THEOREM 4. *If X is a set with three or more elements, then the lattice of topologies on X is not distributive.*

QUESTION. In the lattice of topologies on an infinite point set X, does every complemented topology, which is neither discrete nor trivial, have at least two complements?

References

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On the "indeterminate case" in the theory of a linear functional equation

by

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Dedicated to Professor A. D. Wallace on the occasion of his 60-th birthday

Introduction. In the present paper we are concerned with the linear functional equation of the first order (cf. [3], [6])

$$\varphi[f(x)] = g(x)\varphi(x) + F(x), \tag{1}$$

where $\varphi(x)$ is an unknown function. The values of the functions $\varphi(x)$, $g(x)$, $F(x)$ lie in the field $\mathcal{K}$ of real or complex numbers, x is a real variable, and $f(x)$ is a real-valued function of a real variable.

We shall consider equation (1) in an interval $[a, b)$. The functions $f(x)$, $g(x)$ and $F(x)$ will be subjected to the following conditions:

(i) *The function $f(x)$ is continuous and strictly increasing in $[a, b)$, $a < f(x) < x$ in (a, b), $f(a) = a$.*

(ii) *The function $g(x)$ is continuous in $[a, b)$, $g(x) \neq 0$ in $[a, b)$.*

(iii) *The function $F(x)$ is continuous in $[a, b)$.*

A theory of continuous solutions of equation (1) has been developed in [6] under the condition that $|g(a)| \neq 1$. The case $|g(a)| = 1$ was left as an indeterminate one. In the present paper we are going to investigate the behaviour of continuous solutions of equation (1) in this indeterminate case.

The case where $g(x) \equiv 1$ or $g(x) \equiv -1$ has already been treated more in detail [1], [2], [7], [8].

It is a characteristic feature of functional equations of type (1) that in general their solution depends on an arbitrary function (cf. e.g. [5]). However, the expression "solution depends on an arbitrary function" is not quite clear and therefore it will be given here a precise meaning.

DEFINITION. We say that *equation* (1) *has in an interval I a continuous solution depending on an arbitrary function*, if there exists an interval $J \subset I$ such that every continuous function on J can be extended (not necessarily uniquely) to a continuous solution of equation (1) in I.

In particular, the following result is known (cf. [6]).

LEMMA 1. *Under hypotheses* (i)-(iii) *equation* (1) *has in* (a, b) *a continuous solution depending on an arbitrary function. More precisely, for every* $x_0 \in (a, b)$ *and every function* $\varphi_0(x)$ *continuous in* $[f(x_0), x_0]$ *and fulfilling the condition*

$$\varphi_0[f(x_0)] = g(x_0)\varphi_0(x_0) + F(x_0) \tag{2}$$

there exists exactly one function $\varphi(x)$, *continuous in* (a, b), *satisfying equation* (1) *and such that*

$$\varphi(x) = \varphi_0(x) \quad \textit{for} \quad x \in [f(x_0), x_0]. \tag{3}$$

As we shall see, the above lemma will not remain true if we replace the interval (a, b) by the interval $[a, b)$.

§ 1. Let $f^n(x)$ denote the nth iterate of the function $f(x)$:

$$f^0(x) = x, \quad f^{n+1}(x) = f[f^n(x)], \quad n = 0, 1, 2, \ldots$$

According to (i) the iterates $f^n(x)$ are defined, continuous and strictly increasing in $[a, b)$. Moreover, one can easily prove the following

LEMMA 2. *If the function* $f(x)$ *fulfils hypothesis* (i), *then for every* $x \in (a, b)$ *the sequence* $f^n(x)$ *is strictly decreasing and* $\lim_{n\to\infty} f^n(x) = a$.

We put

$$G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)], \quad n = 1, 2, 3, \ldots \tag{4}$$

The functions $G_n(x)$ are defined and continuous in $[a, b)$. There are the following three possibilities:

(A) The limit

$$G(x) = \lim_{n\to\infty} G_n(x) \tag{5}$$

exists. Moreover, $G(x)$ is continuous in $[a, b)$, $G(x) \neq 0$ in $[a, b)$.

