The complement of a topology
for some topological groups

by

M. P. Berri (New Orleans, Louisiana)

An unsolved problem concerning the lattice of all topologies for a given set $X$ is whether such a lattice is always complemented. If $X$ is finite or $X$ is countably infinite, Hartmanis [2] and Gelfman [1] have given affirmative answers.

In this paper, we wish to give sufficient conditions for the topology of a topological group to be complemented in the lattice of all topologies for a given point set $X$. As a consequence of this theorem, we will see that the real line topology on the set of real numbers is complemented in the lattice of all topologies on the set of real numbers. Finally, in the lattice of all topologies on any infinite set $X$, we shall give a description of a topology, other than the discrete topology or the trivial topology, which is always complemented. Furthermore, it will be shown that the complement for such a topology is not unique.

**Definition 1.** Let $X$ be a fixed point set and let $\mathcal{C}$ be a topology on $X$. A topology $\mathcal{C}'$ on $X$ is said to be a complement for $\mathcal{C}$ if and only if the sup topology of $\mathcal{C}$ and $\mathcal{C}'$ is the discrete topology and the inf topology is the trivial topology.

**Definition 2.** A topological space $(X, \mathcal{C})$ is said to have property ($\ast$) if and only if $X$ has a partition into sets $\{X_\gamma \mid \gamma \in I\}$, where $I$ is some index set such that each $X_\gamma$ is countable and for each non-empty proper subset $J$ of $I$, $\bigcup \{X_\gamma \mid \gamma \in J\}$ is not open in $X$.

**Theorem 1.** If $X$ is an infinite set and if $(X, \mathcal{C})$ possesses property ($\ast$) then $\mathcal{C}$ has a complement in the lattice of all topologies on $X$.

**Proof.** Let $\{X_\gamma \mid \gamma \in I\}$ be a partition on $(X, \mathcal{C})$ satisfying the conditions of property ($\ast$). Let $\mathcal{C}_I$ be the subspace topology on $X_\gamma$. Since $X_\gamma$ is countable, then by [1], $\mathcal{C}_I$ has a complement in the lattice of all topologies on $X_\gamma$. Let $\mathcal{C}_I'$ be such a complement. Since each $\mathcal{C}_I'$ is a col-

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of the topology, one can easily see that this topology satisfies \((\ast)\) and thus is complemented. But this is really no help to prove the non-uniqueness of its complement. So we shall actually construct one complement and modify it slightly to construct another complement.

**Theorem 3.** If \(X\) is an arbitrary infinite point set, then there exists a topology on \(X\) which has a non-unique complement to the lattice of topologies on \(X\), namely, the topology
\[
\mathcal{T} = \{A \subseteq X \mid X - A = \text{finite} \} \cup \{\emptyset\}.
\]

**Proof.** Define a topology \(\mathcal{T}'\) on \(X\) in the following way. If \(\lambda\) is the cardinal number of \(X\), partition \(X\) into \(\lambda\)-many countable subsets \(\{Y_\alpha \mid \alpha \in \Lambda\}\). In each set \(Y_\alpha\), enumerate all the elements and well-order them naturally in the form \(a_1 < a_2 < a_3 < \ldots \). Denote the ordering on \(Y_\alpha\) by \(\leq\) and put the topology of finite sections on each \(Y_\alpha\). Call this topology \(\mathcal{T}_\alpha\). Let \(\mathcal{T}'\) be the topology on \(X\) with the subbase
\[
\bigcup \{Y_\alpha \mid \alpha \in \Lambda\}.
\]

We claim that \(\mathcal{T}'\) is a complement of \(\mathcal{T}\). We shall first show that \(\mathcal{T} \subseteq \mathcal{T}'\) and that \(\mathcal{T}' \subseteq \mathcal{T}\). Now suppose that \(\mathcal{T} \subseteq \mathcal{T}'\). Then \(\mathcal{T} \subseteq \mathcal{T}_\alpha\) for each \(\alpha \in \Lambda\). Thus \(\mathcal{T}' \subseteq \mathcal{T}\).

