number ϱ such that (1) the square disc $\{(x_1, x_2) \in I^2: |x_1 - z_1| \leqslant 2\varrho, |x_2 - z_2| \leqslant 2\varrho\}$ does not intersect K and (2) the interval $\{x_1 \in I: |x_1 - z_1| \leqslant 2\varrho\}$ is between $\pi_1(P)$ and $\pi_1(Q)$. Let S_1 and S_2 denote respectively the strips

$$\{(x_1, x_2) \in I^2: z_1 - 2\varrho \leqslant x_1 \leqslant z_1 - \varrho,\ 0 \leqslant x_2 \leqslant z_2\},$$
$$\{(x_1, x_2) \in I^2: z_1 + \varrho \leqslant x_1 \leqslant z_1 + 2\varrho,\ z_2 \leqslant x_2 \leqslant 1\}.$$

Then $K \cap S_1$ and $K \cap S_2$ are closed subsets of $I^2 - C$ that do not intersect Γ and consequently $\pi_1(K \cap S_1)$ and $\pi_1(K \cap S_2)$ are both of cardinality less than $\mathfrak{c}$. Therefore there are points u and v of I such that $z_1 - 2\varrho \leqslant u \leqslant z_1 - \varrho$, $z_1 + \varrho \leqslant v \leqslant z_1 + 2\varrho$, and u does not belong to $\pi_1(K \cap S_1)$ and v does not belong to $\pi_1(K \cap S_2)$. However $\{(u, x_2): 0 \leqslant x_2 \leqslant z_2\} \cup \{(x_1, z_2): u \leqslant x_1 \leqslant v\} \cup \{(v, x_2): z_2 \leqslant x_2 \leqslant 1\}$ is an arc which separates P from Q in I^2 and does not intersect k, which is a contradiction.

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Locally Hamiltonian and planar graphs

by

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1. Introduction. In this paper we consider finite graphs which contain no loops and no parallel edges. By *a graph* we mean an ordered pair $\langle V, E\rangle$ where V is a finite non-empty set (the *set of vertices*) and E is a set of two-element subsets of V (the *set of edges*). Thus a graph is a zero- or one-dimensional simplicial complex and sometimes, when misunderstandings are improbable we will identify it to its topological realization. By a *circuit* we mean a graph whose topological realization is a simple closed curve. A graph is called *planar* if it has a homeomorphism into the two-sphere S^2.

Two vertices $x, y \in V$ are called *adjacent* (*neighbours*) in $G = \langle V, E\rangle$ if $\{x, y\} \in E$. A graph $H = \langle U, D\rangle$ is called a *subgraph* of a graph $G = \langle V, E\rangle$, or G is said to contain H, if $U \subseteq V$ and $D \subseteq E$. A subgraph H of G is said to be *spanned* by a set $U \subseteq V$ if $H = \langle U, \{e: e \in E \text{ and } e \subseteq U\}\rangle$. The subgraph of $G = \langle V, E\rangle$ spanned by a set of vertices adjacent to a vertex $x \in V$, i.e. by the set $\{y: \{x, y\} \in E\}$, is denoted by $G(x)$.

A graph G is called *Hamiltonian* if it has a *Hamiltonian circuit*, i.e. a circuit whose set of vertices is all the set V. G is called *locally Hamiltonian* if for every $x \in V$ the graph $G(x)$ exists and is Hamiltonian. Obviously a 1-skeleton of any triangulation of a closed surface is a connected and locally Hamiltonian graph. G is called a *triangulation graph* if it is the 1-skeleton of a triangulation.

The main theorem of this paper is the following

THEOREM 1. *If a connected and locally Hamiltonian graph G has n vertices, m edges and $m \leqslant 3n-6$ then G is an S^2 triangulation graph.*

Remark 1. Clearly the converse implication is also valid and an S^2 triangulation graph with n vertices has $3n-6$ edges.

Remark 2. Other easy characterizations of the S^2 triangulation graphs exist, e.g. such is every connected locally Hamiltonian and planar graph or every planar graph with $n \geqslant 4$ vertices and $3n-6$ edges (the last assertion and its generalization to the case of other 2-manifolds follows immediately from [10], pp. 24 and 61); for other characterization, see [1].

Theorem 1 and above Remarks yield the following corollary.

