Dimension lowering monotone non-compact mappings of E^n

by

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1. The following paper is related to the well-known and still unsolved monotone mapping problem. That is, if f is a monotone map of E^n onto E^n, then is f a compact map [7]? Whyburn [6] has shown that for $n = 2$ any monotone mapping of E^2 onto E^2 is necessarily a compact mapping. E. H. Connell [1] has shown that if f is a monotone mapping of E^n onto E^n such that for $p \in E^n$, $H_n[f^{-1}(p)] = 0$ for $n = 1, 2, \ldots$, then f is compact.

The close relationship between upper semi-continuous decompositions and the monotone mapping theorem are well known ([5], [7]). In fact the theorems of Proizvolov [4] would give the monotone mapping problem as an immediate corollary if all the results were correct. In [2], I point out the existence of particularly nice counter examples to the two main theorems and to two corollaries of Proizvolov. Lemma 2, below, also furnishes a counter example, but the mapping there is not as simple and does not yield as much further information as in [3].

The main result is that for $m > n \geqslant 3$ there exists a monotone non-compact mapping of E^m onto E^n. Also we get that for $m > n \geqslant 3$ there exists a monotone non-compact mapping of E^m onto S^3 and this generalizes easily to a monotone non-compact mapping of E^m onto $S^3 \times E^k$, where $3 + k < m$.

2. The standard terminology E^n, S^n will denote euclidean n-space and the n-sphere, respectively. E^n_+ will denote the closed half space of E^n given by $\{x \in E^n |\ x = (x_1, \ldots, x_n)$ and $x_n \geqslant 0\}$.

A mapping f taking X onto Y is *monotone* if counter images of points in Y are continua (that is, for $y \in Y$, $f^{-1}(y)$ is a compact connected set) ([4], [6]). A mapping f taking X into Y is *compact* if for each compact set $A \subset Y$, $f^{-1}(A)$ is compact. We will call a map f of X into Y *non-compact* if f is not compact (that is, if there exists a compact set $A \subset Y$ such that $f^{-1}(A)$ is not a compact subset of X).

A compact set X is an *absolute retract*, denoted AR, if and only if for each homeomorphism of X onto a subset $h(X)$ of a compact space Y,

$h(X)$ is a retract of Y (that is, there exists a continuous mapping $r: Y \to h(X)$ such that $r(y) = y$ for each point $y \in h(X)$).

3. Lemma 1. *For each $m > n \geqslant 1$ there exists a continuous mapping f of E^m onto E^n_+. Furthermore, for each point $y \in E^n_+$ we have the following:*

$$f^{-1}(y) = \begin{cases} \text{point} \in E^m & \text{if} \quad y \in E^{n-1} \times \{0\} \subset E^n_+, \\ \text{an } (m-n)\text{-sphere} \subset E^m & \text{if} \quad y \in E^n_+ - (E^{n-1} \times \{0\}). \end{cases}$$

Proof of Lemma 1. The mapping f is immediate if we consider E^m as $E^n_+ \times S^{m-n}$ with $y \times S^{m-n}$ identified to a point for each $y \in E^{n-1} \times \{0\} \subset E^n_+$. Hence for $x \in E^m$, x corresponds to a point $(y, p) \in E^n_+ \times S^{m-n}$ and we define f taking E^m onto E^n_+ by letting $f(x) = y$.

Lemma 2. *There exists a 1-1 continuous map g taking E^3_+ onto E^3.*

Proof of Lemma 2. Let $U = \operatorname{int} I^3$, the unit cube in E^3. Let D denote the front face of I^3. Since $U + \operatorname{int} D$ is homeomorphic to E^3_+, it will suffice to show there exists a 1-1 continuous map g taking $U + \operatorname{int} D$ onto an open subset V of E^3 which is homeomorphic to E^3. The map g will be gotten by describing a sequence of 1-1 continuous maps $\{g_i\} = 0, 1, 2, \ldots$, where g_i deforms the image of g_{i-1} by only moving points in an open set N_{i-1} of the image g_{i-1}, so that $g_i(N_{i-1}) \subset$ open set $M_i \subset E^3$, and the diameters of N_{i-1} and $M_i \to 0$ as $i \to \infty$. The first map g_0 will deform $V + \operatorname{int} D$ into an open solid torus T plus an open annulus $W \subset \operatorname{Bd} T$ where W bounds the "hole" of T. The remainder of the maps will give us a process of "filling" the "hole" which will be the set V.