(B) There exists an interval $I \subset (a, b)$ such that $\lim_{n\to\infty} G_n(x) = 0$ uniformly in I.

(C) Neither (A) nor (B) occurs.

The above three possibilities determine the behaviour of continuous solutions of the homogeneous equation

$$\varphi[f(x)] = g(x)\varphi(x) \tag{6}$$

in $[a, b)$. Namely, we have the following

THEOREM 1. *Let hypotheses* (i) *and* (ii) *be fulfilled. In case* (A) *equation* (6) *has a one-parameter family of continuous solutions in* $[a, b)$. *For every number* $c \in \mathcal{K}$ *there exists a unique function* $\varphi(x)$, *continuous in* $[a, b)$, *satisfying equation* (6) *in* $[a, b)$ *and fulfilling the condition* $\varphi(a) = c$. *This solution is given by the formula*

$$\varphi(x) = \frac{c}{G(x)},$$

where $G(x)$ *is defined by* (4) *and* (5).

In case (B) *equation* (6) *has in* $[a, b)$ *a continuous solution depending on an arbitrary function. Every continuous solution of equation* (6) *in* $[a, b)$ *fulfils then the condition*

$$\varphi(a) = 0. \tag{7}$$

In particular, if there exists an $x_0 \in (a, b)$ *such that* $[f(x_0), x_0] \subset I$, *then for every function* $\varphi_0(x)$, *continuous in* $[f(x_0), x_0]$ *and fulfilling the condition*

$$\varphi_0[f(x_0)] = g(x_0)\varphi_0(x_0)$$

there exists exactly one function $\varphi(x)$ *continuous in* $[a, b)$, *satisfying equation* (6) *in* $[a, b)$ *and fulfilling condition* (3).

In case (C) *the function* $\varphi(x) \equiv 0$ *is the only continuous solution of equation* (6) *in* $[a, b)$.

The proof of the above theorem does not differ from that given in [9] (theorem 9.1) for the particular case $g(x) = s/f'(x)$ and is therefore omitted.

Theorem 1 is valid independently of the value of $g(a)$. If $|g(a)| \neq 1$, then we have the case considered in [6]. It is easily seen that if $|g(a)| < 1$, then case (B) occurs, and if $|g(a)| > 1$, then (C) is the occurring case.

Case (A) may occur only if

$$g(a) = 1. \tag{8}$$

Only in this case the product $\prod g[f^i(x)]$ may converge (cf. (ii) and lemma 2). But condition (8) alone does not guarantee that case (A) occurs. If (8) holds, then we may have all the three cases (A), (B), (C) (cf. [9]). Below we prove a theorem which gives a sufficient condition for case (A).

Suppose that the functions $f(x)$ and $g(x)$ fulfil the following conditions.

(iv) *There exists a constant* ϑ,

$$0 < \vartheta < 1, \tag{9}$$

such that the inequality

$$|f(x)-a| \leqslant \vartheta |x-a| \tag{10}$$

holds in a neighbourhood of the point a (1).

(v) *There exist constants $M > 0$ and $\mu > 0$ such that*

$$|g(x)-1| \leqslant M|x-a|^{\mu} \tag{11}$$

in a neighbourhood of the point a.

THEOREM 2. *If hypotheses* (i), (ii), (iv), (v) *are fulfilled, then case* (A) *occurs.*

Proof. We shall prove that the sequence $G_n(x)$ converges uniformly in every interval $[a, d]$, $a < d < b$, to a function $G(x) \neq 0$. For this purpose it is enough to show that the series

$$\sum_{n=0}^{\infty} |g[f^n(x)]-1| \tag{12}$$

uniformly converges in $I = [a, d]$ (cf. [4], § 51).

