Connectivity functions and images on Peano continua

by

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The concept of a *connectivity* function is intermediate to that of a *continuous* function and that of a Darboux function and has proved useful and interesting in the study of fixed point properties of continua. For example, Hamilton [1] answered affirmatively the question [5] as to whether every connectivity map $f\colon I^n \to I^n$ has a fixed point. Stallings [7] augmented this argument, extended the result to polyhedra, and presented the notion of *almost continuous* function and some relations between these and connectivity functions. Hildebrand and Sanderson [2] established, among other things, that every connectivity retract of a (finite) polyhedron which has the fixed point property also has the fixed point property.

In this paper there are presented three results concerning connectivity functions and answers to two of the questions raised in [7]. In § 2 it is shown that every connected separable metric space is the image under a connectivity mapping of the closed number interval $[0, 1] = I$. In contrast to this it is shown in § 3 that a certain connected separable metric space (an explosion set in the plane) is *not* the image under a connectivity mapping of the square disc $I \times I = I^2$. In even stronger contrast to § 2, it is shown in § 4 that each connectivity retract of a unicoherent Peano continuum is a unicoherent Peano continuum. Using methods similar to those of § 2, there is described in § 5 a connectivity function $f\colon I \to I$ that is not almost continuous and which, considering I embedded in $I \times I$ as $I \times 0$, can not be extended to a connectivity map $I \times I \to I$. This answers negatively Questions 1 and 2 of [7], p. 261.

Added in proof: After this paper was submitted, a similar example by J. H. Roberts appeared in *Zero-dimensional sets blocking connectivity functions*, Fund. Math. 57 (1965), pp. 173-179.

1. Preliminaries.

DEFINITION 1. If X and Y are topological spaces and $f\colon X \to Y$ is a transformation, the *graph* of f is $\{(x, f(x))\colon x \in X\}$ considered as a subset of the topological product space $X \times Y$.

DEFINITION 2. The statement that f is a *connectivity function* means that for C a connected subset of X, $\{(x, f(x))\colon x \in C\}$ is a connected subset of $X \times Y$.

Since the projection map of $X\times Y\to Y$ is continuous, it follows immediately that for C connected in X, $f(C)$ is connected in Y (i.e. f is a Darboux function). The first lemma is an alteration of a lemma due to Stallings [7]. The proof here is essentially that of Stallings and is given because of the importance of the lemma to this paper. See also [6].

STALLINGS' LEMMA. *Suppose that X is a compact metric semi-locally-connected space and $f\colon X\to Y$ is a connectivity function where Y is a T_1 space. Then if C is a closed subset of Y and G' denotes the collection of components of $f^{-1}(C)$, the set G consisting of G' together with all the degenerate subsets of $X-f^{-1}(C)$ is a monotone uppersemicontinuous decomposition of X, and G' as a subset of the decomposition space is totally disconnected.*

(A compact metric space X is *semi-locally-connected* if X is connected and for each point x of X, every open set containing x contains an open set V containing x such that $X-V$ has only a finite number of components. A *monotone uppersemicontinuous decomposition* of a compact metric space X is a collection G of mutually exclusive closed and connected subsets of X whose sum is X such that if g is in G and U is an open set containing g there is an open set V containing g such that every element of G that intersects V is a subset of U.)

It is immediate that the elements of G are mutually exclusive and connected. Each element g of G is closed since if P is a limit point of g not belonging to g, then g is a component of $f^{-1}(C)$, P does not belong to $f^{-1}(C)$, and $g\cup\{P\}$ is connected but $f(g\cup\{P\})$ is not. Suppose that there is an open set U containing an element g of G such that every open set containing g intersects an element of G that is not a subset of U. Then for $\varepsilon>0$, there is an element of G which contains a point of $X-U$ and a point at a distance less than ε from a point of g. From compactness of X, one may select a sequence $\{g_n\}_{n=1}^{\infty}$ of elements of G which converges to a continuum Q that contains a point of g and a point of $X-U$. Then Q contains two points and is not a subset of an element of G (or G') and consequently contains a point z of $X-f^{-1}(C)$ and a point w distinct from X. Since X is semi-locally-connected, there is an open set N containing z but not w such that $X-N$ has only a finite number of components. There is a component E of $X-N$ and a subsequence $\{g_{n_i}\}_{i=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ such that each g_{n_i} intersects both E and N and is consequently a member of G' and a subset of $f^{-1}(C)$. Let $D=E\cup\{z\}\cup\bigcup_{n=1}^{\infty}g_{n_i}$. Then D is connected, but $\{(x,f(x))\colon x\in D\}$ is not since the open subset $N\times(Y-C)$ of $X\times Y$ contains only the point $(z,f(z))$ of that set. This is a contradiction and G is a monotone uppersemicontinuous decomposition of X.