Now for some details. In order to describe the maps g_i it will be convenient to lable certain points of $\operatorname{Bd} D$. Let E denote the top edge of $\operatorname{Bd} D$, with left end point e and right end point c. Let f and d denote the left and right end points, respectively, of the bottom edge of $\operatorname{Bd} D$. Let $[u_i, s_i, t_i, v_i]$ be a countable infinite collection of 4-tuples of points of E (always ordered as given from left to right) with the following properties:

(1) the interval $[u_0, v_0]$ of E lies in the interior of E;

(2) for $i \geqslant 1$, $[u_i, v_i] \subset (u_{i-1}, s_{i-1})$;

(3) diameter of $[u_i, v_i] < \dfrac{1}{4^{i+1}}$, $i \geqslant 0$.

Also, for each i, let us denote the semi-circular open arc in $\operatorname{int} D$, having u_i and v_i as limit points, by A_i and the region of $\operatorname{int} D$ bounded by $A_i \cup [u_i, v_i]$ by R_i. Finally, let ab be a vertical closed spanning segment of D so that the top point a of ab lies in E between e (the left end point of E) and u_0.

The map g_0 is gotten as follows. g_0 will fold $U + \operatorname{int} D$ around, preserving levels, so that the right vertical edge cd of $\operatorname{Bd} D$ matches up with ab and the right face of I^3 matches up with the region of $\operatorname{int} D$ bounded by $\widehat{ea} + \widehat{ab} + \widehat{fb} + \widehat{ef}$, say D_0. Then $g_0(U + D_0)$ will be the open solid torus T and $g_0(D - D_0)$ will be the open annulus W bounding the "hole" in T.

In describing the maps g_i $(i \geqslant 1)$, we will only modify the image of g_{i-1} in a small neighborhood N_{i-1} (in the image of g_{i-1}) of $g_{i-1}(R_{i-1})$. g_1 will make use of N_0 to fill most of the "hole" of T—in so doing form another such torus (except now with a smaller curved "hole"). g_2 will make use of N_1 to fill most of the new "hole" and again form another such torus with a smaller curved "hole" yet. The size and shape of these "holes" will be of the magnitude of A_{i-1} for each g_i.

g_1 will fold over $g_0(R_0)$ (folding forward) so that the part of the top edge of $g_0(R_0)$ corresponding to the open interval (s_0, t_0) of E is matched up with A_0 and points of N_0 are used to fill in the resulting "cylinder" corresponding to $g_0(K_0)$, where K_0 is the region of $\operatorname{int} D$ bounded by $\widehat{ab} + \widehat{au_0} + A_0 + \widehat{v_0 c} + \widehat{cd} + \widehat{bd}$. The annulus W_1 bounding the "hole" of the new torus formed will be given by $g_1(g_0(R_0 + A_0))$, where W_1 curves around in a similar manner as A_0. $g_1(N_0)$ will be contained in the "hole" of T; this will correspond to the open set M_1 in E^3 as promised.

The maps g_i $(i \geqslant 2)$ will now merely repeat the above procedure folding over the decreasingly smaller regions corresponding to R_{i-1}, and using N_{i-1} to fill in the resulting decreasingly smaller "cylinders" formed. A neighborhood of the region in E^3 filled in will be our M_i's of the first paragraph of the proof. Clearly by taking the foldings as nice as possible we can insure that N_i has diameter $< 1/4^i$ (diameter $R_i < 1/4^{i+1}$) and that M_i has diameter $< 1/4^{i-1}$. Each g_i is 1-1 and continuous, and only moves points in N_{i-1} with $g_i(N_{i-1}) \subset M_i$. Also the sets N_{i-1} and $M_i \to 0$ as $i \to \infty$ and in fact they will converge to the point which will correspond to $\bigcap_{i=1}^{\infty} [u_i, v_i]$ which does not lie in the desired set $V = T$ plus "filled hole".

If we define $h_i = g_i\Big(g_{i-1}\big(\ldots\big(g_0(U + \operatorname{int} D)\big)\ldots\big)\Big)$, then $g = \lim_{i \to \infty} h_i$. Note $g = h_i$ outside of $h_i^{-1}(N_{i-1})$ $(i \geqslant 1)$.