Let N be a positive integer such that inequalities (10) and (11) hold for $x \in I_N = [a, f^N(d)]$. Such an N exists in view of lemma 2. Now, for $x \in I$ we have $f^N(x) \in I_N$ and hence $f^n(x) \in I_N$ for $n \geqslant N$, $x \in I$ (cf. Lemma 2). So we have by (v)

$$|g[f^n(x)]-1| \leqslant M|f^n(x)-a|^{\mu}, \quad n \geqslant N,\ x \in I,$$

and by (iv)

$$|f^n(x)-a| \leqslant \vartheta^{n-N}|f^N(x)-a|, \quad n \geqslant N,\ x \in I,$$

whence

$$|g[f^n(x)]-1| \leqslant M(\vartheta^{\mu})^{n-N}|f^N(x)-a|^{\mu} \leqslant M(\vartheta^{\mu})^{n-N}|f^N(d)-a|^{\mu}$$

for $n \geqslant N$, $x \in I$. The above inequality shows, in view of (9) and since $\mu > 0$, that series (12) uniformly converges in I. It follows by (i) and (ii) that function (5) is continuous in $[a, b)$.

The above theorem generalizes a result of G. Szekeres [10] (theorem 1a; cf. also [9], theorem 9.2).

We shall also give a sufficient condition for case (B). Suppose that:

(vi) *There exist constants $M > 0$ and $\mu > 0$ such that*

$$|f(x)-a-x| \leqslant Mx^{1+\mu} \tag{13}$$

in a neighbourhood of the point a.

(1) Throughout this paper by a neighbourhood of the point a will be meant an interval $[a, a+\delta] \subset [a, b)$, $\delta > 0$.

(vii) *There exists a $\varkappa > 0$ such that*

$$|g(x)| \leqslant e^{-\varkappa(x-a)^{\mu}} \tag{14}$$

in a neighbourhood of the point a. (μ is the constant occurring in (13).)

THEOREM 3. *If hypotheses* (i), (ii), (vi), (vii) *are fulfilled, then case* (B) *occurs.*

Proof. Let us choose an $x_0 \in (a, b)$ such that inequalities (13) and (14) hold in $[a, x_0]$ and put $I = [f(x_0), x_0]$, $x_n = f^n(x_0)$. Then we have $f^n(x) \geqslant x_{n+1}$ for $x \in I$. Further, it was proved in [11] (cf. also [10], § 5) that there exists a constant $K > 0$ such that $x_n - a \geqslant Kn^{-1/\mu}$ for n sufficiently large. Hence

$$f^n(x)-a \geqslant K(n+1)^{-1/\mu} \tag{15}$$

for $x \in I$ and large n, and it follows by (14) that

$$|g[f^n(x)]| \leqslant e^{-\varkappa K^{\mu}(n+1)^{-1}} \tag{16}$$

for $x \in I$ and n sufficiently large. (16) implies that sequence (4) uniformly converges to zero in I.

We conclude this section with an example showing that in case (B) the sequence $G_n(x)$ need not tend to zero in any interval of the form $[f(x_0), x_0]$.

EXAMPLE 1. Take $[a, b) = [0, 1)$, $f(x) = px$, $0 < p < 1$, $g(x) = 1 + \frac{\sin(2\pi \log_p x)}{1+\log_p x}$ for $x \in (0, 1)$, $g(0) = 1$. We have $f^n(x) = p^n x$ and $g[f^n(x)] = 1 + \frac{u}{n+v+1}$, where $v = \log_p x$, $u = \sin 2\pi v$. Hence

$$G_n(x) = \prod_{i=1}^{n-1}\left(1+\frac{u}{i+v+1}\right).$$

Now, for $x \in (p^{k+1/2}, p^k)$, where k is an integer, we have $u > 0$ and $\lim G_n(x) = +\infty$. For $x \in (p^{k+1}, p^{k+1/2})$ we have $u < 0$ and $\lim G_n(x) = 0$. (The convergence is uniform in every compact subinterval.) Lastly, for $x = p^{k/2}$ we have $u = 0$ and $\lim G_n(x) = 1$. Thus the convergence $G_n(x) \to 0$ does not occur in any interval $[f(x_0), x_0] = [px_0, x_0]$.

