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Inductive compactness as a generalization of semicompactness *

by

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1. Introduction. All spaces under discussion are separable metrizable spaces.

In [14] L. Zippin introduced the notion of a semicompact space. Namely, a space is *semicompact* if each point in the space has arbitrarily small neighborhoods with compact boundaries. Spaces which are semicompact are sometimes called *rim compact* or *peripherally compact*. In 1942, J. de Groot proved that a space is semicompact if and only if it can be compactified by adding a set of dimension no higher than zero [2]. This result is implicitly contained in H. Freudenthal's paper [3]. Freudenthal generalizes his results in [4]. For further generalizations, see K. Morita [10] and P. S. Aleksandrov and V. I. Ponomarëv [1]. The notion of semicompact space and the above characterization of such spaces suggest some other concepts. One concept is that of an "inductive compactness" analogous to inductive dimension. Another concept is the compactification of a space by adding a set of minimal dimension. Let us formalize these two concepts.

DEFINITION. INDUCTIVE COMPACTNESS. A space X is said to have *compactness* -1 if X is compact. A space X is said to have *compactness less than or equal to* n $(n \geqslant 0)$ if each point of X has arbitrarily small neighborhoods whose boundaries have compactness less than or equal to $n-1$. We use the notation $\operatorname{cmp} X \leqslant n$. A space X is said to have *compactness equal to* ∞ if $\operatorname{cmp} X \leqslant n$ is false for each integer n.

DEFINITION. A compact space Y is called an *n-compactification* of a space X if X is dense in Y and $\dim(Y \setminus X) = n$. By the *deficiency*

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of a space X, we mean the least integer n such that X has an n-compactification. We denote this integer by $\operatorname{def} X$. (Of course, we allow n to be ∞.)

Now, we have the following:

MAIN THEOREM. *Let $n = -1$ or 0. Then* $\operatorname{cmp} X \leqslant n$ *if and only if* $\operatorname{def} X \leqslant n$.

Hence, the main theorem gives an internal necessary and sufficient condition on a space X so that X can be compactified by adding a set of dimension at most n, when $n = -1$ or 0.

In this paper we investigate the following three questions:

1. Can the above theorem be proved for all $n \geqslant -1$? That is, does $\operatorname{cmp} X$ give rise to an internal characterization of $\operatorname{def} X$?

2. What properties do $\operatorname{cmp} X$ and $\operatorname{def} X$ possess?

3. What similarities exist between inductive compactness and inductive dimension?

These questions are not answered completely. Special cases of question 1 are established in section 2. Since question 1 is not settled, question 2 is of interest and we establish some properties of $\operatorname{cmp} X$ and $\operatorname{def} X$ which tend to support an affirmative answer to question 1 in section 4. Section 3 concerns question 3. Finally, regarding question 1, we mention that it has been posed in the book by J. Isbell [6] as a research problem.

Throughout the remainder of this paper, we will use A^0, $\bar{A}$ and A^* for the interior, closure and boundary of a subset A of a space X, respectively.

2. Extension of the main theorem.

In this section we give the status of the extension of the main theorem established for semicompact spaces. To this end, we have

THEOREM. $\operatorname{def} X = n$ *if and only if* $\operatorname{cmp} X = n$ *provided one of the following conditions hold*:

(a) $n \leqslant 0$;

(b) $\dim X = \operatorname{cmp} X$;

(c) $\operatorname{def} X \leqslant 1$;

(d) $\dim X \leqslant 1$;

(e) *X is a subset of a two-dimensional manifold;*

(f) *X is an extremely disconnected space; i.e., every quasi-component of X is a point.*

Part (a) is the main theorem mentioned in the introduction. We will prove each of the remaining parts separately.

2.1. Proofs of parts (b), (c), (d), (e) and (f).

Let us first prove a fundamental inequality.

THEOREM 2.1.1. $\operatorname{cmp} X \leqslant \operatorname{def} X \leqslant \dim X$.

Proof. Suppose $\dim X = n$. Then by [5], Theorem V 6, there is a compact space Y such that $\dim Y = n$ and X is dense in Y. Hence $\dim(Y\backslash X) \leqslant n$. Consequently, $\operatorname{def} X \leqslant \dim X$.

We prove by induction that $\operatorname{def} X \leqslant n$ implies $\operatorname{cmp} X \leqslant n$. The implication certainly holds for $n = -1$. Assume the induction proposition holds for all integers k, $-1 \leqslant k < n$, and let $\operatorname{def} X \leqslant n$ $(-1 < n < \infty)$. Let Y be a compactification of X with $\dim(Y\backslash X) \leqslant n$. Suppose that $x \in X$ and U is any neighborhood of x in Y. Then by [5], B), page 34, there is a neighborhood V of x in Y such that $V \subset U$ and the boundary of V in Y meets $Y\backslash X$ in a set of dimension $\leqslant n-1$. Therefore the boundary of $V \cap X$ in X has an m-compactification, $m \leqslant n-1$. By the induction hypothesis, we conclude that each point of X has arbitrarily small neighborhoods whose boundaries have inductive compactness less than or equal to $n-1$. Hence $\operatorname{cmp} X \leqslant n$. Thus for all finite n we have $\operatorname{def} X \leqslant n$ implies $\operatorname{cmp} X \leqslant n$. If $n = \infty$ then trivially $\operatorname{cmp} X \leqslant \infty$. The proof of the theorem is now complete.

Remark 2.1.2. One might wonder if each space X has an n-compactification Y so that $\dim X = \dim Y$ and $n = \operatorname{def} X$. The answer to this question is in the negative. (See [11] and [6], p. 118.)

THEOREM 2.1.3. *If* $\operatorname{cmp} X = \dim X$, *then* $\operatorname{cmp} X = \operatorname{def} X$.

THEOREM 2.1.4. *If* $\operatorname{def} X \leqslant 1$ *then* $\operatorname{cmp} X = \operatorname{def} X$.

Proof. If $\operatorname{cmp} X \leqslant 0$ then by part (a), $\operatorname{cmp} X = \operatorname{def} X$. If $\operatorname{cmp} X \geqslant 1$ then $1 \leqslant \operatorname{cmp} X \leqslant \operatorname{def} X \leqslant 1$.

THEOREM 2.1.5. *If* $\dim X \leqslant 1$ *then* $\operatorname{cmp} X = \operatorname{def} X$.

THEOREM 2.1.6. *If X is a subset of a two-dimensional manifold then* $\operatorname{cmp} X = \operatorname{def} X$.

Proof. The closure M of X in the manifold is locally compact and hence M can be compactified by the addition of at most one point. Clearly, the dimension of what has been added to X to compactify it does not exceed one. Hence, by part (c), we have $\operatorname{cmp} X = \operatorname{def} X$.

THEOREM 2.1.7. *If X is extremely disconnected and* $1 \leqslant \dim X$, *then* $\operatorname{cmp} X = \dim X$.

