number \( p \) such that (1) the square disc \( \{ (x_1, x_2) \in P : |x_1 - x_2| \leq 2p \} \)

\[ |x_1 - x_2| \leq 2p \] is between \( \pi_1(P) \) and \( \pi_2(Q) \). Let \( S_1 \) and \( S_2 \) denote respectively the strips

\[ \{ (x_1, x_2) \in P : 0 < |x_2 - x_1| \leq 2p \} \]

\[ \{ (x_1, x_2) \in P : 2p < |x_2 - x_1| < \} \]

Then \( K \subset S \) and \( K \cap S \setminus S_2 \) are closed subsets of \( P \setminus C \) that do not intersect \( I \) and consequently \( \pi_1(K \cap S) \) and \( \pi_2(K \cap S_2) \) are both of cardinality less than \( \pi \). Therefore there are points \( u \) and \( v \) of \( I \) such that \( x_1 - 2p < u < x_1 - p < v < x_1 + 2p \), and \( w \) does not belong to \( \pi_1(K \cap S_2) \)

and \( v \) does not belong to \( \pi_2(K \cap S) \). However, \( \{ (u, x_2) : 0 < x_2 < x_1 \} \cup \{ (x_1, x_3) : x_1 < x_3 < x_1 \} \cup \{ (x_2, x_3) : x_1 < x_1 < x_1 \} \) is an arc which separates \( P \) from \( Q \) in \( P \) and does not intersect \( K \), which is a contradiction.

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Reçu par la Rédaction le 19.2. 1965

Locally Hamiltonian and planar graphs

by

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1. Introduction. In this paper we consider finite graphs which contain no loops and no parallel edges. By a graph we mean an ordered pair \( (V, E) \), where \( V \) is a finite non-empty set (the set of vertices) and \( E \) is a set of two-element subsets of \( V \) (the set of edges). Thus a graph is a zero- or one-dimensional simplicial complex and sometimes, when misunderstandings are improbable, we will identify it to its topological realization. By a drawing we mean a graph whose topological realization is a simple closed curve. A graph is called planar if it has a homeomorphic immersion into the two-sphere \( S^2 \).

Two vertices \( x, y \in V \) are called adjacent (neighbours) if \( x, y \in E \). A graph \( H \subseteq (V, E) \) is called a subgraph of a graph \( G \) if \( G \subseteq (V, E) \), or \( G \) is said to contain \( H \) if \( U \subseteq V \) and \( D \subseteq E \). A subgraph \( H \) of \( G \) is said to be spanned by a set \( U \subseteq V \) if \( \subseteq \subseteq U \subseteq V \) and \( e \subseteq U \). The subgraph of \( G \) spanned by a set of vertices adjacent to a vertex \( x \in V \), i.e., the set \( \{ y : x, y \in E \} \), is denoted by \( G(x) \).

A graph \( G \) is called Hamiltonian if it has a Hamiltonian circuit, i.e., a circuit whose set of vertices is all the set \( V \). A graph \( G \) is called locally Hamiltonian if for every \( x \in V \) the graph \( G(x) \) exists and is Hamiltonian. Obviously a 1-skeleton of any triangulation of a closed surface is a connected and locally Hamiltonian graph. \( G \) is called a triangulation graph if it is the 1-skeleton of a triangulation.

The main theorem of this paper is the following

**Theorem 1.** If a connected and locally Hamiltonian graph \( G \) has \( n \) vertices, \( m \) edges and \( m < 3n - 6 \) then \( G \) is an \( S^3 \) triangulation graph.

**Remark 1.** Clearly the converse implication is also valid and an \( S^3 \) triangulation graph graph with \( n \geq 4 \) vertices and \( 3m - 6 \) edges (the last assertion and its generalization to the case of other 2-manifolds follows immediately from [10], pp. 24 and 61); for other characterization, see [1].

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Corollary 1. A connected and locally Hamiltonian graph having \( n \) vertices has at least \( 3n - 6 \) edges and more than \( 3n - 6 \) edges if and only if it is not planar.