If K is a component of G', then the sum of the elements of K is a connected subset of $f^{-1}(C)$ and is therefore a subset of an element of G' so that K is degenerate.

The following lemma and corollary are used in §§ 4 and 5.

LEMMA 1. *If K is a subset of a unicoherent Peano continuum H and $H-K$ is not connected, there is a continuum C lying in K such that $H-C$ is not connected.*

Since H is metric, there is a closed subset K' of K that separates H. Since H is a Peano continuum, there is a closed subset C of K' that separates $x\in H$ from $y\in H$ and is minimal relative to this property. If C is not connected, C is the sum of two disjoint closed sets C_1 and C_2 neither of which separates H; from the "Phragmen-Brouwer Property" of unicoherent Peano continua ([9], p. 47), $C_1\cup C_2=C$ does not separate H, which is a contradiction.

COROLLARY 1. *If W is a unicoherent Peano continuum which has no cut point, no totally disconnected subset of W separates W.*

2. Connectivity image of an interval.

THEOREM 1. *If Y is a connected separable metric space, there is a connectivity function with domain $I=[0,1]$ and range Y.*

In order for a function f with domain I to be a connectivity function, it is necessary and sufficient that the graph of f be connected. Therefore, to prove Theorem 1 it is only necessary to construct a connected subset Γ of $I\times Y$ such that for each t in I, $\pi_1^{-1}(t)$ contains only one point of Γ and such that $\pi_2(\Gamma)$ is Y. (π_1 and π_2 denote the projections of $I\times Y$ onto I and Y, respectively.) The procedure used is analogous to one used in [3] to construct certain real valued functions.

Let H denote the collection of all closed subsets h of $I\times Y$ such that the cardinality of $\pi_1(h)$ is $\mathfrak{c}$, the cardinality of the continuum. Since I and Y are separable metric, $I\times Y$ is separable metric, hence completely separable, and it follows that the cardinality of H is $\mathfrak{c}$. Assume H to be well ordered into a sequence $h_1, h_2, \ldots, h_\alpha, \ldots$ such that no element of H has $\mathfrak{c}$ predecessors. Let P_1 denote a point of h_1, and for the process of transfinite induction, for each ordinal α less than $\mathfrak{c}$ for which P_β has been defined for all ordinals $\beta<\alpha$, let P_α denote a point of h_α such that $\pi_1(P_\alpha)$ does not belong to $\bigcup_{\beta<\alpha}\pi_1(P_\beta)$. The fact that $\pi_1(h_\alpha)$ has cardinality $\mathfrak{c}$ and h_α does not have $\mathfrak{c}$ predecessors implies that such a point P_α may be selected. Let y_0 denote a specific point of Y and let

$$\Gamma=\bigcup_{\alpha<\mathfrak{c}}P_\alpha\cup\{(x,y_0)\colon x\in[I-\bigcup_{\alpha<\mathfrak{c}}\pi_1(P_\alpha)]\}.$$

It is possible that the induction process will exhaust the points of I in which case Γ is simply $\bigcup_{\alpha<\mathfrak{c}} P_\alpha$. The second set is used to insure the condition that $\pi_1(\Gamma)$ is I; that π_1/Γ is one-to-one is obvious from the construction. Since for y in Y, $I\times\{y\}$ belongs to H, it is immediate that $\pi_2(\Gamma)$ is Y, and it only remains to show that Γ is connected.