Proof. We prove inductively the following proposition: *If X is extremely disconnected and* $\dim X \geqslant n$, *then* $\operatorname{cmp} X \geqslant n$ $(n \geqslant 1)$.

In [8] A. Lelek has proved that if every quasi-component Q of a semicompact space X is locally compact and has the $\dim Q \leqslant 0$, then $\dim X \leqslant 0$. Consequently, our proposition above is true for $n = 1$. Suppose that the proposition is true for all integers k, $1 \leqslant k < n$. Let X be an extremely disconnected space with $\dim X \geqslant n$. Then there is a point $x \in X$ and a neighborhood U of x so that every neighborhood V of x contained in U has $\dim V^* \geqslant n-1$. Since V^* is extremely disconnected, we have $\operatorname{cmp} V^* \geqslant n-1$. Hence $\operatorname{cmp} X \geqslant n$. Thus the induction is completed.

The theorem is now easily proved.

THEOREM 2.1.8. *If X is extremely disconnected, then* $\operatorname{cmp} X = \operatorname{def} X$.

Proof. If $\dim X \leqslant 1$ then part (c) implies $\operatorname{cmp} X = \operatorname{def} X$. If $1 \leqslant \dim X$ then the preceding theorems implies $\operatorname{cmp} X = \operatorname{def} X$.

The proof of the main theorem of section 2 is now complete. For further examples for which $\operatorname{cmp} X = \operatorname{def} X$, see succeeding sections.

Remark 2.1.9. The idea of requiring that a point has arbitrarily small neighborhoods whose boundaries possess a certain property has been used many times. For example, dimension theory, semicompactness, regular curves and rational curves ([13], p. 82). As we have already mentioned in the introduction, the notion of semicompact is sometimes referred to as peripherally compact or rim compact. The latter two names are very descriptive. Using the descriptive word "peripheral', we might say instead of $\dim X \leqslant n$, peripheral dimension less than or equal to $n-1$; or instead of regular curve, we might say peripherally finite curve; or instead of rational curve, we might say peripherally countable curve. With these examples in mind, we give the following

DEFINITION. The *peripheral deficiency* of a space X is $\leqslant n$ (notation: $\text{p-def} X \leqslant n$) if each point of X has arbitrarily small neighborhoods whose boundaries have deficiency $\leqslant n-1$.

It is clear that $\text{p-def} X = 0$ if and only if $\operatorname{cmp} X \leqslant 0$. Theorem 2.1.1 and [5] B), p. 34, imply

THEOREM 2.1.10. $\operatorname{cmp} X \leqslant \text{p-def} X \leqslant \operatorname{def} X$, *when* $\operatorname{def} X \geqslant 0$.

We conjecture that the reverse inequalities hold.

3. Inductive compactness.

In this section we examine the similarities and differences between inductive dimension and inductive compactness. Most of the remarks, theorems and examples are motivated by known facts about dimension.

3.1. Existence of spaces X with $\operatorname{cmp} X = n$.

We first consider the problem of finding spaces whose inductive compactness is n. In [9] Mazurkiewicz exhibits extremely disconnected spaces of every dimension. Hence, by theorem 2.1.8, we have

THEOREM 3.1.1. *There are spaces with inductive compactness n for every $n \geqslant -1$.*

We next investigate the values $\operatorname{cmp} X$ taken for subsets X of Euclidean n-space E^n. A space X is called *totally imperfect* if it contains no uncountable compact subsets. We first prove

LEMMA 3.1.2. *If a compact, n-dimensional space X ($n < \infty$) is a union of two disjoint totally imperfect sets X_1 and X_2, then $\operatorname{cmp} X_1$ is n or $n-1$.*

Proof. We prove inductively the following proposition: *If a compact space X is the union of two disjoint totally imperfect sets X_1 and X_2 and $\infty > \dim X \geqslant n$ then $\operatorname{cmp} X_1 \geqslant n-1$.*

If $n \leqslant 1$, then the proposition is obvious. Suppose that the proposition is true for all integers k, $-1 \leqslant k < n$, and let X be the union of two disjoint totally imperfect sets X_1 and X_2 with $n \leqslant \dim X < \infty$. Since X is compact, by [5] corollary, p. 95, there is a point $x \in X_1$ and a neighborhood U of x such that, for any neighborhood V of x with $V \subset U$, we have $\dim V^* \geqslant n-1$. Clearly, V^* is compact and $V^* \cap X_1$ and $V^* \cap X_2$ are disjoint totally imperfect sets whose union is V^*. Hence $\operatorname{cmp}(V^* \cap X_1) \geqslant n-2$. Therefore $\operatorname{cmp} X_1 \geqslant n-1$ (see theorem 3.2.2). The induction is now complete.

The inequalities $n = \dim X \geqslant \dim X_1 \geqslant \operatorname{cmp} X_1 \geqslant n-1$ establish the lemma.

Let $S^n = \{x \in E^{n+1} | \|x\| = 1\}$. It is known that S^n can be written as the union of two totally imperfect sets X_1 and X_2. Hence $\operatorname{cmp} X_1 \geqslant n-1$. The following lemma is easily proved. (See theorem 2.1.6.)

LEMMA 3.1.3. *If X is a subset of an n-dimensional manifold then* $\operatorname{def} X \leqslant n-1$.

Consequently, $\operatorname{cmp} X_1 = n-1$. Since the one-point compactification of E^n is homeomorphic to S^n, we have that E^n contains a set X so that $\operatorname{cmp} X = n-1$. Now the following theorem is easily established.

THEOREM 3.1.4. *For each k ($-1 \leqslant k \leqslant n-1$) there is a subset X of E^n with $\operatorname{cmp} X = k$. There is no subset X of E^n with $\operatorname{cmp} X \geqslant n$.*

The examples which have been found so far are quite pathological. Let us give a few examples which are not so pathological. By the open ball B in E^3 we mean the set $B = \{x \in E^3 | \|x\| < 1\}$.

EXAMPLE 3.1.5. Suppose that X is the open ball B with an equator of rational points added on the surface of the ball. Then $\operatorname{cmp} X = 1 = \operatorname{def} X$.

EXAMPLE 3.1.6. Suppose that X is the open ball B with an open "disc" D on the surface and a countable dense set F on the edge of D. Then $\operatorname{cmp} X = 1 = \operatorname{def} X$. (Note that the space in example 3.1.5 is a "doubling" of this example about the set $F \cup D$.)

That $\operatorname{cmp} X \geqslant 1$ for either example is easily proved from theorem 3.2.1 and example 3.3.1. A relatively complicated construction gives a one-compactification for each space. For instance, in example 3.1.6, one first shows X is homeomorphic to a dense subset Z of $B \cup D \cup C$ where C is a "Cantor set" on the edge of D. Then it is a matter of forming an upper-semicontinuous decomposition of the closure of Z in E^3 so that the quotient topology on the decomposition yields a one-compactification. One can easily construct the one-compactification Y so that $Y\backslash X$ is a dense subset of a dendrite. For further examples, see the examples in the succeeding sections.