This reminds of the classical theorem of Kuratowski [7] (see also [3] and [9]) which says that a graph is not planar if and only if it contains topologically \( K_5 \) or \( K_{3,3} \) (\( K_n \) is the unique graph with \( n \) vertices and \( \binom{n}{2} \) edges and \( K_{3,3} \) is a graph with 9 edges and 6 vertices three of which are adjacent to each of the remaining three). Now numerous characterizations of planar graphs exist (see [3], [9], [5], [7], [4], [3], [12] and [13]; see also [2], [11] and [11] for methods for determining whether a given graph is planar).

Fig. 1a. The graph \( K_5 \)

Fig. 1b. The graph \( K_{3,3} \)

Remark 3. The supposition of Theorem 1 and Corollary 1 that \( G \) be connected is essential since the graph \( K_5 + K_4 \), where \( K_5 \) and \( K_4 \) are disjoint, is non-planar but satisfies the other conditions.

We will also prove an easier theorem.

Theorem 2. If a locally Hamiltonian graph \( G = \langle V, E \rangle \) has no subgraph homeomorphic with \( K_5 \) then for every \( x \in V \) there is a single Hamiltonian circuit in \( G(x) \).

Remark 4. The converse implication is not valid since the triangulation graphs of closed surfaces may contain \( K_5 \). In Fig. 3 is shown a minimal graph \( P \) with property that it contains topologically \( K_5 \) and has a single Hamiltonian circuit of \( P(x) \) for every vertex \( x \) of \( P \).

Theorem 2 yields the following

Corollary 2. If the graph \( G \) is an \( S^3 \) triangulation graph, then the simplicial complex, whose 1-skeleton is \( G \) and whose realization is \( S^3 \), is uniquely determined.

2. Further definitions and lemmas. Sometimes we will write \( x \in G \) if \( x \) is a vertex of the graph \( G \), and \( H \subseteq G \) if \( H \) is a subgraph of \( G \).

The degree \( d(x) = d(x, G) \) of the vertex \( x \) in \( G \) is the number of its neighbours in \( G \).

L1. Each graph containing exactly \( n \) vertices and \( m < 3n \) edges has a vertex of degree \( \leq 5 \).

Clearly, a graph \( G \) containing \( n \) vertices of degree \( \geq 6 \) has \( m \) edges where

\[
m = \frac{1}{2} \sum_{x \in V} d(x, G) > \frac{1}{2} \cdot 6n = 3n.
\]

Therefore lemma L1 holds.

The following lemma is obvious:

L2. If for some \( x \in G \) there exists a Hamiltonian circuit of \( G(x) \) then \( d(x, G) \geq 3 \).

Let \( LH \) denote the class of connected and locally Hamiltonian graphs, and let \( LH(n, m) \) be the subset of \( LH \) consisting of all graphs with \( n \) vertices and \( m \) edges. One can easily prove the following lemma:

L3. If a vertex \( x \) is of degree \( 3 \) in a graph \( G \in LH \) with \( n \geq 4 \) vertices, then each neighbour of \( x \) in \( G \) has a degree \( \geq 4 \).

Let \( S \) denote the class of 1-skeletons of triangulations of \( S^3 \). Obviously, by Euler’s formula, the following holds.

L4. If \( G_1 \in S \) then \( G_1 \in LH(n, 3n - 6) \) for some \( n \geq 4 \).

If \( x_i \neq x_j \) for \( i, j = 0, \ldots, n \), \( i \neq j \) and \( \{i, j\} \neq \{0, n\} \) we put

\[
P = \{x_0, x_1, \ldots, x_n\} = \{x_0, x_1, x_2, x_3\}, \{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}\}
\]

Each such a graph \( P \) is called a path with ends \( x_0 \) and \( x_n \) if \( x_0 \neq x_n \) and is a circuit if \( x_n = x_0 \) and \( n \geq 3 \). A path \( P \) with ends \( x_0 \) and \( x_n \) is denoted also by \( P[x_0, x_n] \) or \( P[x_n, x_0] \). The length \( l(P) \) of the path (circuit) \( P \) is the number of its edges. A path \( P = [y, z] \) with \( l(P) = 1 \) is also called an edge.

A topological realization \( G \) of an abstract planar graph in the two-sphere \( S^2 \) is called a plane graph \( G \). The union of all \( G \)'s simplices is denoted by \( \{G\} \). Each component of the set \( S^2 \backslash \{G\} \) is said to be a face of the plane graph \( G \). Two faces \( D_1 \) and \( D_2 \) of \( G \) are adjacent if there exists an edge \( E \subseteq G \) such that \( |E| \subseteq D_1 \cap D_2 \).