Suppose that Γ is not connected. Then Γ is the sum of two mutually separated sets A and B and since $I\times Y$ is metric, there exists mutually exclusive open sets α and β in $I\times Y$ containing A and B respectively. Let K denote $I\times Y-(\alpha\cup\beta)$. Then K is closed and does not intersect Γ, and since Γ intersects every element of H, K does not belong to H. Consequently, the cardinality of $\pi_1(K)$ is less than $\mathfrak{c}$ and it follows that $I-\pi_1(K)$ is dense in I. However, π_1 is an open mapping and since $\alpha\cup\beta$ contains Γ, $\pi_1(\alpha)$ and $\pi_1(\beta)$ are open sets which cover I and therefore $\pi_1(\alpha)\cap\pi_1(\beta)$ must exist and is open and contains an element t of $I-\pi_1(K)$. Then $\pi_1^{-1}(t)$ is a connected subset of $\alpha\cup\beta$ which intersects both α and β and this is a contradiction.

3. Connectivity image of a square disc. In view of Theorem 1, it appears that for very simple domain spaces X, the range Y of a connectivity function $f\colon X\to Y$ may be quite varied. However, it is found that by complicating the domain slightly, a greater restriction is placed on Y. Perhaps this is due to the increase in the number of or types of connected sets in the domain.

Let E denote an explosion point set in the plane with explosion point e. (I.e. E is a nondegenerate connected subset of the plane which contains a point e such that $E-e$ is totally disconnected. Such a set was described by Knaster and Kuratowski [4].) Then E is separable metric and connected and by Theorem 1 is the range of a connectivity function with domain an interval. However,

THEOREM 2. *There does not exist a connectivity function with domain $I\times I=I^2$ and range E.*

Suppose that f is a connectivity function which maps I^2 onto E. Let G' denote the collection of components of $f^{-1}(e)$ and let G denote G' together with the degenerate subsets of $I^2-f^{-1}(e)$. From Stallings' Lemma it follows that G is a monotone uppersemicontinuous decomposition of I^2 and G' is totally disconnected. From a theorem of Whyburn ([8], p. 172) it follows that each true cyclic element of the decomposition space (also denoted by G) is either a 2-sphere or a 2-cell. (A true cyclic element of a (compact) semi-locally-connected metric continuum M is a nondegenerate connected subset of M which has no cut point and which is not a proper subset of a connected subset of M which has no cut point. Every point of M either belongs to a true cyclic element of M or is a cut point of M or has arbitrarily small neighbourhoods with degenerate boundary ([8], p. 64).)

It is also true that G is a Peano continuum and there is an arc α in G. Since G' is totally disconnected, there is a cut point k of α that does not belong to G'. Then k is a degenerate subset of I^2, does not separate I^2 and therefore is not a cutpoint of G, and since "small" neighborhoods of k must have two boundary points in α, it follows that k must belong to a true cyclic element W of G.

Since W is either a 2-sphere or a 2-cell, and G' is totally disconnected, it follows that $W-G'$ is connected and since G is monotone, the set $(W-G')^*$ (the sum of the elements of $W-G'$) is a connected subset of I^2. Observe that $(W-G')^*$ does not intersect $f^{-1}(e)$ and therefore $f[(W-G')^*]$ is a subset of the totally disconnected set $E-e$. Since f preserves connected sets, $f[(W-G'^*])$ is a single point e'. However, W is a connected set and since G is monotone, W^* is a connected subset of I^2. It may be seen that $f(W^*)$ is $f[(W-G')^*]\cup f[(G')^*]$ which consists of two points (e and e') and is therefore not connected. This is a contradiction and Theorem 2 is proved.

4. Connectivity retracts of Peano continua.

DEFINITION 3. If Y is a subspace of a topological space X, then Y is a *connectivity retract* of X if there is a connectivity function $f\colon X\to Y$ such that for each point x in Y, $f(x)=x$.