3.2. Monotone property. Since every subset of a space of dimension -1 is a space of dimension -1, it is easily proved that dimension is a monotone function. That is, if $A \subset X$ then $\dim A \leqslant \dim X$. It is easy to find spaces with inductive compactness -1 whose subspaces are not necessarily of inductive compactness -1. Hence a general monotone property is not possible. But it is true that closed subspaces of a space with inductive compactness -1 have inductive compactness -1. Hence we have the following theorem.

THEOREM 3.2.1. *If A is a closed subspace of X then* $\operatorname{cmp} A \leqslant \operatorname{cmp} X$.

Proof. We prove by induction the following proposition: *If A is closed in X and* $\operatorname{cmp} X \leqslant n$ *then* $\operatorname{cmp} A \leqslant n$. If $n = -1$ and A is closed in X then obviously $\operatorname{cmp} A \leqslant -1$. Suppose for all integers k, $-1 \leqslant k < n$, that the induction proposition is true. Suppose that $\operatorname{cmp} X \leqslant n$ and A is closed in X. Let $x \in A$ and U be any neighborhood in X of x. Then there is a neighborhood V of x in X so that $\operatorname{cmp} V^* \leqslant n-1$ and $V \subset U$. Let B be the boundary in A of $V \cap A$. Since A is closed in X, B is closed in V^*. Hence $\operatorname{cmp} B \leqslant n-1$. That is $\operatorname{cmp} A \leqslant n$. This concludes the induction. If $n = \infty$, the proposition is obvious. Thus the theorem is proved.

We next prove a theorem which will be useful later. (See [5] A), p. 27, for the dimension analogue.)

THEOREM 3.2.2. *A subspace X' of a space X has inductive compactness $\leqslant n$ if and only if every point of X' has arbitrarily small neighborhoods U in X so that* $\operatorname{cmp}(U^* \cap X') \leqslant n-1$ $(n \geqslant 0)$.

Proof. Suppose that $\operatorname{cmp} X' \leqslant n$, $x \in X'$ and V is a neighborhood in X of x. Then there is a neighborhood W in X' of x so that the boundary B of W in X' has $\operatorname{cmp} B \leqslant n-1$ and $W \subset V$. Since X is completely normal, there is a neighborhood U in X of x so that $U \subset V$ and $B \supset U^* \cap X'$. $U^* \cap X'$ and B are closed in X'. Hence $\operatorname{cmp}(U^* \cap X') \leqslant n-1$, by theorem 3.2.1.

Conversely, suppose that every point of X' has arbitrarily small neighborhoods U in X so that $\operatorname{cmp}(U^* \cap X') \leqslant n-1$. Let $x \in X'$ and V' be any neighborhood of x in X'. Then there is a neighborhood V in X so that $V' = V \cap X'$. Hence there is a neighborhood U of x in X so that $U \subset V$ and $\operatorname{cmp}(U^* \cap X') \leqslant n-1$. Let $U' = U \cap X'$. Then $U' \subset V'$. Let B' be the boundary of U' in X'. Then $B' \subset U^* \cap X'$, B' is closed in X' and $U^* \cap X'$ is closed in X'. Hence, by theorem 3.2.1, $\operatorname{cmp} B' \leqslant \operatorname{cmp}(U^* \cap X') \leqslant n-1$. Consequently, $\operatorname{cmp} X' \leqslant n$. The theorem is now proved.

The next theorem is obvious.

THEOREM 3.2.3. *If A is open in X and* $\operatorname{cmp} X \geqslant 0$ *then* $\operatorname{cmp} A \leqslant \operatorname{cmp} X$.

3.3. Sum theorem. In dimension theory, there are two sum theorems. Namely,

(A) *If* $X = A \cup B$ *then* $\dim X \leqslant \dim A + \dim B + 1$.

(B) *If* $X = \bigcup_{i=1}^{\infty} A_i$ *where each A_i is closed in X and* $\dim A_i \leqslant n$, *then* $\dim X \leqslant n$. (See [5].)

We will discuss inductive compactness with these two theorems in mind.

Let us consider theorem (A) first. One might hope to replace dimension by inductive compactness. But this cannot be done as the following example shows.

EXAMPLE 3.3.1. Let $D = \{x \in E^2 \mid \|x\| < 1\}$, p be a point in the boundary of D in the plane and $X = D \cup \{p\}$. Then it is easily shown that $\operatorname{cmp} X = 1 = \operatorname{def} X$. Clearly, $\operatorname{cmp} D = 0$ and $\operatorname{cmp}\{p\} = -1$. Hence $\operatorname{cmp} X > \operatorname{cmp} D + \operatorname{cmp}\{p\} + 1$.

In fact, from theorem 3.2.3, we see that inductive compactness is not lowered by adding a closed set to a space unless the space is locally compact. Hence by adding a compact set to a space one cannot lower inductive compactness by more than one and this can occur only when the space is locally compact.

From the above discussion we find that the following theorem is the best sum theorem analogous to theorem (A).

THEOREM 3.3.2. *If* $X = A \cup B$ *then* $\operatorname{cmp} X \leqslant \operatorname{cmp} A + \dim B + 1$.

Proof. We perform an induction on $\dim B$. If $\dim B = -1$, then the inequality is valid for all A. Suppose that the inequality is valid for all $X' = A' \cup B'$ with $\dim B' < n$. Let $X = A \cup B$ and $\dim B = n$. By [5] B), p. 34, each point $x \in X$ has arbitrarily small neighborhood U so that $\dim(U^* \cap B) \leqslant n-1$. Since $U^* \cap A$ is closed in A, theorem 3.2.1 implies $\operatorname{cmp}(U^* \cap A) \leqslant \operatorname{cmp} A$. Hence $\operatorname{cmp} U^* \leqslant \operatorname{cmp} A + \dim B$. Therefore $\operatorname{cmp} X \leqslant \operatorname{cmp} A + \dim B + 1$, and the induction is completed.

If $\dim B = \infty$ then the inequality is obvious. The theorem is now proved.

It would be interesting to know what additional conditions on A and B will make the analogue of theorem (A) true. For an answer to this question, see theorem 3.3.4 below.

Let us next consider theorem (B). Clearly, the inductive compactness analogue does not hold as a countably infinite space exhibits for $n=-1$. Hence one might hope to prove a finite sum theorem. But, this is not the case as the following example shows.

EXAMPLE 3.3.3. Let X be the same as in example 3.3.1. Let

$$A = \{x \in D \mid 4(2n)^{-1} \geqslant \|x-p\| \geqslant 4(2n+1)^{-1}, \text{ for some } n \geqslant 1\} \cup \{p\},$$

$$B = \{x \in D \mid 4(2n+1)^{-1} \geqslant \|x-p\| \geqslant 4(2n+2)^{-1}, \text{ for some } n \geqslant 1\} \cup \{p\}.$$

It is easy to show $\operatorname{cmp} A = \operatorname{cmp} B = 0$ and A and B are closed. Hence a finite sum theorem does not hold.

The next theorem gives a positive result. (Note the similarity to theorem (A).)