A sum \( G_1 + G_2 \) of two graphs \( G_1 = \langle V_1, E_1 \rangle \) and \( G_2 = \langle V_2, E_2 \rangle \) is a graph

\[
G_1 + G_2 = \langle V_1 \cup V_2, E_1 \cup E_2 \rangle.
\]

If \( x \in G = \langle V, E \rangle \), then \( G \backslash x \) will denote a subgraph of \( G \) spanned by a set \( V \backslash \{x\} \).

(*) \( \overline{D} \) denotes a closure of \( D \).
3. Proof of Theorem 2. Let us assume that, for some $x \in G$, there exist two distinct Hamiltonian circuits $H_1$ and $H_2$ of $G(x)$. Now there exists an edge $[t, u]$ such that $[t, u] \subset H_1$ and $[t, u] \subset H_2$. Hence $H_2$ is divided by $t$ and $u$ into two paths $P_2$ and $P_3$, both with ends $t$ and $u$ and both of length $> 1$. Therefore there exists in $H_1$ an edge $[y_1, y_2]$ such that $y_1 \in P_2$ and $y_2 \in P_3$, and $[y_1, y_2] \cap [t, u] = \emptyset$. So $G$ contains topologically a $K_4$ (Fig. 2), contrary to the supposition.

![Fig. 2. The subgraph of G](image)

![Fig. 3. The graph F](image)

The following lemma is obvious (see Corollary 2).

L8. If, for any $x_1 \in G_1 \in \mathcal{A}$, a Hamiltonian circuit of $G_1(x_1)$ contains an edge $[x_1, x_2]$, then $G_1$ has a face whose boundary is the circuit $[x_1, x_2, x_3, x_4]$.

4. Proof of Theorem 1. Since $G \in LH$ then $G$ has at least 4 vertices, therefore it suffices to prove that if $G \in LH(n, m)$, where $n > 4$ and $m < 3n - 6$, then $G \in \mathcal{A}$. We proceed by induction with respect to $n$.

Let us assume that the theorem is valid for $n > 4$. Let $G$ be any graph of the class $LH(n + 1, m)$, where $m < 3(n + 1) - 6$. Let for any $y \in G$ the symbol $H(y)$ denote some fixed Hamiltonian circuit of $G(y)$. For the given $G$ we construct an auxiliary graph $G_1$ belonging to the class $LH(n, m_1)$, where $m_1 < 3n - 6$. For any $y \in G_1$, the symbol $H(y)$ will denote some Hamiltonian circuit of $G_1(y)$.

Let $x$ denote a vertex of $G$ with minimal degree in $G$. By virtue of L1 and L2, the inequalities $3 < d(x) < 5$ hold. We consider three main cases: $d(x) = 3, 4$ or 5.

Case I: $d(x) = 3$. Let $G_1 = G \setminus x$. Let $t$, $u$ and $w$ be all the neighbours of $x$ in $G$. Each of circuits $H(t), H(u), H(w)$ contains, by L3, more than 3 vertices. Obviously $[w, x, u] \subset H(t)$. We can put $H_1(t) = H(t) \setminus x + [w, x, w]$. Analogously we can define $H_1(u)$ and $H_1(w)$. We can put $H_1(y) = H(y)$ for any $y$ such that $y \in G$ and $y \neq t, u, w$. Therefore $G_1 \in LH(n, m_1)$, where $m_1 = m - 3 < 3n - 6$. By the induction hypothesis $G_1 \in \mathcal{A}$. Since

![Fig. 4. The graph G(x) + St(x) (*)](image)

Let $G_1 = G \setminus x + [t, u]$. We can prove that $G_1 \in LH(n, m_0)$, where $m_0 = m - 3 < 3n - 6$. In particular we can put $H_1(t) = H(t) \setminus x + [w, x, w, y]$. Now, by the induction hypothesis and L5, we have: $G_1 \in \mathcal{A}$ and any plane graph $G_1$ has two adjacent faces with boundaries $[t, u, w, f]$ and $[t, u, y, g]$, respectively. Therefore $G \in \mathcal{A}$.