Hildebrand and Sanderson ([2], Theorem 3.13) have shown that every connectivity retract of a (finite) polyhedron which has the fixed point property also has the fixed point property. (A continuum M has the fixed point property if for every continuous function $f\colon M\to M$, there is a point x in M such that $f(x)=x$.) Furthermore, they describe a Peano continuum M in the plane which has a connectivity retract which is the closure of the graph of $y=\operatorname{Sin}1/x$, $0<x\leqslant 1$. There was a natural suggestion that investigation of connectivity retracts of a square disc might yield some results relating to the long standing unsettled question: Does every nonseparating plane continuum have the fixed point property? However, one of the consequences of the next theorem is that such a program will not yield new results.

THEOREM 3. *Every connectivity retract of a unicoherent Peano continuum is a unicoherent Peano continuum.*

Suppose that X is a unicoherent Peano continuum, Y is a subspace of X and $f\colon X\to Y$ is a connectivity function such that for each x in Y, $f(x)=x$. It has already been shown ([2], Theorem 3.5) that Y is a continuum. It will be shown first that Y is locally connected and then that Y is unicoherent.

Y is locally connected. Suppose that P is a point of Y at which Y is not locally connected. Then there are open sets R and D containing P such that $\bar{D}$ (closure) is a subset of R and there is a sequence $M_1, M_2, \ldots$ of components of $Y \cap \bar{D}$ which converges to a nondegenerate continuum M which contains P but no point of $\bigcup_{n=1}^{\infty} M_n$ and such that no component of $Y \cap \bar{R}$ intersects two elements of $M_1, M_2, \ldots$ Let G' denote the collection of all components of $f^{-1}(Y-R)$ and let G denote G' together with all of the degenerate subsets of $f^{-1}(R)$. From Stallings' Lemma, G is a monotone uppersemicontinuous decomposition of X and G' is totally disconnected. Since each monotone decomposition of a unicoherent Peano continuum yields a unicoherent Peano continuum ([8], p. 138), G is a unicoherent Peano continuum and since unicoherence is cyclicly reducible ([8], p. 82), each cyclic element of G is unicoherent. Furthermore, each cyclic element of G is a Peano continuum ([8], p. 69).

Let $T\colon X \to G$ denote the continuous transformation associated with the uppersemicontinuous decomposition G of X. Observe that the sequence $T(M_1), T(M_2), \ldots$ converges to $T(M)$ in G and that since M is a subset of $R \cap Y$ and f is the identity on Y, M does not intersect $f^{-1}(Y-R)$ and it follows that T is one-to-one on M and consequently $T(M)$ is nondegenerate. From Theorem 4.1, page 70 of [8], there is a true cyclic element W of G containing $T(M)$ such that the sequence $W \cap T(M_1), W \cap T(M_2), \ldots$ converges to $T(M)$. Since G' is totally disconnected, it follows from Corollary 1 that $W-G'$ is connected.

Since $W \cap T(M_1), W \cap T(M_2), \ldots$ converges to $T(M)$, W obviously intersects infinitely many $T(M_n)$ and since M_n is a subset of R and f is the identity on M_n (a subset of Y), M_n does not intersect $f^{-1}(Y-R)$ so that $T(M_n)$ does not intersect G'. Consequently $W-G'$ intersects infinitely many $T(M_n)$ and because $W-G'$ is connected and T is monotone, $T^{-1}(W-G')$ is connected and must intersect infinitely many of $M_1, M_2, \ldots$ Observe also that $T^{-1}(W-G')$ is a subset of R. Now, $f[T^{-1}(W-G')]$ must be connected and a subset of $R \cap Y$, but since f is the identity on Y and no component of $R \cap Y$ intersects two of $M_1, M_2, \ldots$, $f[T^{-1}(W-G')]$ intersects infinitely many components of $R \cap Y$ and this is a contradiction.