THEOREM 3.3.4. *If $X = A \cup B$ and A and B are closed in X, then* $\operatorname{cmp} X \leqslant \operatorname{cmp} A + \operatorname{cmp} B + 1$.

Proof. The proof is by induction on $\operatorname{cmp} A$ and $\operatorname{cmp} B$. If $\operatorname{cmp} A = -1 = \operatorname{cmp} B$ then the inequality is valid. Suppose that the inequality is valid if $\operatorname{cmp} A = -1$ and $\operatorname{cmp} B' < n$. Let $\operatorname{cmp} B = n$. If $x \in A \setminus B$ then x has arbitrarily small neighborhoods U such that $\operatorname{cmp} U^* = -1 \leqslant \operatorname{cmp} A + \operatorname{cmp} B$, since B is closed. If $x \in B$ then, by theorem 3.2.2, there are arbitrarily small neighborhoods U of x so that $\operatorname{cmp}(U^* \cap B) \leqslant n-1$. Clearly, $\operatorname{cmp}(U^* \cap A) = -1 = \operatorname{cmp} A$. Therefore $\operatorname{cmp} U^* \leqslant \operatorname{cmp} A + \operatorname{cmp} B$. Hence, if $x \in X$, there are arbitrarily small neighborhoods of x so that $\operatorname{cmp} U^* \leqslant \operatorname{cmp} A + \operatorname{cmp} B$. That is, $\operatorname{cmp} X \leqslant \operatorname{cmp} A + \operatorname{cmp} B + 1$. Finally, suppose that the inequality is valid for $\operatorname{cmp} A' \leqslant n$ and $\operatorname{cmp} B' < m$ or $\operatorname{cmp} A' < n$ and $\operatorname{cmp} B' \leqslant m$ and let $\operatorname{cmp} A = n$ and $\operatorname{cmp} B = m$. If $x \in A$ then, by theorem 3.2.2, there are arbitrarily small neighborhoods U of x such that $\operatorname{cmp}(U^* \cap A) \leqslant n-1$. Since U^* is closed, $\operatorname{cmp}(U^* \cap B) \leqslant m$. Hence $\operatorname{cmp} U^* \leqslant \operatorname{cmp} A + \operatorname{cmp} B$. By symmetry, every point of B has arbitrarily small neighborhoods U such that $\operatorname{cmp} U^* \leqslant \operatorname{cmp} A + \operatorname{cmp} B$. Hence $\operatorname{cmp} X \leqslant \operatorname{cmp} A + \operatorname{cmp} B + 1$. Thus the induction is completed.

If $\operatorname{cmp} A = \infty$ or $\operatorname{cmp} B = \infty$, then the inequality is obvious. Thus the theorem is proved.

Example 3.3.3 shows that theorem 3.3.4 is best possible. We have the following obvious corollary.

COROLLARY 3.3.5. *Let $X = \bigcup_{i=0}^{n} X_i$, each X_i be closed in X and $\operatorname{cmp} X_i = 0$. Then $\operatorname{cmp} X \leqslant n$.*

In dimension theory, theorems (A) and (B) give rise to the decomposition theorem. Namely, $\dim X \leqslant n$ if and only if X is the union of $n+1$ sets of dimension $\leqslant 0$. Corollary 3.3.5 might lead one to believe

that such a decomposition theorem exists for inductive compactness. That is, $\operatorname{cmp} X \leqslant n$ if and only if X is the union of $n+1$ closed subsets of inductive compactness $\leqslant 0$. This is not true as the following example shows.

EXAMPLE 3.3.6. Let Q be the set of rational numbers and R be the set of real numbers. Then $X = Q \times R$ has inductive compactness one. If $X = A \cup B$, where A and B are closed, then either $\operatorname{cmp} A$ or $\operatorname{cmp} B$ is one. Hence a decomposition theorem does not hold.

Next, we find a sufficient condition for the analogue of theorem (B) to hold. We first give two lemmas, the first of which is obvious.

LEMMA 3.3.7. *If A and B are separated then* $\operatorname{cmp}(A \cup B) = \max\{\operatorname{cmp} A, \operatorname{cmp} B\}$.

LEMMA 3.3.8. *If A is compact and B is closed in $X = A \cup B$, then* $\operatorname{cmp} X = \operatorname{cmp} B$.

Proof. Since B is closed in X, $\operatorname{cmp} X \geqslant \operatorname{cmp} B$. The reverse inequality follows from theorem 3.3.4.

Since a finite sum theorem holds for inductive compactness -1, we take the added condition that the intersection of two distinct sets in the sum be compact. Then we have

THEOREM 3.3.9. *Suppose $X = A \cup B$ where A and B are closed and $A \cap B$ is compact. If $\operatorname{cmp} A \leqslant n$ and $\operatorname{cmp} B \leqslant n$ then $\operatorname{cmp} X \leqslant n$.*

Proof. Since A and B are closed and $\operatorname{cmp} A \leqslant n$ and $\operatorname{cmp} B \leqslant n$, we have, by lemma 3.3.7, that $\operatorname{cmp}(X \setminus A \cap B) \leqslant n$ and $X \setminus A \cap B$ is open. Let $x \in A \cap B$ and U be a neighborhood of x. Then by theorem 3.2.2, there is a neighborhood V_1 of x such that $\operatorname{cmp}(V_1^* \cap A) \leqslant n-1$ and $V_1 \subset U$. Applying the same theorem again, we find a neighborhood V_2 of x such that $\overline{V}_2 \subset V_1$ and $\operatorname{cmp}(V_2^* \cap B) \leqslant n-1$. Let $W = V_2 \cup (V_1 \setminus B)$. Clearly, W is a neighborhood of x and $W \subset U$. It is easily seen that $W^* \subset (V_1^* \cap A) \cup (A \cap B) \cup (V_2^* \cap B) = X'$. By lemma 3.3.7, $\operatorname{cmp}[(V_1^* \cap A) \cup (V_2^* \cap B)] \leqslant n-1$. Hence, by lemma 3.3.8, we have $\operatorname{cmp} X' \leqslant n-1$. W^* is closed in X' and hence $\operatorname{cmp} W^* \leqslant n-1$. Therefore $\operatorname{cmp} X \leqslant n$. The proof of the theorem is now complete.

The following two lemmas are easily proved.

LEMMA 3.3.10. *Let $\mathfrak{U} = \{U\}$ be a family of subsets of X such that $\operatorname{cmp} U \leqslant n$ $(n \geqslant 0)$ for all $U \in \mathfrak{U}$ and $X = \bigcup \{U^0 \mid U \in \mathfrak{U}\}$. Then $\operatorname{cmp} X \leqslant n$.*

LEMMA 3.3.11. *If $X = \bigcup_{i=1}^{n} A_i$, where each A_i is locally compact and closed, then X is locally compact.*

With the aid of the above two lemmas, theorems 3.2.1, 3.2.3 and 3.3.9, we derive the following slight generalization of theorem 3.3.9.

