# The complement of a topology for some topological groups

by

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An unsolved problem concerning the lattice of all topologies for a given set $X$ is whether such a lattice is always complemented [1]. If $X$ is finite or $X$ is countably infinite, Hartmanis [2] and Gaifman [1] have given affirmative answers.

In this paper, we wish to give sufficient conditions for the topology of a topological group to be complemented in the lattice of all topologies for a given point set $X$. As a consequence of this theorem, we will see that the real line topology on the set of real numbers is complemented in the lattice of all topologies on the set of real numbers. Finally, in the lattice of all topologies on any infinite set $X$, we shall give a description of a topology, other than the discrete topology or the trivial topology, which is always complemented. Furthermore, it will be shown that the complement for such a topology is not unique.

DEFINITION 1. Let $X$ be a fixed point set and let $\mathfrak{T}$ be a topology on $X$. A topology $\mathfrak{T}'$ on $X$ is said to be a *complement* for $\mathfrak{T}$ if and only if the sup topology of $\mathfrak{T}$ and $\mathfrak{T}'$ is the discrete topology and the inf topology is the trivial topology.

DEFINITION 2. A topological space $(X, \mathfrak{T})$ is said *to have property* $(*)$ if and only if $X$ has a partition into sets $\{X_\gamma |\ \gamma \in I\}$, where $I$ is some index set such that each $X_\gamma$ is countable and for each non-empty proper subset $J$ of $I$, $\bigcup\{X_\gamma |\ \gamma \in J\}$ is not open in $X$.

THEOREM 1. *If $X$ is an infinite set and if $(X, \mathfrak{T})$ possesses property $(*)$ then $\mathfrak{T}$ has a complement in the lattice of all topologies on $X$.*

Proof. Let $\{X_\gamma |\ \gamma \in I\}$ be a partition on $(X, \mathfrak{T})$ satisfying the conditions of property $(*)$. Let $\mathfrak{T}_\gamma$ be the subspace topology on $X_\gamma$. Since $X_\gamma$ is countable, then by [1], $\mathfrak{T}_\gamma$ has a complement in the lattice of all topologies on $X_\gamma$. Let $\mathfrak{T}'_\gamma$ be such a complement. Since each $\mathfrak{T}'_\gamma$ is a col-

(1) Added in proof: This problem is now solved. The reader is referred to A. K. Steiner, The topological complementation problem, Bull. Am. Math. Soc. 72 (1966), pp. 125—127.

lection of subsets of $X$, then $\{\mathcal{T}'_\gamma |\ \gamma \in I\}$ is a subbase (indeed, a base) for some topology $\mathcal{T}'$ on $X$.

We claim that $\mathcal{T}'$ is a complement of $\mathcal{T}$. We shall first show that $\sup(\mathcal{T}, \mathcal{T}')$ is the discrete topology. Take $x \in X$. It suffices to show that $\{x\} \in \sup(\mathcal{T}, \mathcal{T}')$. Now there exists $\gamma \in I$ such that $x \in X_\gamma$. Since $\mathcal{T}_\gamma$ is the complement of $\mathcal{T}_\gamma$ on $X_\gamma$, then there exist $U \in \mathcal{T}_\gamma$, $V \in \mathcal{T}'_\gamma$ such that $U \cap V = \{x\}$. Since $U \in \mathcal{T}_\gamma$, then there exists $W \in \mathcal{T}$ such that $U = W \cap X_\gamma$. By definition of $\mathcal{T}'$, $\mathcal{T}'_\gamma \subset \mathcal{T}'$.

Thus $V \in \mathcal{T}'$. Hence $W \cap V \in \operatorname{Sup}(\mathcal{T}, \mathcal{T}')$.

But $W \cap V = W \cap (X_\gamma \cap V) = (W \cap X_\gamma) \cap V = U \cap V = \{x\}$. Hence $\{x\} \in \sup(\mathcal{T}, \mathcal{T}')$.