§ 2. Since the difference of two continuous solutions of equation (1) is a continuous solution of equation (6), from theorem 1 results immediately the following

THEOREM 4. *Let hypotheses* (i), (ii), (iii) *be fulfilled. In case* (A) *equation* (1) *has a one-parameter family of continuous solutions in* $[a, b)$, *or none. If equation* (1) *has a continuous solution* $\varphi_0(x)$ *in* $[a, b)$, *then the general continuous solution in* $[a, b)$ *is given by the formula*

$$\varphi(x) = \varphi_0(x) + \frac{c}{G(x)}, \tag{17}$$

where $c \in \mathcal{K}$ *is an arbitrary constant and* $G(x)$ *is defined by* (4) *and* (5).

In case (B) *equation* (1) *has in* $[a, b)$ *a continuous solution depending on an arbitrary function or it has no continuous solution in* $[a, b)$.

In case (C) *equation* (1) *has in* $[a, b)$ *exactly one continuous solution, or none.*

Thus as we see, the problem remains to decide whether equation (1) has at least one continuous solution in $[a, b)$. In the sequel we shall deal with this problem. We shall also give some criteria of the existence of solutions.

§ 3. In this section we shall assume that case (A) occurs.

THEOREM 5. *Let hypotheses* (i), (ii), (iii) *be fulfilled, and suppose that case* (A) *occurs. Then equation* (1) *has a continuous solution in* $[a, b)$ *if and only if the series*

$$\varphi_0(x) = -\sum_{n=0}^{\infty} \frac{F[f^n(x)]}{G_{n+1}(x)} \tag{18}$$

converges to a continuous function in $[a, b)$. *The general continuous solution of equation* (1) *is then given by formula* (17).

Proof. 1. Suppose that (18) defines a continuous function $\varphi_0(x)$ in $[a, b)$. Put

$$\varphi_n(x) = -\sum_{i=0}^{n-1} \frac{F[f^i(x)]}{G_{i+1}(x)}. \tag{19}$$

We have

$$\varphi_n[f(x)] = g(x)\varphi_{n+1}(x) + F(x).$$

Passing to the limit as $n \to \infty$ we see that the function $\varphi_0(x)$ $(= \lim_{n\to\infty} \varphi_n(x))$ satisfies equation (1). Formula (17) results then from theorem 4.

2. If there exists a continuous solution $\varphi(x)$ of equation (1) in $[a, b)$, then

$$\varphi(x) - \frac{\varphi[f^n(x)]}{G_n(x)} = \varphi_n(x),$$

where $\varphi_n(x)$ is given by (19). (This may be shown by induction.) Hence

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x) - \frac{c}{G(x)}, \tag{20}$$

where $c = \varphi(a)$. This means that series (18) converges in $[a, b)$ and its sum (20) is continuous in $[a, b)$.

Of course it may happen that series (18) diverges. Then equation (1) has no continuous solution in $[a, b)$. It is so e.g. in the following example.

EXAMPLE 2. Take $[a, b) = [0, 1)$ and consider the equation

$$\varphi\left(\frac{x}{x+1}\right) = (1-x^2)\varphi(x) + x. \tag{21}$$

Here

$$f^n(x) = \frac{x}{1+nx}, \qquad G_n(x) = (1-x)\frac{1+nx}{1+(n-1)x}$$

and $\lim_{n\to\infty} G_n(x) = 1-x$. Consequently case (A) occurs. On the other hand, series (18) becomes

$$-\frac{x}{1-x}\sum_{n=0}^{\infty}\frac{1}{1+(n+1)x}$$

and thus evidently diverges. Consequently equation (21) has no continuous solution in $[0, 1)$.

The below theorem gives a sufficient condition of the existence of a continuous solution of equation (1) in $[a, b)$. Suppose that the function $F(x)$ fulfils the following condition.

(viii) *There exist constants* $L > 0$ *and* $\varkappa > 0$ *such that*

$$|F(x)| \leqslant L|x-a|^{\varkappa} \tag{22}$$

in a neighbourhood of the point a.