COROLLARY 1. *A connected and locally Hamiltonian graph having n vertices has at least $3n-6$ edges and more than $3n-6$ edges if and only if it is not planar.*

This reminds of the classical theorem of Kuratowski [5] (see also [3] and [4]) which says that *a graph is not planar if and only if it contains topologically K_5 or $K_{3,3}$* (K_n is the unique graph with n vertices and $\binom{n}{2}$ edges and $K_{3,3}$ is a graph with 9 edges and 6 vertices three of which are adjacent to each of the remaining three). Now numerous characterizations of planar graphs exist (see [3], [4], [5], [6], [7], [8], [12] and [13]; see also [2], [9] and [11] for methods for determining whether a given graph is planar).

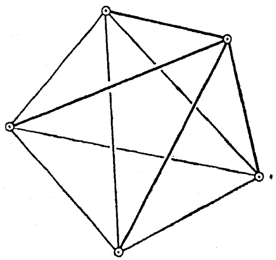

Fig. 1a. The graph K_5

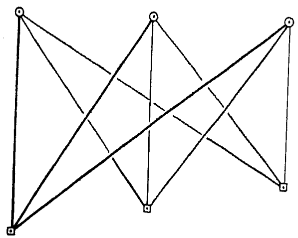

Fig. 1b. The graph $K_{3,3}$

Remark 3. The supposition of Theorem 1 and Corollary 1 that G be connected is essential since the graph K_5+K_4, where K_5 and K_4 are disjoint, is non-planar but satisfies the other conditions.

We will also prove an easier auxiliary theorem.

THEOREM 2. *If a locally Hamiltonian graph $G=\langle V, E\rangle$ has no subgraph homeomorphic with K_5 then for every $x \in V$ there is a single Hamiltonian circuit in $G(x)$.*

Remark 4. The converse implication is not valid since the triangulation graphs of closed surfaces may contain K_5. In Fig. 3 is shown a minimal graph F with property that it contains topologically K_5 and has a single Hamiltonian circuit of $F(x)$ for every vertex x of F.

Theorem 2 yields the following

COROLLARY 2. *If the graph G is an S^2 triangulation graph, then the simplicial complex, whose 1-skeleton is G and whose realization is S^2, is uniquely determined.*

2. Further definitions and lemmas. Sometimes we will write $x \in G$ if x is a vertex of the graph G, and $H \subseteq G$ if H is a subgraph of G.

The *degree* $d(x) = d(x, G)$ of the vertex x in G is the number of its neighbours in G.

L1. *Each graph containing exactly n vertices and $m < 3n$ edges has a vertex of degree $\leqslant 5$.*

Clearly, a graph G containing n vertices of degree $\geqslant 6$ has m edges where

$$m = \tfrac{1}{2}\sum_{x \in G} d(x, G) \geqslant \tfrac{1}{2}\cdot 6n = 3n\,.$$

Therefore lemma L1 holds.

The following lemma is obvious:

L2. *If for some $x \in G$ there exists a Hamiltonian circuit of $G(x)$ then* $d(x, G) \geqslant 3$.

Let **LH** denote the class of connected and locally Hamiltonian graphs, and let $\mathbf{LH}(n, m)$ be the subset of **LH** consisting of all graphs with n vertices and m edges. One can easily prove the following lemma:

L3. *If a vertex x is of degree 3 in a graph $G \in \mathbf{LH}$ with $n > 4$ vertices, then each neighbour of x in G has a degree $\geqslant 4$.*

Let $\Delta = \Delta(S^2)$ denote the class of 1-skeletons of triangulations of S^2. Obviously, by Euler's formula, the following lemma holds.

L4. *If $G_1 \in \Delta$ then $G_1 \in \mathbf{LH}(n, 3n-6)$ for some $n \geqslant 4$.*

If $x_i \neq x_j$ for $i, j = 0, \ldots, n$, $i \neq j$ and $\{i, j\} \neq \{0, n\}$ we put

$$P = [x_0, x_1, \ldots, x_n] = \big\langle \{x_0, x_1, \ldots, x_n\}, \big\{\{x_0, x_1\}, \{x_1, x_2\}, \ldots \{x_{n-1}, x_n\}\big\}\big\rangle\,.$$

Such a graph P is called a *path with ends x_0 and x_n* if $x_0 \neq x_n$ and is a circuit if $x_0 = x_n$ and $n \geqslant 3$. A path P with ends x_0 and x_n is denoted also by $P[x_0, x_n]$ or $P[x_n, x_0]$. The *length* $l(P)$ of the path (circuit) P is the number of its edges. A path $P = [y, z]$ with $l(P) = 1$ is also called an *edge*.