We will next show that $\inf(\mathcal{T}, \mathcal{T}')$ is the trivial topology. Take $H \in \inf(\mathcal{T}, \mathcal{T}')$. It suffices to show that $H = X$ or $H = \emptyset$. Let $H_\gamma = H \cap X_\gamma$. Since $H \in \mathcal{T}'$, then $H_\gamma \in \mathcal{T}'_\gamma$. Also $H_\gamma \in \mathcal{T}_\gamma$ since $H \in \mathcal{T}$. Thus for each $\gamma \in I$, $H_\gamma = \emptyset$ or $H = X_\gamma$. Now

$$H = H \cap X = H \cap \bigcup \{X_\gamma |\ \gamma \in I\} = \bigcup \{H \cap X_\gamma |\ \gamma \in I\} = \bigcup \{H_\gamma |\ \gamma \in I\}.$$

If for each $\gamma \in I$, $H_\gamma = \emptyset$, then $H = \emptyset$. So suppose that there exists some $\gamma_0 \in I$ such that $H_{\gamma_0} = X_{\gamma_0}$. Since $(X, \mathcal{T})$ satisfies property $(*)$ and $H$ is open and non-empty, then $H_\gamma = X_\gamma$ for all $\gamma \in I$. Thus $H = \bigcup \{X_\gamma |\ \gamma \in I\} = X$. Hence $\inf(\mathcal{T}, \mathcal{T}')$ is the trivial topology.

THEOREM 2. *Let $(X, \mathcal{T})$ be an infinite topological group and let $H$ be dense, non-open, countable subgroup of $X$. Then $(X, \mathcal{T})$ satisfies property $(*)$. Hence $T$ is complemented in the lattice of topologies on $X$.*

Proof. Let $\mathcal{F} = \{yH\}$ be a partition of $X$ by distinct left cosets. Let $I = \{y \in X |\ yH \in \mathcal{F}\}$ and let $J$ be a non-empty subset of $I$ such that $K = \bigcup \{yH |\ y \in J\}$ is open. In order to show that $(X, \mathcal{T})$ satisfies property $(*)$, it suffices to prove that $J = I$, or equivalently, $K = X$.

So take and fix $z \in X$. Since $H$ is dense in $X$, then $zH$ is dense in $X$. Since $K$ is open and non-empty, then $zH \cap K \neq \emptyset$. Hence there exists $y_0 \in J$ such that $zH \cap y_0 H \neq \emptyset$. Thus $zH = y_0 H$. Hence $z \in y_0 H \subset K$. Since $z$ is an arbitrary element of $X$, then $X = K$. Thus $(X, \mathcal{T})$ satisfies property $(*)$ and by theorem 1, $(X, \mathcal{T})$ is complemented.

COROLLARY. *If $(X, \mathcal{T})$ is the space of real numbers with the natural topology, then $\mathcal{T}$ is complemented in the lattice of topologies on the set of real numbers.*

Proof. Since the subspace of rational numbers form a dense, non-open, countable subgroup of the group $(X, \mathcal{T})$, then by theorem 2, $\mathcal{T}$ is complemented.

The next theorem gives us a recipe for finding a topology on an infinite set $X$ which is not uniquely complemented. In the description of the topology, one can easily see that this topology satisfies $(*)$ and thus is complemented. But this is really no help to prove the non-uniquennes of its complement. So we shall actually construct one complement and modify it slightly to construct another complement.



THEOREM 3. *If $X$ is an arbitrary infinite point set, then there exists a topology on $X$ which has a non-unique complement in the lattice of topologies on $X$, namely, the topology*

$$\mathcal{T} = \{A \subset X |\ X - A \text{ is finite}\} \cup \{\emptyset\}.$$

Proof. Define a topology $\mathcal{T}'$ on $X$ in the following way. If $\lambda$ is the cardinal number of $X$, partition $X$ into $\lambda$-many countable subsets $\{Y_x |\ x \in X\}$. In each set $Y_x$, enumerate all the elements and well-order them naturally in the form $x_1 < x_2 < x_3 < \ldots$ Denote the ordering on $Y_x$ by $\leqslant_x$ and put the topology of finite sections on each $Y_x$. Call this topology $\mathcal{T}_x$. Let $\mathcal{T}'$ be the topology on $X$ with the subbase

$$\bigcup \{T_x |\ x \in X\}.$$

We claim that $\mathcal{T}'$ is a complement of $\mathcal{T}$. We shall first show that $\sup(\mathcal{T}, \mathcal{T}')$ is the discrete topology on $X$. Take $z \in X$. We wish to show that $\{z\} \in \sup(\mathcal{T}, \mathcal{T}')$. Now there exists $x \in X$ such that $z \in Y_x$. Thus $V = \{y \in Y_x |\ y \leqslant_x z\}$ is finite and open in $\mathcal{T}'$. Since $V$ is finite, then $X - V \in \mathcal{T}$. Hence $(X - V) \cup \{z\} \in \mathcal{T}$. Thus