Y is unicoherent. Suppose that Y is not unicoherent. Since unicoherence is cyclicly extensible ([8], p. 82), there is a cyclic element V of Y that is not unicoherent. Then V does not have the "Phragmen-Brouwer Property" ([9], pp. 48, 49) so there are two points P and Q of V and two mutually exclusive closed subsets A and B of V neither of which separates P from Q in V, but such that $A \cup B$ does separate P from Q in V. Notice also that $A \cup B$ separates P from Q in Y, for otherwise there would be an arc in $Y-A \cup B$ with endpoints P and Q and each such arc is a subset of V. Let G' denote the collection of components of $f^{-1}(A \cup B)$ and let G denote G' together with all of the degenerate subsets of $X-f^{-1}(A \cup B)$. Since f preserves connected sets, each element of G' is a component of $f^{-1}(A)$ or of $f^{-1}(B)$. As before, G is a monotone uppersemicontinuous decomposition of X, G' is totally disconnected, and the decomposition space is a unicoherent Peano continuum and each cyclic element of G is a unicoherent Peano continuum. Also, $T\colon X \to G$ will denote the associated continuous transformation.

It will next be shown that $T(P)$ and $T(Q)$ belong to the same cyclic element of G. Using an alternate definition of cyclic element, two points of a Peano continuum M belong to the same cyclic element of M if no point separates those two points in M ([8], p. 66). Consequently, if no cyclic element of G contains $T(P)$ and $T(Q)$, there is a "point" g of G that separates $T(P)$ from $T(Q)$ in G. Either g is degenerate, $\{x\}$, or g belongs to G'. If g is $\{x\}$, $V-x$ is connected and $T(V-x)$ is a connected subset of $G-g$ that contains $T(P)$ and $T(Q)$. If g belongs to G' and is a component of $f^{-1}(A)$, then $g \cap V$ is a subset of A and does not separate P from Q in V; consequently g does not separate $T(P)$ from $T(Q)$ in $T(V)$, or in G. A similar contradiction is reached if g is a component of $f^{-1}(B)$.

Let W denote the cyclic element of G that contains $T(P)$ and $T(Q)$. Now, G' is totally disconnected and W is a unicoherent Peano continuum which has no cut point. From Corollary 1, $W-G'$ is connected. Since T is monotone, $T^{-1}(W-G')$ is connected and since f preserves connected sets, $f[T^{-1}(W-G')]$ is connected, but $f[T^{-1}(W-G')]$ is a connected subset of $Y-A \cup B$ that contains P and Q and this is a contradiction.

5. A connectivity function $f\colon I \to I$ that is not almost continuous.

DEFINITION 4. A transformation $f\colon X \to Y$ between the topological spaces X and Y is *almost continuous* if every open subset of $X \times Y$ that contains the graph of f also contains the graph of a continuous function from X to Y.

It follows from Corollary 1 of [7] that every real valued connectivity map on I^n ($n \geqslant 2$) is almost continuous and Question 1 at the end of the paper asks (in a slightly more general setting) whether the same is true for $n = 1$. Question 2 is an alternate way of visualizing the problem and asks whether, considering I embedded in I^2 as $I \times 0$, a connectivity map $I \to X$ can be extended to a connectivity map $I^2 \to X$. That the answer to both questions is *no* may be seen by the following example.

The procedure is to first describe a Cantor set C in I^2 which intersects the graph of every continuous function from I to I. Then using a slight alteration of the methods of § 2, the graph, Γ, of a connectivity function $f\colon I\to I$ is described as a subset of I^2-C. Obviously then I^2-C is an open subset of I^2 that contains the graph of f and does not contain the graph of a continuous function from I to I so that f is not almost continuous.

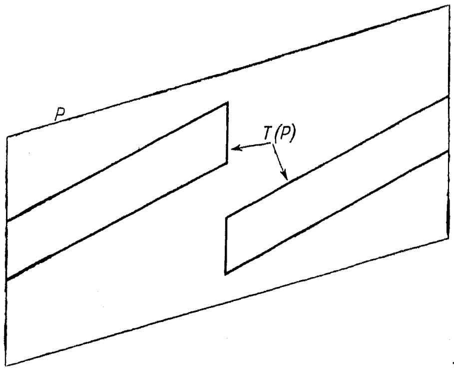

Fig. 1

Description of C. See Figure 1. For $P=\langle(a,b),(a,b+w),(c,d),(c,d+w)\rangle$ $(w>0)$ a parallelogram disc in the plane, $T(P)$ will denote the set consisting of the two parallelogram discs