A topological realization G of an abstract planar graph in the two-sphere S^2 is called a *plane graph G*. The union of all G's simplexes is denoted by $|G|$. Each component of the set $S^2\setminus|G|$ is said to be a *face* of the plane graph G. Two faces D_1 and D_2 of G are *adjacent* if there exists an edge $P \subseteq G$ such that $|P| \subseteq \overline{D}_1 \cap \overline{D}_2$ (1).

A *sum* G_1+G_2 of two graphs $G_1 = \langle V_1, E_1\rangle$ and $G_2 = \langle V_2, E_2\rangle$ is a graph

$$G_1+G_2 = \langle V_1 \cup V_2, E_1 \cup E_2\rangle\,.$$

If $x \in G = \langle V, E\rangle$, then $G\setminus x$ will denote a subgraph of G spanned by a set $V\setminus\{x\}$.

(1) $\overline{D}$ denotes a closure of D.

3. Proof of Theorem 2. Let us assume that, for some $x \in G$, there exist two distinct Hamiltonian circuits H_1 and H_2 of $G(x)$. Now there exists an edge $[t, u]$ such that $[t, u] \subset H_1$ and $[t, u] \subsetneq H_2$. Hence H_2 is divided by t and u into two paths P_1 and P_2 both with ends t and u and both of length >1. Therefore there exists in H_1 an edge $[y_1, y_2]$ such that $y_1 \in P_1$, $y_2 \in P_2$ and $\{y_1, y_2\} \cap \{t, u\} = \emptyset$. So G contains topologically a K_5 (Fig. 2), contrary to the supposition.

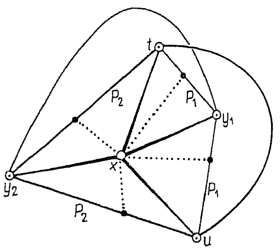

Fig. 2. The subgraph of G

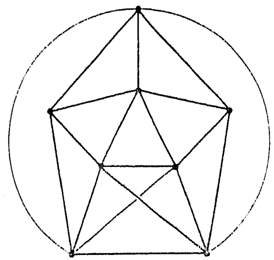

Fig. 3. The graph F

The following lemma is obvious (see Corollary 2).

L5. *If, for any $x_1 \in G_1 \in \Delta$, a Hamiltonian circuit of $G_1(x_1)$ contains an edge $[x_2, x_3]$ then G_1 has a face whose boundary is the circuit $[x_1, x_2, x_3, x_1]$.*

4. Proof of Theorem 1. Since if $G \in \mathbf{LH}$ then G has at least 4 vertices, therefore it suffices to prove that *if $G \in \mathbf{LH}(n, m)$, where $n \geqslant 4$ and $m \leqslant 3n-6$, then $G \in \Delta$.* We proceed by induction with respect to n. If $n = 4$ it is easily seen that $G = K_4 \in \Delta$.

Let us assume that the theorem is valid for $n \geqslant 4$. Let G be any graph of the class $\mathbf{LH}(n+1, m)$, where $m \leqslant 3(n+1)-6$. Let for any $y \in G$ the symbol $H(y)$ denote some fixed Hamiltonian circuit of $G(y)$. For the given G we construct an auxiliary graph G_1 belonging to the class $\mathbf{LH}(n, m_1)$, where $m_1 \leqslant 3n-6$. For any $y \in G_1$, the symbol $H_1(y)$ will denote some Hamiltonian circuit of $G_1(y)$.

Let x denote a vertex of G with minimal degree in G. By virtue of L1 and L2, the inequalities $3 \leqslant d(x) \leqslant 5$ hold. We consider three main cases: $d(x) = 3$, 4 or 5.

Case I: $d(x) = 3$. Let $G_1 = G \setminus x$. Let t, u and w be all the neighbours of x in G. Each of circuits $H(t)$, $H(u)$, $H(w)$ contains, by L3, more than 3 vertices. Obviously $[w, x, u] \subset H(t)$. We can put $H_1(t) = H(t)\setminus x + [w, u]$. Analogously we can define $H_1(u)$ and $H_1(w)$. We can put $H_1(y) = H(y)$ for any y such that $y \in G_1$ and $y \neq t, u, w$. Therefore $G_1 \in \mathbf{LH}(n, m_1)$, where $m_1 = m-3 \leqslant 3n-6$. By the induction hypothesis $G_1 \in \Delta$. Since $[w, u] \subset H_1(t)$, any plane graph G_1 has, by L5, a face with boundary $[t, u, w, t]$. So it is easily seen that $G \in \Delta$.