THEOREM 6. *Let hypotheses* (i), (ii), (iii), (iv), (v) *and* (viii) *be fulfilled. Then equation* (1) *has a one-parameter family of continuous solutions in* $[a, b)$, *given by formula* (17) *with* (18).

Proof. It follows from theorem 2 that case (A) occurs. Let us fix a $d \in (a, b)$. It was shown in the proof of theorem 2 that the sequence $G_n(x)$ uniformly converges to a continuous function $G(x) \neq 0$ in $I = [a, d]$. Consequently there exists a constant $K > 0$ and an index N such that

$$|G_n(x)| \geqslant K \quad \text{for} \quad n \geqslant N,\ x \in I.$$

Further we may assume that N has been chosen so large that inequalities (10) and (22) hold in $I_N = [a, f^N(d)]$. Since $f^n(x) \in I_N$ for $x \in I$ and $n \geqslant N$, we have

$$|F[f^n(x)]| \leqslant L|f^n(x)-a|^{\varkappa} \leqslant L(\vartheta^{\varkappa})^{n-N}|f^N(d)-a|^{\varkappa} \quad \text{for} \quad x \in I,\ n \geqslant N,$$

and

$$\left|\frac{F[f^n(x)]}{G_{n+1}(x)}\right| \leqslant \frac{L}{K}(\vartheta^{\varkappa})^{n-N}|f^N(d)-a|^{\varkappa} \quad \text{for} \quad x \in I,\ n \geqslant N.$$

Thus series (18) uniformly converges in $[a, d]$ (cf. (9)) for every $d \in (a, b)$. By (i), (ii), (iii) its sum is continuous in $[a, b)$ and the theorem results from theorem 5.

§ 4. Now we pass to case (B). Put

$$F_c^*(x) = F(x) + c[g(x) - 1], \tag{23}$$

where $c \in \mathcal{K}$ is a constant [2] and

$$H_n(x) = \sum_{i=0}^{n-2} \prod_{j=i+1}^{n-1} g[f^j(x)] F_c^*[f^i(x)]. \tag{24}$$

THEOREM 7. *Let hypotheses* (i), (ii), (iii) *be fulfilled and suppose that case* (B) *occurs. In order that equation* (1) *possess in* $[a, b)$ *a continuous solution fulfilling the condition*

$$\varphi(a) = c \tag{25}$$

it is necessary that $F_c^*(a) = 0$ *and the sequence* $H_n(x)$ *tend to zero uniformly in* I. *On the other hand, if there exists a* c *such that* $F_c^*(a) = 0$ *and an* $x_0 \in (a, b)$ *such that*

$$\lim_{n\to\infty} G_n(x) = \lim_{n\to\infty} H_n(x) = 0 \quad \text{uniformly in} \quad [f(x_0), x_0], \tag{26}$$

then equation (1) *has in* $[a, b)$ *a continuous solution depending on an arbitrary function: to every function* $\varphi_0(x)$ *continuous in* $[f(x_0), x_0]$ *and fulfilling condition* (2) *there exists exactly one function* $\varphi(x)$, *continuous in* $[a, b)$, *satisfying equation* (1) *and fulfilling condition* (3). *All these solutions fulfil* (25).

Proof. Put

$$\psi(x) = \varphi(x) - c. \tag{27}$$

If $\varphi(x)$ is a continuous solution of equation (1) in $[a, b)$ fulfilling condition (25), then $\psi(x)$ is a continuous solution of the equation

$$\psi[f(x)] = g(x)\psi(x) + F_c^*(x) \tag{28}$$

such that $\psi(a) = 0$, and conversely. Therefore we may confine ourselves to the study of equation (28).

1. Suppose that $\lim G_n(x) = 0$ uniformly in an interval $I \subset (a, b)$. We may assume that I is closed. Further suppose that equation (28) has in $[a, b)$ a continuous solution $\psi(x)$ such that $\psi(a) = 0$. By induction we obtain from (28)

$$H_n(x) = \psi[f^n(x)] - F_c^*[f^{n-1}(x)] - G_n(x)\psi(x). \tag{29}$$

[2] Concerning the choice of c, cf. the remark after theorem 7.