THEOREM 3.3.12. *Suppose* $X = \bigcup_{i=1}^{m} A_i$ *where each* A_i *is closed and* $A_i \cap A_j$ *is locally compact,* $i \neq j$. *If* $\operatorname{cmp} A_i \leqslant n$, $i = 1, \ldots, m$, *then* $\operatorname{cmp} X \leqslant n$.

Since locally compact spaces have inductive compactness $\leqslant 0$, one might hope to replace local compactness of the intersection by inductive compactness $\leqslant 0$. This is not possible as the following example shows.

EXAMPLE 3.3.13. Let X, A and B be as in example 3.3.3. There, $\operatorname{cmp} A \cap B = 0$ and $A \cap B$ is not locally compact.

But we do have

THEOREM 3.3.14. *If* A *and* B *are closed in* $A \cup B$, $\operatorname{cmp} A \leqslant n$, $\operatorname{cmp} B \leqslant n$ *and* $\operatorname{cmp} A \cap B \leqslant m$ *then* $\operatorname{cmp} A \cup B \leqslant n+m+1$.

The proof of the above theorem follows easily from theorems 3.2.2, 3.3.9 and induction on m. Obviously, theorem 3.3.14 implies theorem 3.3.4. Finally, let us remark that the finite union in theorems 3.3.12 and 3.3.14 can be replaced by a union of a locally finite collection.

3.4. Product theorem. In dimension theory we have the product theorem $\dim A \times B \leqslant \dim A + \dim B$, where A or B is not void. The inductive compactness analogue of the product theorem is false as witnessed by example 3.3.6. We give in this section some positive relationships using both dimension and inductive compactness.

First we discuss the effect of the "doubling" process on inductive compactness.

DEFINITION 3.4.1. Let A be a nonempty closed subset of X and E^1 be the real line. By the *double of* X *modulo* A we mean the subspace of $X \times E^1$ defined as follows:

$$[X, A] = \{(x, d(x, A)) \mid x \in X\} \cup \{(x, -d(x, A)) \mid x \in X\},$$

where d is a metric on X and $d(x, A)$ is the usual distance from a point x to a set A.

LEMMA 3.4.2. *Let* A *be a nonempty closed subset of* X. *Then* $\operatorname{cmp} X = \operatorname{cmp}[X, A]$.

Proof. Since X is homeomorphic to the closed subset $\{(x, d(x, A)) \mid x \in X\}$ of $[X, A]$ we have $\operatorname{cmp}[X, A] \geqslant \operatorname{cmp} X$. We prove the reverse inequality by induction.

If $\operatorname{cmp} X = -1$ then clearly $\operatorname{cmp}[X, A] = -1$. Assume for $n > -1$ that whenever A' is a nonempty closed subset of X' with $\operatorname{cmp} X' < n$ we have $\operatorname{cmp}[X', A'] < n$ $(n < \infty)$. Let A be a nonempty closed subset of X where $\operatorname{cmp} X \leqslant n$. If $(\bar{x}, \bar{r}) \in [X, A]$ and $\bar{r} \neq 0$ then one can easily find arbitrarily small neighborhoods of $(\bar{x}, \bar{r})$ whose boundaries have inductive compactness $\leqslant n-1$. Consider $(\bar{x}, 0) \in [X, A]$. There are arbitrarily small neighborhoods U of $\bar{x}$ in X with $\operatorname{cmp} U^* \leqslant n-1$. Clearly, $V = \{(x, d(x, A)) \mid x \in U\} \cup \{(x, -d(x, A)) \mid x \in U\}$ is a neighborhood of $(\bar{x}, 0)$ in $[X, A]$ with boundary $B = B_+ \cup B_-$, where $B_+ = \{(x, d(x, A)) \mid x \in U^*\}$ and $B_- = \{(x, -d(x, A)) \mid x \in U^*\}$. If $A \cap U^* = \emptyset$ then $B_+ \cap B_- = \emptyset$. Since B_+ and B_- are closed and hence separated, $\operatorname{cmp} B \leqslant n-1$. If $A \cap U^* \neq \emptyset$ then $[U^*, A \cap U^*]$ is homeomorphic to B. Hence $\operatorname{cmp} B \leqslant n-1$. Therefore $\operatorname{cmp}[X, A] \leqslant n$. The case $n = \infty$ is obvious. The proof of the lemma is now completed.

We are now able to prove

THEOREM 3.4.3. (1) *If* $n \geqslant 1$ *then* $\operatorname{cmp} A \times E^n \leqslant \operatorname{cmp} A + n$.

(2) *If* B *is a closed subset of* E^n *then* $\operatorname{cmp} A \times B \leqslant \operatorname{cmp} A + n$.

(3) *If* B *is a compact space then* $\operatorname{cmp} A \times B \leqslant \operatorname{cmp} A + 2\dim B + 1$.

(4) *If* $\dim B = 0$ *and* B *is compact then* $\operatorname{cmp} A \times B \leqslant \operatorname{cmp} A$.

(5) *If* $\operatorname{cmp} A \leqslant 0$ *and* B *is a locally compact space then* $\operatorname{cmp} A \times B \leqslant \dim B$.

Proof. Part (1) is obvious for $n = 1$ by lemma 3.4.2. By induction we have (1) for all $n \geqslant 1$. Part (2) follows from (1) since $A \times B$ is closed in $A \times E^n$. Since each compact space of dimension n $(n < \infty)$ can be embedded in E^{2n+1}, [5] Theorem V5, we have (3) from (2). Part (4) follows from induction on $\operatorname{cmp} A$. Finally, part (5) follows from theorem 4.4.1 below.

3.5. Separation properties. In inductive dimension one can define a dimension in terms of separation properties, [5] p. 34. Namely, a space has *dimension* -1 if it is empty. A space has *dimension* $\leqslant n$ if every pair of disjoint closed sets can be separated by a closed set of dimension $\leqslant n-1$. We use $\operatorname{Dim} X \leqslant n$ for this definition. It is well known that $\dim X = \operatorname{Dim} X$. (Note: We are dealing only with separable metrizable spaces.) In this section we will analogously define inductive compactness using separation properties.

We first discuss the case $-1 \leqslant n \leqslant 0$. If $\operatorname{cmp} X \leqslant 0$, then one might hope that every pair of disjoint closed subsets A and B of X can be separated by a closed set C where $\operatorname{cmp} C = -1$. That is, C is a compact set such that $X \setminus C = X_1 \cup X_2$, X_1 and X_2 are disjoint and open and $A \subset X_1$ and $B \subset X_2$. That such a separation can be made for all X with $\operatorname{cmp} X \leqslant 0$ is easily seen to be false by considering the example of an open disc in the plane. But such a separation can almost be made as the following theorem illustrates.

THEOREM 3.5.1. *If* $\operatorname{cmp} X \leqslant 0$, A *and* B *are disjoint closed subsets of* X *then there is a closed locally compact subset* C *of* X *which separates* A *and* B.