Remark 1. In [3] it is shown that the 1-point compactification of $g(\operatorname{int} D)$ is a compact AR (in fact, general conditions are given so that this is always true). Also the explicit compact AR is given.

Lemma 3. *There exists a 1-1 continuous map G taking E^n_+ onto E^n $(n \geqslant 3)$.*

Proof of Lemma 3. Considering E^n_+ as $E^3_+ \times E^{n-3}$, $G(x) = G(p, q) = \big(g(p), q\big)$ where $p \in E^3_+$, $q \in E^{n-3}$ and g is the map of Lemma 2.

Remark 2. In [3] it is shown that no such map exists for $n = 1$ or 2.

LEMMA 4. *There exists a 1-1 continuous map h taking E^3_+ onto S^3.*

Proof of Lemma 4. The description of h here will be much simpler than that of g in Lemma 2. The author discovered the map h first and in trying to prove you could not get a 1-1 continuous of E^3_+ onto E^3 discovered the map g also.

In order to describe h, let us consider a solid closed cone with the disk forming the base of the cone removed. Since this is homeomorphic to E^3_+, it will suffice to get a map h of this set onto S^3. Also for $p \in$ int (cone), if we can get a 1-1 continuous of the above set $-\{p\}$ onto E^3 so that the map is the identity in a neighborhood of p, we can also get our desired h. Or, another way of looking at this is to consider p as the point at infinity and then it will suffice to get a 1-1 continuous map of E^3—{open solid cone C + closed disk E forming base of $\bar{C}$} $\equiv M$ onto E^3. The boundary of M is an open disk D and under the 1-point compactification of $E^3 = E^3 + \{\omega\}$, $M + \{\omega\}$ is topologically equivalent to E^3_+.

The mapping $\hat{h}$ of M onto E^3 will be described by making use of the construction of a contractible 2-complex known as the dunce hat. The dunce hat is the decomposition space gotten by identyfying a line segment L corresponding to a radius of a circular closed disk F with its boundary. That is, the dunce hat is formed from F by wrapping L around BdF in a smooth manner so that under the resultant identification, the end points of L have been identified, along with L to BdF.

We can repeat this same identification on the open disk D of BdM except using a segment L in D as a half-open line segment. Let M' denote M under such an identification (in fact M' can be gotten from M in E^3 by merely deforming D and hence a neighborhood of D in M so that D becomes a dunce hat). Let h' be the natural 1-1 continuous map from M onto M'. M' is topologically equivalent to E^3—{open solid cone C' + open disk E' forming base of $\bar{C}'$}. Here, however, the cone C' is twisted so as to lie "inside" the dunce hat. That is, $\bar{C}'$ will contain the dunce hat. Let h'' correspond to the 1-1 continuous map which results by pushing up points near the base of C' in M' to fill the hole and hence get E^3. Then the desired $\hat{h}$ taking M onto E^3 is merely $\hat{h} = h'' \circ h'$. The map h taking E^3_+ onto S^3 results by extending $\hat{h}$ to take ∞ onto ∞.

LEMMA 5. *There exists a 1-1 continuous map 1-1 taking E^n_+ onto $S^3 \times E^{n-3}$ $(n \geqslant 3)$.*

Proof of Lemma 5. Considering E^n_+ as $E^3_+ \times E^{n-3}$, $H(x) = H(y, p) = (h(y), p)$ where $y \in E^3_+$, $p \in E^{n-3}$ and h is the map of Lemma 4.

THEOREM 1. *For $m > n \geqslant 3$ there exists a monotone non-compact map F taking E^m onto E^n.*

Proof of Theorem 1. By Lemma 1 there exist a continuous map f of E^m onto E^n_+ so that inverses of points in E^n_+ are points or $(m-n)$-spheres. By Lemma 3 there exists a 1-1 continuous map G taking E^n_+ onto E^n. Then $F = G \circ f$. F is monotone since inverses of points are points or $(m-n)$-sphere. F is non-compact since G is (that is, consider any ball in E^n intersecting $G(E^{n-1} \times \{0\})$).

THEOREM 2. *For $m > n \geqslant 3$ there exists a monotone non-compact map F taking E^m onto $S^3 \times E^{n-3}$.*

Proof of Theorem 2. Let f be as in the proof of Theorem 1. By Lemma 5 there exists a 1-1 continuous map H taking E^n_+ onto $S^3 \times E^{n-3}$. Then $F = H \circ f$.

References

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