On setting $x = a$ in (28) we obtain $F_c^*(a) = 0$. Hence in view of lemma 2 the sequences $\psi[f^n(x)]$ and $F_c^*[f^{n-1}(x)]$ tend to zero uniformly in I. Thus $H_n(x)$ also tends to zero uniformly in I.

2. Now suppose that (26) is fulfilled and c is chosen so that $F_c^*(a) = 0$. The function $F_c^*(a)$ fulfils hypothesis (iii). Consequently by lemma 1 to every function $\psi_0(x)$ continuous in $[f(x_0), x_0]$ and fulfilling the condition

$$\psi_0[f(x_0)] = g(x_0)\psi_0(x_0) + F_c^*(x_0) \tag{30}$$

there exists a unique function $\psi(x)$ continuous and satisfying equation (28) in (a, b) and such that $\psi(x) = \psi_0(x)$ in $[f(x_0), x_0]$. If we extend $\psi(x)$ by putting $\psi(a) = 0$, then $\psi(x)$ satisfies (28) in $[a, b)$. It remains to prove that

$$\lim_{x\to a+0} \psi(x) = 0. \tag{31}$$

Since $\psi(x)$ satisfies (28), we have in view of (29)

$$\psi[f^n(x)] = H_n(x) + F_c^*[f^{n-1}(x)] + G_n(x)\psi(x). \tag{32}$$

Put $K = \sup_{[f(x_0), x_0]} |\psi_0(x)|$. (If $\psi_0(x) \equiv 0$, we take $K = 1$.) Given $\varepsilon > 0$, we can find an N such that

$$|G_n(x)| < \varepsilon/3K \quad \text{for} \quad x \in [f(x_0), x_0], \quad n \geqslant N, \tag{33}$$

$$|H_n(x)| < \varepsilon/3 \quad \text{for} \quad x \in [f(x_0), x_0], \quad n \geqslant N, \tag{34}$$

$$|F_c^*(x)| < \varepsilon/3 \quad \text{for} \quad x \in [a, f^{N-1}(x_0)]. \tag{35}$$

This is possible in view of (26) and of the fact that $F_c^*(x)$ is continuous at a, $F_c^*(a) = 0$.

Let us fix an arbitrary $x \in (a, f^N(x_0))$. There exists an $\bar{x} \in [f(x_0), x_0]$ and an $n \geqslant N$ such that $x = f^n(\bar{x})$. (32) gives then

$$\psi(x) = G_n(\bar{x})\psi_0(\bar{x}) + H_n(\bar{x}) + F_c^*[f^{n-1}(\bar{x})],$$

whence in view of (33), (34) and (35)

$$|\psi(x)| < \varepsilon. \tag{36}$$

Since x has been chosen arbitrarily in $(a, f^N(x_0))$, (36) holds generally in $(a, f^N(x_0))$, which proves relation (31). Now, if $\varphi_0(x)$ is an arbitrary function continuous in $[f(x_0), x_0]$ and fulfilling condition (2), then the function $\psi_0(x) = \varphi_0(x) - c$ is continuous in $[f(x_0), x_0]$ and fulfils condition (30). $\psi_0(x)$ can be extended, on account of what has just been proved, to a continuous solution $\psi(x)$ of equation (28) in $[a, b)$. Therefore the function $\varphi(x)$ obtained from (27) is a continuous solution of equation (1) in $[a, b)$ fulfilling condition (3). The last statement of the present theorem follows from theorem 1. This completes the proof.

Remark. Setting $x = a$ in (1) we obtain with notation (25)

$$c = g(a)c + F(a). \tag{37}$$

If $g(a) \neq 1$, (37) allows us to determine c uniquely and this is the only possible c in theorem 7. On the other hand, if $g(a) = 1$, then (37) gives only $F(a) = 0$ as a necessary condition of the existence of a solution of equation (1) in $[a, b)$. If this condition is fulfilled, then $F_c^*(a) = 0$ with every choice of c. But it follows from theorem 1 (compare in particular relation (7)) that all the continuous solutions of equation (1) in $[a, b)$ take on the same value at $x = a$. Consequently condition (26) may be fulfilled for at most one value of c.