Proof. If $\operatorname{cmp} X = -1$ then C is easily found. Suppose $\operatorname{cmp} X = 0$. Then $\operatorname{def} X = 0$. Let Y be a 0-compactification of X. Then A and B are separated sets in Y. The common part of the closures A and B in Y is a closed subset of $Y \setminus X$. Call this closed subset D and let $Z = Y \setminus D$. If $M = Z \setminus X$ then $\dim M \leqslant 0$. By [5] B), p. 34, there is a closed set C which separates A and B in Z and $C \cap M = \emptyset$. Hence C is a closed locally compact subset of X which separates A and B in X. The theorem is now proved.

Encouraged by the previous theorem, we define a "separation compactness", called $\operatorname{Cmp} X$.

DEFINITION 3.5.2. A space X has $\operatorname{Cmp} X = 0$ if and only if $\operatorname{cmp} X \leqslant 0$. For $n \geqslant 1$, a space X has $\operatorname{Cmp} X \leqslant n$ if and only if every closed subset of X has arbitrarily small neighborhoods whose boundaries have $\operatorname{Cmp} \leqslant n-1$.

The nontrivial part of the following theorem is proved essentially in the same manner as theorem 2.1.1.

THEOREM 3.5.3. *For* $\operatorname{def} X \geqslant 0$, $\operatorname{cmp} X \leqslant \operatorname{Cmp} X \leqslant \operatorname{def} X$.

Since it is not known whether $\operatorname{cmp} X = \operatorname{def} X$, an intermediate problem would be to prove some of the above inequalities are equalities.

The corresponding Cmp analogues of theorems 3.2.1, 3.2.2 and 3.3.2 are valid. With the aid of the following two lemmas, one can establish the Cmp analogue of theorem 3.3.12.

LEMMA 3.5.4. *Suppose that X' is a subspace of X and A and B are disjoint closed subsets of X. Then there is a closed set C in X which separates A and B such that*

(i) *$C \cap X'$ is locally compact if* $\operatorname{Cmp} X' = 0$,

or

(ii) $\operatorname{Cmp}(C \cap X') \leqslant \operatorname{Cmp} X' - 1$ *if* $\operatorname{Cmp} X' > 0$.

LEMMA 3.5.5. *If A is locally compact and closed and B is closed then* $\operatorname{Cmp}(A \cup B) = \operatorname{Cmp} B$.

The Cmp analogue of theorem 3.3.12 can then be used to prove the Cmp analogue of theorem 3.3.14, which, of course, implies the Cmp analogue of theorem 3.3.4. See section 4.5 for similarities between the definition of $\operatorname{Cmp} X$ and the separation properties of $\operatorname{def} X$.

4. The deficiency of a space.

In this section we investigate properties of $\operatorname{def} X$. We will divide our investigation into subsections which correspond to the subsections of section 3. In general the theorems concerning $\operatorname{cmp} X$ are true for $\operatorname{def} X$. The proofs in many cases are quite different. The results in this section, in general, support the validity of the equality between $\operatorname{cmp} X$ and $\operatorname{def} X$.

4.1. Existence of spaces X with $\operatorname{def} X = n$. The problem of existence is easily settled by the discussion of section 3.1 and the main theorem of section 2. We consider here the problem of finding relatively simple examples for which $\operatorname{def} X = n$.

Let I and J be respectively the closed and open unit intervals in the real line, I^n and J^n be their n-fold Cartesian products. Let $X_n = (I \times I^n) \setminus (\{1\} \times J^n)$, $n \geqslant 1$. Then we have the following theorem. (See [6], p. 121.)

THEOREM 4.1.1. $\operatorname{def} X_n = n$, $n \geqslant 1$.

Proof. Since the proof for the case $n = 2$ is typical, we will only prove this case. It is easily seen that

$$X = X_2 = ([0,1) \times [0,1] \times [0,1]) \cup (\{1\} \times S^1),$$

$$\text{where} \quad S^1 = \{0,1\} \times [0,1] \cup [0,1] \times \{0,1\}$$

is clearly a one sphere. For convenience, let $D = X \setminus \{1\} \times S^1$. By lemma 3.1.3, we have $\operatorname{def} X \leqslant 2$. Let Y be an n-compactification of X. Suppose $n < 2$. Then $Y \setminus D = (Y \setminus X) \cup (\{1\} \times S^1)$ has dimension equal to one and D is a dense open set in Y. Since $\dim(Y \setminus D) = 1$, the identity mapping f of $\{1\} \times S^1$ onto S^1 can be extended to a mapping g of $Y \setminus D$ onto S^1, [5] Theorem V14. Let h be the obvious extension of f on $\{1\} \times S^1$ to $[0,1] \times S^1$. Then h and g give a mapping k of $(Y \setminus D) \cup ([0,1] \times S^1)$. Since S^1 is an absolute neighborhood retract, there is a neighborhood U of $(Y \setminus D) \cup ([0,1] \times S^1)$ and a mapping j of U into S^1 which extends k. Now, $\bigcup_{p<1} ([0,p] \times I \times I) = D$. Consequently, there is $p \in J$ so that $\{p\} \times I \times I \subset U$. Hence j maps $\{p\} \times I \times I$ onto S^1 and is the identity mapping on $\{p\} \times S^1$. This, of course, is not possible by [5] A), p. 97. Hence for every n-compactification Y of X, we have $n \geqslant 2$. The theorem is now proved.

It is easily shown that $\operatorname{cmp} X_1 = 1$. It can be proved with some effort that $\operatorname{cmp} X_2 = 2$. For $n > 2$, it is not known whether $\operatorname{cmp} X_n \geqslant n$.

4.2. Monotone property. In this section we show that theorems 3.2.1 and 3.2.3 have their analogues in terms of $\operatorname{def} X$.

THEOREM 4.2.1. *If A is a closed subspace of X then* $\operatorname{def} A \leqslant \operatorname{def} X$.

Proof. Let Y be an n-compactification of X where $n = \operatorname{def} X$. Let Z be the closure of A in Y. Since A is closed in X, we have $Z \setminus A \subset Y \setminus X$. Hence $\operatorname{def} A \leqslant \dim(Z \setminus A) \leqslant \dim(Y \setminus X) = \operatorname{def} X$.

THEOREM 4.2.2. *If A is open in X and* $\operatorname{def} X \geqslant 0$ *then* $\operatorname{def} A \leqslant \operatorname{def} X$.

Proof. Suppose that $\operatorname{def} A > -1$ and let Y be an n-compactification of X where $n = \operatorname{def} X$. Let U be an open set in Y such that

$X \cap U = A$. Since U is locally compact, let Z be the one-point compactification of U. Then

$$Z \setminus A = (Z \setminus U) \cup (U \setminus A) = (Z \setminus U) \cup (U \setminus X \cap U) = (Z \setminus U) \cup [(Y \setminus X) \cap U],$$

where $Z \setminus U$ is a one-point set. Hence $\operatorname{def} A \leqslant \dim(Z \setminus A) \leqslant \dim(Y \setminus X) = \operatorname{def} X$.