Case III: $d(x) = 5$. Let $K_4$ denote a complete graph with 5 vertices such that $G(x) \subseteq K_4$. Let $G_1 = G \setminus x + K_4$. One can prove that $G_1 \in LH(n, m_0)$, where $m_0 = 3n - 6$, contrary to the induction hypothesis and L4. Hence case III is impossible.

Case III: $d(x) = 5$. Let $K_4$ denote a complete graph with 5 vertices such that $G(x) \subseteq K_4$. Let $G_1 = G \setminus x + K_4$. One can prove similarly as before that $G_1 \in LH(n, m_0)$, where $m_0 = 3n - 6$. Therefore $G_1 \in \mathcal{A}$. On the other hand, since $K_4 \subseteq G_1$, $G_1$ is non-planar. This contradiction proves that the case III is impossible.

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(*) St(x) denotes the closed star of $x$ in $G$. 

Locally Hamiltonian graphs
Case IIIId: \([x, w, y] \subseteq H(t)\). In this case we put \(G_i = G\setminus x + [x, w] + [z, w]\). Analogously as in IIIId1 we can show that \(G \not\in \mathcal{A}\).

Case IIIId2: \([w, z, y, t] \subseteq H(t)\) and \([z, x, w] \subseteq H(t)\). Using the same method as in IIIId1 or as in IIIId2 one can prove that \(G \not\in \mathcal{A}\).

Remark. One can show that the cases IIIId1 and IIIId2 are impossible.

Case IIIe: \(G(x)\) has 5 or 6 edges (see Fig. 5, where the intermittent line represents eventual 6th edge of \(G(x)\)). Let \(G_i = G\setminus x + [z, y] + [w, t]\) or \(G_i = G\setminus x + [z, w] + [t, y]\). Analogously as in the previous cases we can show that \(G \not\in \mathcal{A}\).

Fig. 8. The graph \(G(x) + S_t(x)\)

All possible cases have been examined. Thus Theorem 1 is proved.

This paper contains the main results of my doctoral dissertation.

In conclusion I wish to express my thanks to Professors J. Górski and S. Golab and specially to J. Mycielski for their kind interest and many valuable advices.

Added in proof. The first of the propositions mentioned in Remark 2 is proved in [14].

References

Inductive compactness as a generalization of semicompactness

by

J. de Groot and T. Nishiura ** (Amsterdam and Detroit, Mich.)

1. Introduction. All spaces under discussion are separable metrizable spaces.

In 1942, J. de Groot introduced the notion of a semicompact space. Namely, a space is semicompact if each point in the space has arbitrarily small neighborhoods with compact boundaries. Spaces which are semicompact are sometimes called rim compact or peripherally compact. In 1942, J. de Groot proved that a space is semicompact if and only if it can be compactified by adding a set of dimension no higher than zero [2]. This result is implicitly contained in H. Freudenthal's paper [3]. Freudenthal generalizes his results in [4]. For further generalizations, see K. Morita [10] and P. S. Aleksandrov and V. I. Ponomarev [1]. The notion of semicompact space and the above characterization of such spaces suggest some other concepts. One concept is that of an "inductive compactness" analogous to inductive dimension. Another concept is the compactification of a space by adding a set of minimal dimension. Let us formalize these two concepts.

DEFINITION. INDUCTIVE COMPACTNESS. A space $X$ is said to have compactness $n$ if $X$ is compact. A space $X$ is said to have compactness less than or equal to $n$ ($n \geq 0$) if each point of $X$ has arbitrarily small neighborhoods whose boundaries have compactness less than or equal to $n-1$. We use the notation $\text{cmp} X \leq n$. A space $X$ is said to have compactness equal to $\infty$ if $\text{cmp} X < n$ is false for each integer $n$.

DEFINITION. A compact space $Y$ is called an $n$-compactification of a space $X$ if $X$ is dense in $Y$ and $\dim (Y \setminus X) = n$. By the deficiency

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* Some of the results in this paper were proved in a seminar conducted by J. de Groot at Purdue University in the Spring of 1960. The authors wish to acknowledge the participants: Professors M. Henle, C. Neugebauer, A. Pecold, Jr., R. McDowell, and Dr. G. Day and Mr. A. Rasson. Helpful contributions were also obtained from Professors J. Isbell and M. E. Rudin.

** The second author was partially supported by the National Science Foundation Grants NSF-G14634 and NSF-GP3554.