Of course, it may happen that in case (B) equation (1) has no continuous solution in $[a, b)$ as it may be seen from the following example.

Example 3. Take $[a, b) = [0, 1)$ and consider the equation

$$\varphi\left(\frac{x}{x+1}\right) = (1-x)\varphi(x) + x^{\alpha}, \qquad 0 < \alpha < 1. \tag{38}$$

Since $1 - x < e^{-x}$, case (B) occurs in virtue of theorem 3. We have moreover

$$H_n(x) = \frac{x^{\alpha}}{1+(n-1)x} \sum_{i=0}^{n-2} (1+ix)^{1-\alpha} - \frac{c(n-1)x}{1+(n-1)x}.$$

Further

$$\sum_{i=0}^{n-2} (1+ix)^{1-\alpha} \geqslant 1 + \int_0^{n-2} (1+tx)^{1-\alpha} dt = 1 + \frac{1}{(2-\alpha)x}\left([1+(n-2)x]^{2-\alpha} - 1\right),$$

whence

$$H_n(x) \geqslant \frac{x^{\alpha-1}}{2-\alpha} \cdot \frac{[1+(n-2)x]^{2-\alpha}}{1+(n-1)x} - \frac{c(n-1)x}{1+(n-1)x}.$$

Consequently $\lim_{n\to\infty} H_n(x) = +\infty$ for every $x \in (0, 1)$, independently of the choice of c. Thus equation (38) has no continuous solution in $[0, 1)$.

On the other hand, in this case it is difficult to find a sufficient condition for the existence of continuous solutions of equation (1) which would be of a satisfactory generality. Since

$$H_n(x) = G_n(x) \sum_{i=0}^{n-2} \frac{F_c^*[f^i(x)]}{G_{i+1}(x)}, \tag{39}$$

it is sufficient that $\lim_{n\to\infty} G_n(x) = 0$ uniformly in $[f(x_0), x_0]$ and the series $\sum_{n=0}^{\infty} F_c^*[f^n(x)]/G_{n+1}(x)$ converges (or, more generally: does not diverge too

fast) in $[f(x_0), x_0]$ in order that condition (26) be fulfilled. This may be realized in various ways; unfortunately we could find no natural conditions of f, g and F ensuring the fulfillment of (26) and sufficiently general as to cover a variety of cases.

We conclude this section with one more example.

Example 4. Take $[a, b) = [0, 1)$ and consider the equation

$$\varphi\left(\frac{x}{x+1}\right) = (1-x)\varphi(x) + 2x + x^2. \tag{40}$$

By theorem 3 case (B) occurs. We have moreover $G_n(x) = \dfrac{1-x}{1+(n-1)x}$, $F_c^*(x) = (2-c)x + x^2$ and

$$H_n(x) = (2-c)\frac{(n-1)x}{1+(n-1)x} + \frac{x^2}{1+(n-1)x} \sum_{i=0}^{n-2} \frac{1}{1+ix}.$$

Consequently (26) is fulfilled if and only if $c = 2$. Equation (40) has in $[0, 1)$ a continuous solution depending on an arbitrary function. Every continuous solution $\varphi(x)$ of equation (40) in $[0, 1)$ fulfils the condition $\varphi(0) = 2$.

§ 5. Finally we are going to study case (C). In this case we know very little about the behaviour of the sequence $G_n(x)$. Therefore, in order to be able to prove some theorems concerning the existence of continuous solutions of equation (1), we shall assume additionally that the sequence $1/G_n(x)$ is bounded in $[a, b)$:

(ix) *There exists a constant $M > 0$ such that*

$$\left|\frac{1}{G_n(x)}\right| \leqslant M \tag{41}$$

for $x \in [a, b)$ and $n = 1, 2, 3, \ldots$

Theorem 8. *Let hypotheses* (i), (ii), (iii), (ix) *be fulfilled and suppose that case* (C) *occurs. Then equation* (1) *has a continuous solution in* $[a, b)$ *if and only if there exists a c such that the series* [3]

$$\varphi_0(x) = -\sum_{n=0}^{\infty} \frac{F_c^*[f^n(x)]}{G_{n+1}(x)} \tag{42}$$