Finally, we prove

THEOREM 4.2.3. *If $A \subset X$ then* $\operatorname{def} A \leqslant \operatorname{def} X + \dim(X \setminus A) + 1$.

Proof. Let Y be an n-compactification where $n = \operatorname{def} X$. Suppose that B is the closure of A in Y. Then $B \setminus A = [(B \setminus A) \cap (Y \setminus X)] \cup [(B \setminus A) \cap (X \setminus A)]$. Hence $\operatorname{def} A \leqslant \dim(B \setminus A) \leqslant \dim(Y \setminus X) + \dim(X \setminus A) + 1 \leqslant \operatorname{def} X + \dim(X \setminus A) + 1$.

Remark 4.2.4. The last theorem gives an upper bound for the deficiency of a subspace in terms of the deficiency of the space and the dimension of its complement. This theorem can be improved for spaces which are locally compact. Namely, the deficiency of a proper subspace of a locally compact space does not exceed the dimension of its complement. The first two theorems give conditions which insure that the deficiency of a subspace does not exceed the deficiency of the space.

4.3. Sum theorems. In section 3.3 two sum theorems are proved. The closest analogue to theorem 3.3.2 has been proved in section 4.2 as theorem 4.2.3. We devote this section to proving the following analogue of theorem 3.3.12.

THEOREM 4.3.1. *Suppose that $X = \bigcup_{i=1}^{m} A_i$ where each A_i is closed in X and $A_i \cap A_j$ is locally compact for $i \neq j$. If $\operatorname{def} A_i \leqslant n$ for $i = 1, \ldots, m$, then $\operatorname{def} X \leqslant n$.*

The theorem will follow from induction if we establish the case $m = 2$. To prove this theorem we first discuss disjoint topological sum of two spaces.

Suppose that A and B are closed in $X = A \cup B$. Consider the disjoint topological sum $A + B$ of A and B. That is, $A + B = A \times \{0\} \cup B \times \{1\}$. There is a natural projection map P from $A + B$ onto $X = A \cup B$ defined by $P(x, t) = x$. Clearly, P is continuous and closed. Hence the topology on X is the quotient topology on X relative to the map P ([7] theorem 3.8). Also $P^{-1}(x)$ is finite for each $x \in X$. We next prove the following lemma.

LEMMA 4.3.2. *Suppose that $A \subset X$ and A is locally compact and closed in X. Then there is an n-compactification Y of X such that $n = \operatorname{def} X$ and if B is the closure of A in Y then $B \setminus A$ is empty or a one-point subset of $Y \setminus X$ according as A is compact or not.*

Proof. If A is compact then A is closed in any compactification of X. Hence we suppose that A is not compact and let Z be an n-compactification of X where $n = \operatorname{def} X$. Since A is closed in X, $C \setminus A \subset Z \setminus X$ where C is the closure of A in Z. $D = C \setminus A$ is a nonempty compact subset of Z since A is locally compact and not compact. Let Y be the space obtained from Z by identifying the compact set D and giving Y the quotient topology. Since the decomposition is upper-semicontinuous, Y is a metrizable compactification of X ([7] theorem 5.20). Furthermore, $\operatorname{def} X \leqslant \dim(Y \setminus X) \leqslant \dim(Z \setminus X) = \operatorname{def} X$. Hence Y is an n-compactification of X where $n = \operatorname{def} X$ and if B is the closure of A in Y then $B \setminus A$ is a one-point subset of $Y \setminus X$. Thus the lemma is proved.

We now proceed to the proof of theorem 4.3.1. Let $X = A \cup B$ where A and B are closed and $A \cap B$ is locally compact. By lemma 4.3.2 above, there is an n-compactification $\tilde{A}$ of A such that $n = \operatorname{def} A$ and if C_A is the closure of $C = A \cap B$ in $\tilde{A}$ then $C_A \setminus C$ is empty or a one-point subset of $\tilde{A} \setminus A$ according as C is compact or not. Similarly, for B, there is an m-compactification $\tilde{B}$ such that $m = \operatorname{def} B$ and if C_B is the closure of C in $\tilde{B}$ then $C_B \setminus C$ is empty or a one-point subset of $\tilde{B} \setminus B$ according as C is compact or not. Clearly, C_A and C_B are homeomorphic in a natural manner. We assume that C is not compact since the compact case is a simple modification of the non-compact case. Since C_A and C_B are homeomorphic in a natural manner, let us consider $C_A \setminus C$ and $C_B \setminus C$ to be the same one-point set, say $\{\infty\}$.

Consider the disjoint topological sum $\tilde{A} + \tilde{B}$ of $\tilde{A}$ and $\tilde{B}$. $\tilde{A} + \tilde{B}$ has $A + B$ as a dense subspace. There is a natural projection map P from $\tilde{A} + \tilde{B}$ onto the set $\tilde{X} = \tilde{A} \cup \tilde{B}$. Each member of the decomposition $\{P^{-1}(x) \mid x \in \tilde{A} \cup \tilde{B}\}$ of $\tilde{A} + \tilde{B}$ is a compact subset of $\tilde{A} + \tilde{B}$. Let us show this decomposition is upper-semicontinuous. Suppose that $U_1 + U_2$ is an open neighborhood of $P^{-1}(x)$. If $x \in \tilde{A} \setminus \tilde{B}$ then $U_1 \setminus \tilde{A} \cap \tilde{B}$ is an open neighborhood of x in $\tilde{A}$ since $\tilde{A} \cap \tilde{B}$ is a compact subset of $\tilde{A}$. Hence $P^{-1}(x) \subset P^{-1}(U_1 \setminus \tilde{A} \cap \tilde{B}) = (U_1 \setminus \tilde{A} \cap \tilde{B}) + \emptyset \subset U_1 + U_2$. By symmetry, we have the case $x \in \tilde{B} \setminus \tilde{A}$. Suppose $x \in \tilde{A} \cap \tilde{B}$. Then $U_i \cap \tilde{A} \cap \tilde{B}$, $i = 1, 2$, are open neighborhoods of x in $\tilde{A} \cap \tilde{B}$. If W_i is the closure of $U_i \cap \tilde{A} \cap \tilde{B}$ in $\tilde{A} \cap \tilde{B}$ then W_i is a compact subset of $\tilde{A} \cap \tilde{B}$, $i = 1, 2$. Clearly, $W = U_1 \cap U_2 \cap \tilde{A} \cap \tilde{B}$ is an open neighborhood of x in $\tilde{A} \cap \tilde{B}$. Hence $V_i = W_i \setminus W$ is a compact subset of $\tilde{A} \cap \tilde{B}$ and $x \notin V_i$, $i = 1, 2$. Consequently, $U_1 \setminus V_1$ is an open neighborhood of x in $\tilde{A}$ and $U_2 \setminus V_2$ is an open neighborhood of x in $\tilde{B}$. If $x' \in U_i \setminus V_i$ then $P^{-1}(x') \subset (U_1 \setminus V_1) + (U_2 \setminus V_2) \subset U_1 + U_2$. Thus we conclude that the decomposition is upper-semicontinuous. Consequently, P gives a closed continuous mapping from the compact metrizable space $\tilde{A} + \tilde{B}$ onto $\tilde{X} = \tilde{A} \cup \tilde{B}$ ([7] theorem 3.12). Therefore, the quotient topology on $\tilde{X}$ relative to the map P is compact and metrizable ([7] theorem 5.20).