$$\left\langle\left(a,b+\frac{3w}{8}\right),\left(a,b+\frac{5w}{8}\right),\left(\frac{a+c}{2},\frac{b+d}{2}+\frac{5w}{8}\right),\left(\frac{a+c}{2},\frac{b+d}{2}+\frac{7w}{8}\right)\right\rangle$$

and

$$\left\langle\left(\frac{a+c}{2},\frac{b+d}{2}+\frac{w}{8}\right),\left(\frac{a+c}{2},\frac{b+d}{2}+\frac{3w}{8}\right),\left(c,d+\frac{3w}{8}\right),\left(c,d+\frac{5w}{8}\right)\right\rangle.$$

If K is a collection of such parallelogram discs, $T(K)$ will denote $\bigcup_{P\in K}T(P)$.

Let $K_0,K_1,K_2,\ldots$ be such that K_0 is the parallelogram disc $\langle(0,-1/2),(0,1/2),(1,1/2),(1,3/2)\rangle$ and for n a positive integer, K_n is $T(K_{n-1})$. Then C is $\bigcap_{n=1}^{\infty}K_n^*$. Note that since K_n^* $(n=1,2,\ldots)$ intersects the graph of every continuous function from I to I, C does also.

Description of Γ. Let Γ_1 denote the collection of all the centroids of all the parallelograms in $\bigcup_{n=1}^{\infty}K_n$ together with the centroid of K_0. (Γ_1 is a countable subset of Γ.) Let π_1 (π_2) denote the projection map of I^2 onto the first (second) coordinate space. Let H denote the collection of all closed subset h of I^2-C such that the cardinality of $\pi_1(h)$ is $\mathfrak{c}$, the cardinality of the continuum. Using a transfinite induction process similar to that of § 2, a subset Γ_2 of I^2-C may be selected such that (1) Γ_2 intersects each element of H and (2) if x and y belong to Γ_2, $\pi_1(x)$ is not in $\pi_1(\Gamma_1)$ and is not $\pi_1(y)$. Then

$$\Gamma=\Gamma_1\cup\Gamma_2\cup\{(t,1)\colon t\in[[0,1/2]-\pi_1(\Gamma_1\cup\Gamma_2)]\}\cup$$
$$\cup\{(t,0)\colon t\in[[1/2,1]-\pi_1(\Gamma_1\cup\Gamma_2)]\}.$$

The last two sets insure that $\pi_1(\Gamma)$ is $[0,1]$ and are selected so as to avoid points of C. It is obvious that Γ is the graph of a function $f\colon I\to I$ and as previously noted, f is a connectivity function if and only if Γ is connected.

Γ is connected. Suppose that Γ is not connected. Then Γ is the sum of two mutually separated sets A and B and there exist mutually exclusive open subsets α and β of I^2 containing A and B, respectively. Let K denote $I^2-(\alpha\cup\beta)$. Then from Lemma 1, K contains a continuum k that separates I^2, but it is shown in the next paragraph that k is a proper subset of vertical interval in I^2 which presents a contradiction and the argument is complete.

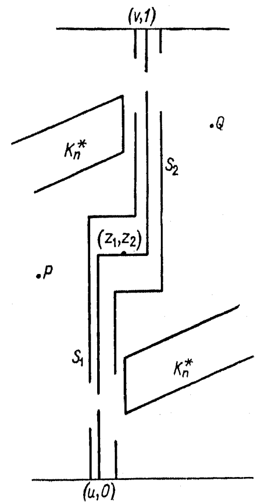

Fig. 2

Since k does not intersect Γ, it is only necessary to show that $\pi_1(k)$ is degenerate. Suppose that P and Q are two points of k such that $\pi_1(P)$ is not $\pi_1(Q)$. Then (Figure 2) there is a point (z_1,z_2) of Γ_1 and a positive number ϱ such that (1) the square disc $\{(x_1, x_2) \in I^2\colon |x_1-z_1| \leqslant 2\varrho, |x_2-z_2| \leqslant 2\varrho\}$ does not intersect K and (2) the interval $\{x_1 \in I\colon |x_1-z_1| \leqslant 2\varrho\}$ is between $\pi_1(P)$ and $\pi_1(Q)$. Let S_1 and S_2 denote respectively the strips