[3] Here $F_c^*(x)$ is given by (23). It follows from theorem 1 that in the present case equation (1) may have at most one solution; consequently series (42) (depending in fact on the constant c) may converge at most for one value of c. The constant c in (23) and (43) must be chosen so that (42) converges. If series (42) diverges for every value of $c \in \mathcal{K}$, then equation (1) has no continuous solution in $[a, b)$.

converges to a continuous function in $[a, b)$. *The only continuous solution of equation* (1) *in* $[a, b)$ *is then given by the formula*

$$\varphi(x) = \varphi_0(x) + c. \tag{43}$$

Proof. If series (42) converges, then we verify as in the proof of theorem 5 that the function $\psi(x) = \varphi_0(x)$ satisfies equation (28). Consequently $\varphi(x) = \varphi_0(x) + c$ is a continuous solution of equation (1) in $[a, b)$. The uniqueness results from theorem 4.

If, on the other hand, $\varphi(x)$ is a continuous solution of equation (1) in $[a, b)$, then the function $\varphi_0(x) = \varphi(x) - c$, where $c = \varphi(a)$, is a continuous solution of equation (28) in $[a, b)$ and $\varphi_0(a) = 0$. Hence (compare formulae (32) and (39))

$$\varphi_0(x) = \frac{\varphi_0[f^n(x)]}{G_n(x)} - \sum_{i=0}^{n-1} \frac{F_c^*[f^i(x)]}{G_{i+1}(x)}$$

and formula (42) results on letting $n \to \infty$.

Remark. Let us note that hypothesis (ix) was used only in the proof of the necessity. And it would be sufficient to assume (here as well as in theorem 9 below) that inequality (41) holds in a neighbourhood of a. The theorem could then be applied to the interval $[a, a+\delta)$ instead of $[a, b)$, and by lemma 1 there is a one-to-one correspondence between continuous solutions of equation (1) in $[a, a+\delta)$ and in $[a, b)$.

THEOREM 9. *Suppose that hypotheses* (i), (ii), (iii), (ix) *are fulfilled and case* (C) *occurs. If there exist a constant* c, *a bounded function* $B(x)$ *and a constant* Θ, $0 < \Theta < 1$, *such that the inequalities*

$$|F_c^*(x)| \leqslant B(x), \tag{44}$$

$$B[f(x)] \leqslant \Theta B(x), \tag{45}$$

hold in a neighbourhood of a, *then equation* (1) *has a unique continuous solution in* $[a, b)$, *given by formulae* (43) *and* (42).

Proof. Let $d \in (a, b)$ be arbitrarily fixed and let N be chosen so that (44) and (45) hold in $[a, f^N(d)]$. Then we have by (41), (44) and (45) for $n \geqslant N$

$$\left|\frac{F_c^*[f^n(x)]}{G_{n+1}(x)}\right| \leqslant MB[f^n(x)] \leqslant M\Theta^{n-N}B[f^N(x)] \leqslant M\Theta^{n-N} \sup_{[a,d]} B(t),$$

which shows that series (42) uniformly converges in $[a, d]$. Since $d \in (a, b)$ has been arbitrary, formula (42) defines in $[a, b)$ a continuous function $\varphi_0(x)$ and the assertion follows from theorem 8.

Let us note that inequalities (44) and (45) (where $B(x) = L|x-a|^{\varkappa}$ and $\Theta = \vartheta^{\varkappa}$) hold if hypotheses (iv) and (viii) (with $F_c^*(x)$ in place of $F(x)$) are fulfilled.

The above theorem generalizes a previous result ([8], theorem 5) concerning the case $g(x) \equiv -1$. But subtle criteria given for that case by M. Bajraktarević [1] are not contained in our theorem 9.

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