Clearly, $P|(A+B)$ is a continuous mapping from $A+B$ onto the subset $X = A \cup B$ with the relative topology induced by $\widetilde{X}$. Let us show that this relative topology on X is the original metric topology on X. To this end, we show that $P|(A+B)$ is a closed mapping from $A+B$. Let F be a set closed in $A+B$ and E be the closure of F in $\widetilde{A}+\widetilde{B}$. Then $P(E) = P(E\setminus F) \cup P(F)$. Since $P^{-1}(A\cup B) = A+B$ and $E\setminus F \subset (\widetilde{A}+\widetilde{B})\setminus(A+B)$, we have $P(E\setminus F) \cap (A\cup B) = \emptyset$. Hence

$$(P|A+B)(F) = P(F) = P(F) \cap (A\cup B) = P(E) \cap (A\cup B).$$

Since P is closed, we have that $(P|A+B)(F)$ is closed in $A\cup B$. Thus $P|(A+B)$ is closed.

Finally,

$$\widetilde{X}\setminus X = P[(\widetilde{A}+\widetilde{B})\setminus(A+B)] = P\{[\widetilde{A}+\emptyset]\setminus[(A\cup\{\infty\})+\emptyset]\} \cup \cup P\{[\emptyset+\widetilde{B}]\setminus[\emptyset+(B\cup\{\infty\})]\} \cup P(\infty+\infty).$$

The first two sets in the last union are separated and homeomorphic to $\widetilde{A}\setminus(A\cup\{\infty\})$ and $\widetilde{B}\setminus(B\cup\{\infty\})$, respectively. Hence

$$\operatorname{def} X \leqslant \dim(\widetilde{X}\setminus X) = \max\{\dim(\widetilde{A}\setminus A), \dim(\widetilde{B}\setminus B)\} = \max\{\operatorname{def} A, \operatorname{def} B\}.$$

That is, if X is the union of two closed sets whose intersection is locally compact and the deficiency of each set is $\leqslant n$ then $\operatorname{def} X \leqslant n$. Thus theorem 4.3.1 is proved.

It is not known whether the deficiency analogue of theorem 3.3.14 is true.

4.4. Product theorem. The product theorem for deficiency in this section seems to be the first serious deviation from the close similarity between inductive compactness and deficiency. That is, we get a stronger inequality for deficiency.

THEOREM 4.4.1. *Suppose that B is a locally compact space. Then*

(1) *if A is locally compact then* $\operatorname{def} A\times B \leqslant 0$;

(2) *if A is not compact then*

$$\operatorname{def} A\times B \leqslant \operatorname{def} A + \dim B.$$

Proof. Part (1): This is obvious since $A\times B$ is locally compact.

Part (2): Suppose $B \neq \emptyset$. Let X be an n-compactification of A where $n = \operatorname{def} A$. Then $X\times B$ is locally compact and

$$0 \leqslant \dim(X\times B\setminus A\times B) = \dim((X\setminus A)\times B) \leqslant \dim(X\setminus A) + \dim B = \operatorname{def} A + \dim B.$$

Hence

$$\operatorname{def} A\times B \leqslant \operatorname{def} A + \dim B.$$

Remark 4.4.2. The deficiency analogue of lemma 3.4.2 holds.

In section 2.1, the notion of peripheral deficiency was introduced. It is not difficult to show the analogues of theorem 4.2.1, 4.2.2, 4.2.3 and 4.3.1 hold for peripheral deficiency.

For further applications of deficiency, see [12].

4.5. Separation properties. In this section we find that $\operatorname{def} X$ has the same properties that the definition of $\operatorname{Cmp} X$ possesses. Also, $\operatorname{def} X$ and $\operatorname{Cmp} X$ have a monotone property and the finite sum theorems in common.

We have

THEOREM 4.5.1. *Let A and B be two disjoint closed subsets of X.*

(1) *If $\operatorname{def} X \leqslant 0$ then there is a closed locally compact subset C of X which separates A and B.*

(2) *If $\operatorname{def} X > 0$ then there is a closed subset C of X which separates A and B and* $\operatorname{def} C \leqslant \operatorname{def} X - 1$.

Proof. Part (1) follows from theorem 3.5.1, since $\operatorname{def} X \leqslant 0$ if and only if $\operatorname{cmp} X \leqslant 0$.

To prove part (2), let Y be an n-compactification of X where $n = \operatorname{def} X$. Then by [5] B), p. 34, there is a closed set D in Y which separates A and B and $\dim(D\cap(Y\setminus X)) \leqslant n-1$. Let $C = X\cap D$. Then $\operatorname{def} C \leqslant n-1 < \operatorname{def} X$. Thus the theorem is proved.

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On relations between some algebraic and topological properties of lattices

by

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Dedicated to Professor A. D. Wallace on the occasion of his 60-th birthday

§ 1. Introduction. Let $\Gamma=(L,\cup,\cap,0,1)$ denote a distributive lattice. Γ is called *Brouwerian* (see [4]) if there is an operation $a-b$ (called *pseudo-difference*) such that

$$(a-b\subset c)\equiv[a\subset(b\cup c)].$$

We shall consider in this paper the following three algebraic (structural) properties of lattices:

1. The property of being *Wallman*, which means that:

(1) $(a\not\subset b)\Rightarrow$ *there is* d *such that* $(0\neq d\subset a)(b\cap d=0)$.

2. The *regularity* of Γ:

(2) $(a\not\subset b)\Rightarrow$ *there are* c *and* d *such that* $(c\cup d=1)(a\not\subset c)(b\cap d=0)$.

2. The *normality* of Γ:

(3) $(a\cap b=0)\Rightarrow$ *there are* c *and* d *such that* $(c\cup d=1)$ $(a\cap c=0=b\cap d)$

Remark. It is easy to see that assuming the lattice to be Brouwerian one can replace the formulas (2) and (3) by the following:

(2′) $(a\not\subset b)\Rightarrow$ *there is* d *such that* $b\cap d=0$ *and* $a\not\subset 1-d$,

(3′) $(a\cap b=0)\Rightarrow$ *there is* d *such that* $b\cap d=0$ *and* $a\cap(1-d)=0$.

The three above defined properties of Γ have algebraic aspect (they have been defined without introducing any topology in Γ). Nevertheless, they origin is topological. In fact, in order that the lattice 2^X of closed subsets of a topological space X be structurally regular (resp. normal) it is necessary and sufficient that the space X be regular (resp. normal) in the usual topological sense. If X is a $\mathcal{T}_1$-space, then 2^X is structurally Wallman (the converse is not true).