**Theorem 2.** Let \((X, \mathcal{T})\) be an infinite topological group and let \(H\) be dense, non-open, countable subgroup of \(X\). Then \((X, \mathcal{T})\) satisfies property \((\ast)\). Hence \(\mathcal{T}\) is complemented in the lattice of topologies on \(X\).

**Proof.** Let \(\mathcal{S} = \{yH \mid y \in Y\}\) be a partition of \(X\) by distinct left cosets. Let \(J = \{y \in X \mid \exists H \in \mathcal{S}\} \) and let \(J\) be a non-empty subset of \(I\) such that \(K = \bigcup\{yH \mid y \in J\}\) is open. In order to show that \((X, \mathcal{T})\) satisfies property \((\ast)\), it suffices to prove that \(J = I\), or equivalently, \(K = X\).

So take and fix \(z \in X\). Since \(H\) is dense in \(X\), then \(zH\) is dense in \(X\). Since \(K\) is open and non-empty, then \(zH \cap K \neq \emptyset\). Hence there exists \(y_H \in \mathcal{S}\) such that \(zH \cap y_H H \neq \emptyset\). Thus \(zH = y_H H\). Hence \(z \in y_H H\). Since \(z\) is a subgroup of \(X\), then \(X = K\). Thus \((X, \mathcal{T})\) satisfies property \((\ast)\) and by theorem 1, \((X, \mathcal{T})\) is complemented.

**Corollary.** If \((X, \mathcal{T})\) is the space of real numbers with the natural topology, then \(\mathcal{T}\) is complemented in the lattice of topologies on the set of real numbers.

**Proof.** Since the subspace of rational numbers form a dense, non-open, countable subgroup of the group \((X, \mathcal{T})\), then by theorem 2, \(\mathcal{T}\) is complemented. The next theorem gives us a recipe for finding a topology on an infinite set \(X\) which is not uniquely complemented. In the description.
Theorem 4. If $X$ is a set with three or more elements, then the lattice of topologies on $X$ is not distributive.

Question. In the lattice of topologies on an infinite point set $X$, does every complemented topology, which is neither discrete nor trivial, have at least two complements?

References


On the “indeterminate case” in the theory of a linear functional equation

by

B. Choczewski (Kraków) and M. Kuczma (Katowice)

Dedicated to Professor A. D. Wallace on the occasion of his 60-th birthday

Introduction. In the present paper we are concerned with the linear functional equation of the first order (cf. [3], [6])

$$
\phi[f(x)] = g(x)\phi(x) + F(x),
$$

where $\phi(x)$ is an unknown function. The values of the functions $\phi(x)$, $g(x)$, $F(x)$ lie in the field $\mathbb{K}$ of real or complex numbers, $x$ is a real variable, and $f(x)$ is a real-valued function of a real variable.

We shall consider equation (1) in an interval $[a, b)$. The functions $f(x)$, $g(x)$ and $F(x)$ will be subjected to the following conditions:

(i) The function $f(x)$ is continuous and strictly increasing in $[a, b)$, $a < f(x) < b$ in $(a, b)$, $f(a) = a$.

(ii) The function $g(x)$ is continuous in $[a, b)$, $g(x) \neq 0$ in $(a, b)$.

(iii) The function $F(x)$ is continuous in $[a, b)$.

A theory of continuous solutions of equation (1) has been developed in [6] under the condition that $|g(x)| \neq 1$. The case $|g(x)| = 1$ was left as an indeterminate one. In the present paper we are going to investigate the behaviour of continuous solutions of equation (1) in this indeterminate case.

The case where $g(x) = 1$ or $g(x) = -1$ has already been treated more in detail [1], [2], [7], [8].

It is a characteristic feature of functional equations of type (1) that in general their solution depends on an arbitrary function (cf. e.g. [5]). However, the expression “solution depends on an arbitrary function” is not quite clear and therefore it will be given here a precise meaning.

Definition. We say that equation (1) has in an interval $I$ a continuous solution depending on an arbitrary function, if there exists an interval $J \subseteq I$ such that every continuous function on $J$ can be extended (not necessarily uniquely) to a continuous solution of equation (1) in $I$. 