consider an x such that $\varphi(x) \neq 0$. Hence there is $r > 0$ such that $r \notin C(x)$, i.e. $f(r) \frown x = 0$. It follows by (3′) that $f(0) \frown x = 0$, i.e. $a \frown x = 0$ (by (1)). Thus (7) is fulfilled.

Next assume $x \subset b$. By (9), $c \frown x = 0$. It follows that $C(x) = \emptyset$. For suppose $r \in C(x)$, i.e. $f(r) \frown x \neq 0$; then by (3′) $f(1) \frown x \neq 0$, i.e. $c \frown x \neq 0$ (by (2)). The identity $C(x) = \emptyset$ yields $\varphi(x) = 1$, which completes the proof of (8).

THEOREM 1. *Let Γ be Brouwerian and Wallman. If Γ is structurally normal, then L is topologically completely regular.*

Proof. Let $a_0 \notin A$ where A is a non-void closed subset of L. We have to define a continuous function $\chi: L \to \mathfrak{J}$ such that

(10) $$\chi(a_0) = 0 \quad \text{and} \quad \chi(x) = 1 \quad \text{for} \quad x \in A.$$

If $a_0 = 0$, we put $\chi(0) = 0$ and $\chi(x) = 1$ for $x \neq 0$; χ is continuous since 0 is an isolated point of L. Thus we may assume that $a_0 \neq 0$. Put $\chi(0) = 1$. Hence we may assume that $0 \notin A$. Finally it may be assumed that A belongs to the closed base of L, i.e. that (cf. § 1 (5)) there exist $b_0, b_1, \ldots, b_n$ all different from 0 and such that

(11) $$(x \in A) \equiv (x \frown b_0 \neq 0) \quad \text{or} \quad (x \subset b_1) \quad \text{or} \ \ldots \ \text{or} \quad (x \subset b_n).$$

As $a_0 \notin A$, we have $a_0 \frown b_0 = 0$. By the corollary (where we replace a by b_0 and b by a_0), there is a continuous $\psi_0: L \to \mathfrak{J}$ such that

(12) $$\psi_0(a_0) = 1, \quad b_0 \frown x \neq 0 \Rightarrow \psi_0(x) = 0.$$

Since $a_0 \not\subset b_i$ for $i = 1, \ldots, n$, there is an a_i (Γ being Wallman) such that $0 \neq a_i \subset a_0$ and $a_i \frown b_i = 0$. According to the Corollary, there is a continuous $\varphi_i: L \to \mathfrak{J}$ such that:

(13) $$a_i \frown x \neq 0 \Rightarrow \varphi_i(x) = 0, \quad \text{hence} \quad \varphi_i(a_0) = 0,$$

(14) $$x \subset b_i \Rightarrow \varphi_i(x) = 1.$$

Put

(15) $$\chi(x) = \max[1 - \psi_0(x), \varphi_1(x), \ldots, \varphi_n(x)].$$

Obviously χ is continuous. Then $\chi(a_0) = 0$ by (12) and (13). Furthermore, if $x \frown b_0 \neq 0$, we have $\chi(x) = 1$ by (12), and if $x \subset b_i$, we have $\chi(x) = 1$ by (14).

It follows by (11) that (10) is satisfied.

Conversely, the following is true.

THEOREM 2. *Under the same assumptions, if L is topologically regular, Γ is structurally normal.*

Proof. Let $a \frown b = 0$, i.e. $a \in J(b)$. As $J(b)$ is open, there is by virtue of the regularity of L an open G such that

(16) $$a \in G \quad \text{and} \quad \overline{G} \subset J(b).$$

We may assume that G belongs to a base of L. Hence we may put (see § 1):

$$G = J(a_0) - I(a_1) - \ldots - I(a_n), \quad \text{where} \quad a_0 \subset a_i.$$

As $a \in G$, it follows that:

(17) $$a \frown a_0 = 0,$$

(18) $$a \not\subset a_i \quad \text{for} \quad 1 \leqslant i \leqslant n,$$

and as $\overline{G} \subset J(b)$ we have by § 2 (0):

(19) $$I(1 - a_0) - J(1 - a_1) - \ldots - J(1 - a_n) \subset J(b).$$

In view of (17) and of § 1 (3′), it remains to be shown that $(1 - a_0) \frown b = 0$, i.e. that $(1 - a_0) \in J(b)$, or that $1 - a_0$ belongs to the left member of (19), which means that $(1 - a_0) \frown (1 - a_i) \neq 0$ for $i = 1, \ldots, n$.

Now, this follows from (17) and (18). For (18) implies $a \frown (1 - a_i) \neq 0$, and by (17), $a = a \frown (1 - a_0)$.

COROLLARY 2. *For Brouwerian and Wallman lattices the conditions of topological regularity and of topological complete regularity are equivalent.*

Finally, let us recall that the structural normality of the lattice Γ can be characterized also by each of the two conditions (see [2], p. 16):

1. *the set $\{(x, y): x \frown y = 0\}$ is open,*
2. *the mapping $x \frown y: L \times L \to L$ is upper semi-continuous.*

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Two theorems on the generation of systems of functions

by

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This paper deals with two basic questions about multiplace functions ("functions of several variables") defined on a finite set $N_m = \{1, \ldots, m\}$. How many functions can k functions generate by composition, and how many functions are needed to generate by composition all p-place functions?

The essential feature of the paper is its algebraic approach to the subject matter in contrast to the traditional treatment of functions in logic (1). Consider e.g. the functions over N_2. By composition, the two basic logical functions, negation and disjunction, do not generate more than eight functions, namely, the four 1-place functions, four of the sixteen 2-place functions and none of the higherplace functions (see Example 2). All that Sheffer's stroke (herein denoted by a frontal A) generates are four of the 2-place functions. The traditional statement that $A(x, y)$ also generates e.g. the 1-place negation $n(x)$ is based on the fact that $n(x) = A(x, x)$. But in so saying one substitutes x for y; and similarly one substitutes $A(y, z)$ for y in saying that $A(x, y)$ generates $A\big(x, A(y, z)\big)$. Substitution of an expression for a variable, however, is not the composition of functions. Nor is it possible to obtain any 1-place of 3-place function from A by compositions.

From our strictly algebraic point of view, we prove that the maximum number of functions that k functions can generate depends upon k but (except for trivial limitations) is independent of the place-numbers of the functions (Corollary 2 of Theorem I). At least p functions are necessary (Corollary 3 of Theorem I), and p properly chosen functions are sufficient (Theorem II), to generate all p-place functions for $p > 1$ with one important exception: the 2-place functions over N_2. Thus while three functions are needed to generate all the 2-place func-

* Theorem I and its Corollaries are due to the first author, Theorem II is the work of the second.

(1) Another algebraic approach to the study of multiplace functions is the Marczewski abstract algebra which, however, stresses the domains of the functions rather than their composition.