Case II: $d(x) = 4$. The subgraph $G(x)$ of G contains 4 vertices and 4, 5 or 6 edges. We consider two cases:

Case IIa: *$G(x)$ has 6 edges.* Let $G_1 = G \setminus x$. There exists $H_1(\bar{x})$ for any $\bar{x} \in G_1$; e.g. we can put $H_1(t) = H(t)\setminus x + [x_1, x_2]$ for any neighbour t of x in G, where x_1 and x_2 denote the neighbours of x in $H(t)$. Therefore $G_1 \in \mathbf{LH}(n, m_1)$, where $m_1 = m-4 < 3n-6$, contrary to induction hypothesis and L4. Thus case IIa is impossible.

Case IIb: *$G(x)$ contains 4 or 5 edges* (see Fig. 4, where intermittent line represents the eventual 5th edge of $G(x)$).

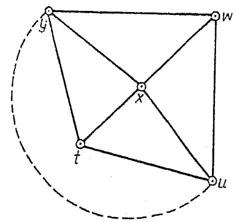

Fig. 4. The graph $G(x) + \mathrm{St}(x)$ [2]

Let $G_1 = G\setminus x + [t, w]$. We can prove that $G_1 \in \mathbf{LH}(n, m_1)$, where $m_1 = m-3 \leqslant 3n-6$. In particular we can put $H_1(t) = H(t)\setminus x + [u, w, y]$. Now, by the induction hypothesis and L5, we have: $G_1 \in \Delta$ and any plane graph G_1 has two adjacent faces with boundaries $[t, u, w, t]$ and $[t, w, y, t]$, respectively. Therefore $G \in \Delta$.

Case III: $d(x) = 5$. $G(x)$ contains 5 vertices and 5, 6, 7, 8, 9 or 10 edges. We consider several cases.

Case IIIa: *$G(x)$ has 10 or 9 edges.* Let K_5' denote a complete graph with 5 vertices such that $G(x) \subseteq K_5'$. Let $G_1 = G\setminus x + K_5'$. One can prove that $G_1 \in \mathbf{LH}(n, m_1)$, where $m_1 < 3n-6$, contrary to the induction hypothesis and L4. Hence case IIIa is impossible.

Case IIIb: *$G(x)$ contains 8 edges.* Let $G(x) \subset K_5'$. Let $G_1 = G\setminus x + K_5'$. One can prove similarly as before that $G_1 \in \mathbf{LH}(n, m_1)$, where $m_1 \leqslant 3n-6$. Therefore $G_1 \in \Delta$. On the other hand, since $K_5' \subseteq G_1$, G_1 is non-planar. This contradiction proves that the case IIIb is impossible.

Case IIIc: *$G(x)$ contains 7 edges and each vertex of $G(x)$ is of degree less than 4 in $G(x)$* (see Fig. 5).

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(2) $\mathrm{St}(x)$ denotes the closed star of x in G.

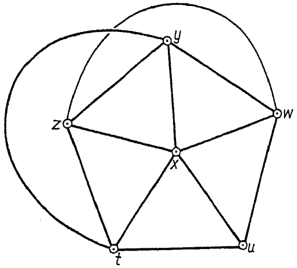

Fig. 5. The graph $G(x)+\mathrm{St}(x)$

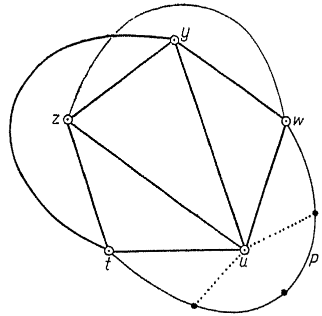

Fig. 6. The subgraph of G_1

Let $G_1 = G\backslash x+[u, z]+[u, y]$. We can show that $G_1 \in \mathbf{LH}(n, m_1)$, where $m_1 \leqslant 3n-6$. Now $G_1 \in \Delta$. On the other hand, graph G_1 contains topologically K_5 (see Fig. 6, where the path $P = P[t, w]$ is a subgraph of $H(u)$ such that the vertices $x, z, y, u \notin P$), contrary to previous conclusion. Thus case IIIc is impossible.