$$\{z\} = V \cap [(X - V) \cup \{z\}] \in \sup(\mathcal{T}, \mathcal{T}').$$

We shall now prove that $\inf(\mathcal{T}, \mathcal{T}')$ is the trivial topology on $X$. Take $V \in \inf(\mathcal{T}, \mathcal{T}')$ such that $V \neq \emptyset$. We wish to show that $V = X$. Since $V \in \mathcal{T}$ and $V \neq \emptyset$, then for each $x \in X$, $V \cap V_x \neq \emptyset$. Now take and fix $x \in X$ and take $z \in Y_x$. We wish to show that $z \in V$. Now there exists $\omega_{xz} \in Y_x$ such that $z \leqslant w_{xz}$ and $w_{xz} \in V$. Since $V \in \mathcal{T}$, we have $V \cap Y_x \in \mathcal{T}_x$. Thus $w_{xz} \in V \cap Y_x$. Hence $\{y \in Y_x |\ y \leqslant_x w_{xz}\} \subset V$. Hence $z \in V$.

Thus $Y_x \subset V$. Since $x$ is arbitrary, we have $X = \bigcup \{Y_x |\ x \in X\} \subset V$. Thus $\inf(\mathcal{T}, \mathcal{T}')$ is the trivial topology. Hence $\mathcal{T}'$ is a complement of $\mathcal{T}$.

Now another complementary topology $\mathcal{T}''$ for $\mathcal{T}$ can be constructed by simply taking one of the $Y_x$'s above and interchanging the position of the first two elements in the ordering for that $Y_x$ and then defining $\mathcal{T}''$ in the same way that $\mathcal{T}'$ is defined. Clearly $\mathcal{T}'' \neq \mathcal{T}'$.

Now Hartmanis [2] has stated that if $X$ is a finite set with three or more elements, then any topology on $X$ which is neither discrete nor trivial has more than one complement.

Thus using this result and theorem 3, we have the following result.

THEOREM 4. *If $X$ is a set with three or more elements, then the lattice of topologies on $X$ is not distributive.*

QUESTION. In the lattice of topologies on an infinite point set $X$, does every complemented topology, which is neither discrete nor trivial, have at least two complements?

## References

[1] H. Gaifman, *The lattice of all topologies on a denumerable set*, (abstract), Am. Math. Soc. Notices 8 (1961), p. 356.

[2] J. Hartmanis, *On the lattice of topologies*, Canadian Journ. Math. 10 (1958), pp. 547-553.

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# On the "indeterminate case" in the theory of a linear functional equation

by

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*Dedicated to Professor A. D. Wallace on the occasion of his 60-th birthday*

**Introduction.** In the present paper we are concerned with the linear functional equation of the first order (cf. [3], [6])

$$\varphi[f(x)] = g(x)\varphi(x)+F(x), \tag{1}$$

where $\varphi(x)$ is an unknown function. The values of the functions $\varphi(x)$, $g(x)$, $F(x)$ lie in the field $\mathcal{K}$ of real or complex numbers, $x$ is a real variable, and $f(x)$ is a real-valued function of a real variable.

We shall consider equation (1) in an interval $[a, b)$. The functions $f(x)$, $g(x)$ and $F(x)$ will be subjected to the following conditions:

(i) *The function $f(x)$ is continuous and strictly increasing in $[a, b)$, $a < f(x) < x$ in $(a, b)$, $f(a) = a$.*

(ii) *The function $g(x)$ is continuous in $[a, b)$, $g(x) \neq 0$ in $[a, b)$.*

(iii) *The function $F(x)$ is continuous in $[a, b)$.*

A theory of continuous solutions of equation (1) has been developed in [6] under the condition that $|g(a)| \neq 1$. The case $|g(a)| = 1$ was left as an indeterminate one. In the present paper we are going to investigate the behaviour of continuous solutions of equation (1) in this indeterminate case.

The case where $g(x) \equiv 1$ or $g(x) \equiv -1$ has already been treated more in detail [1], [2], [7], [8].

It is a characteristic feature of functional equations of type (1) that in general their solution depends on an arbitrary function (cf. e.g. [5]). However, the expression "solution depends on an arbitrary function" is not quite clear and therefore it will be given here a precise meaning.

DEFINITION. We say that *equation* (1) *has in an interval $I$ a continuous solution depending on an arbitrary function*, if there exists an interval $J \subset I$ such that every continuous function on $J$ can be extended (not necessarily uniquely) to a continuous solution of equation (1) in $I$.