$$\{(x_1, x_2) \in I^2\colon z_1-2\varrho \leqslant x_1 \leqslant z_1-\varrho,\ 0 \leqslant x_2 \leqslant z_2\},$$
$$\{(x_1, x_2) \in I^2\colon z_1+\varrho \leqslant x_1 \leqslant z_1+2\varrho,\ z_2 \leqslant x_2 \leqslant 1\}.$$

Then $K \cap S_1$ and $K \cap S_2$ are closed subsets of I^2-C that do not intersect Γ and consequently $\pi_1(K \cap S_1)$ and $\pi_1(K \cap S_2)$ are both of cardinality less than $\mathfrak{c}$. Therefore there are points u and v of I such that $z_1-2\varrho \leqslant u \leqslant z_1-\varrho$, $z_1+\varrho \leqslant v \leqslant z_1+2\varrho$, and u does not belong to $\pi_1(K \cap S_1)$ and v does not belong to $\pi_1(K \cap S_2)$. However $\{(u, x_2)\colon 0 \leqslant x_2 \leqslant z_2\} \cup \cup \{(x_1, z_2)\colon u \leqslant x_1 \leqslant v\} \cup \{(v, x_2)\colon z_2 \leqslant x_2 \leqslant 1\}$ is an arc which separates P from Q in I^2 and does not intersect k, which is a contradiction.

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Locally Hamiltonian and planar graphs

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1. Introduction. In this paper we consider finite graphs which contain no loops and no parallel edges. By *a graph* we mean an ordered pair $\langle V, E\rangle$ where V is a finite non-empty set (the *set of vertices*) and E is a set of two-element subsets of V (the *set of edges*). Thus a graph is a zero- or one-dimensional simplicial complex and sometimes, when misunderstandings are improbable we will identify it to its topological realization. By a *circuit* we mean a graph whose topological realization is a simple closed curve. A graph is called *planar* if it has a homeomorphism into the two-sphere S^2.

Two vertices $x, y \in V$ are called *adjacent* (*neighbours*) in $G = \langle V, E\rangle$ if $\{x, y\} \in E$. A graph $H = \langle U, D\rangle$ is called a *subgraph* of a graph $G = \langle V, E\rangle$, or G is said to contain H, if $U \subseteq V$ and $D \subseteq E$. A subgraph H of G is said to be *spanned* by a set $U \subseteq V$ if $H = \langle U, \{e\colon e \in E$ and $e \subseteq U\}\rangle$. The subgraph of $G = \langle V, E\rangle$ spanned by a set of vertices adjacent to a vertex $x \in V$, i.e. by the set $\{y\colon \{x, y\} \in E\}$, is denoted by $G(x)$.

A graph G is called *Hamiltonian* if it has a *Hamiltonian circuit*, i.e. a circuit whose set of vertices is all the set V. G is called *locally Hamiltonian* if for every $x \in V$ the graph $G(x)$ exists and is Hamiltonian. Obviously a 1-skeleton of any triangulation of a closed surface is a connected and locally Hamiltonian graph. G is called a *triangulation graph* if it is the 1-skeleton of a triangulation.

The main theorem of this paper is the following

THEOREM 1. *If a connected and locally Hamiltonian graph G has n vertices, m edges and $m \leqslant 3n-6$ then G is an S^2 triangulation graph.*

Remark 1. Clearly the converse implication is also valid and an S^2 triangulation graph with n vertices has $3n-6$ edges.

Remark 2. Other easy characterizations of the S^2 triangulation graphs exist, e.g. such is every connected locally Hamiltonian and planar graph or every planar graph with $n \geqslant 4$ vertices and $3n-6$ edges (the last assertion and its generalization to the case of other 2-manifolds follows immediately from [10], pp. 24 and 61); for other characterization, see [1].