Case IIId: *$G(x)$ contains 7 edges and there exists a vertex of degree 4 in $G(x)$* (see Fig. 7). We consider three further cases.

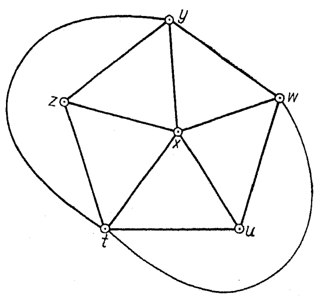

Fig. 7. The graph $G(x)+\mathrm{St}(x)$

Case IIId1: $[u, x, y] \subset H(t)$. Let $G_1 = G\backslash x+[u, y]+[u, z]$. One can prove that $G_1 \in \Delta$. One can put $H_1(u) = H(u)\backslash x+[t, z, y, w]$. From this and L5 it follows that any plane graph G_1 has three faces D_1, D_2 and D_3 with boundaries $[u, t, z, u]$, $[u, z, y, u]$ and $[u, y, w, u]$, respectively. D_1 and D_2 as well as D_2 and D_3 are adjacent. Now it is easy to see that $G \in \Delta$.

Case IIId2: $[z, x, w] \subset H(t)$. In this case we put $G_1 = G\backslash x+[z, u]+[z, w]$. Analogously as in IIId1 we can show that $G \in \Delta$.

Case IIId3: $[u, x, y] \subsetneq H(t)$ *and* $[z, x, w] \subsetneq H(t)$. Using the same method as in IIId1 or as in IIId2 one can prove that $G \in \Delta$.

Remark. One can show that the cases IIId1 and IIId2 are impossible.

Case IIIe: *$G(x)$ has 5 or 6 edges* (see Fig. 8, where the intermittent line represents eventual 6th edge of $G(x)$). Let $G_1 = G\backslash x+[u, y]+[u, z]$ or $G_1 = G\backslash x+[w, t]+[w, z]$. Analogously as in the previous cases we can show that $G \in \Delta$.

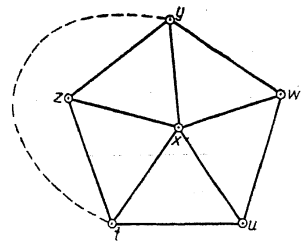

Fig. 8. The graph $G(x)+\mathrm{St}(x)$

All possible cases have been examined. Thus Theorem 1 is proved.

This paper contains the main results of my doctoral dissertation.

In conclusion I wish to express my thanks to Professors J. Górski and S. Gołąb and specially to J. Mycielski for their kind interest and many valuable advices.

Added in proof. The first of the propositions mentioned in Remark 2 is proved in [14].

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Inductive compactness as a generalization of semicompactness *

by

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1. Introduction. All spaces under discussion are separable metrizable spaces.

In [14] L. Zippin introduced the notion of a semicompact space. Namely, a space is *semicompact* if each point in the space has arbitrarily small neighborhoods with compact boundaries. Spaces which are semicompact are sometimes called *rim compact* or *peripherally compact*. In 1942, J. de Groot proved that a space is semicompact if and only if it can be compactified by adding a set of dimension no higher than zero [2]. This result is implicitly contained in H. Freudenthal's paper [3]. Freudenthal generalizes his results in [4]. For further generalizations, see K. Morita [10] and P. S. Aleksandrov and V. I. Ponomarëv [1]. The notion of semicompact space and the above characterization of such spaces suggest some other concepts. One concept is that of an "inductive compactness" analogous to inductive dimension. Another concept is the compactification of a space by adding a set of minimal dimension. Let us formalize these two concepts.

DEFINITION. INDUCTIVE COMPACTNESS. A space X is said to have *compactness* -1 if X is compact. A space X is said to have *compactness less than or equal to* n $(n \geqslant 0)$ if each point of X has arbitrarily small neighborhoods whose boundaries have compactness less than or equal to $n-1$. We use the notation $\operatorname{cmp} X \leqslant n$. A space X is said to have *compactness equal to* ∞ if $\operatorname{cmp} X \leqslant n$ is false for each integer n.

DEFINITION. A compact space Y is called an *n-compactification* of a space X if X is dense in Y and $\dim(Y \setminus X) = n$. By the *deficiency*

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