Bimeasurable functions *

by

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1. Introduction. Let f be a real-valued continuous function on the unit interval. We will say that f is *bimeasurable* if the image under f of each Borel set in the unit interval is itself a Borel set. Lusin ([2], p. 178) gave a sufficient condition that the range of a continuous function defined on a Borel set be Borel: that each value in the range of the function is taken on at most countably many times. This condition is hereditary, in the sense that if a function satisfies it, so does the restriction of the function to any Borel set in its domain. Thus the hypothesis of Lusin's theorem is sufficient for the function f to be bimeasurable.

A slightly weaker condition, also sufficient for bimeasurability, is that there are at most countably many values in the range of the function which are taken on uncountably often. What follows is a proof that this condition is also necessary.

2. Definitions and main theorem. By *Borel set* is meant a set in a complete separable metric space which belongs to every sigma-field containing the open sets. By a *Borel function* is meant a function whose domain is a Borel set, which takes its values in some complete separable metric space and which is Borel measurable in the usual sense. A Borel function is *bimeasurable* if the image of every Borel set in its domain is a Borel set.

Baire space is the product space of all sequences of positive integers, the integers endowed with the discrete topology. Throughout, this space will be denoted by J. A non-empty subset of a complete separable metric space is said to be *analytic* if it is the range of some continuous function defined on J. The empty set is taken to be analytic.

THEOREM. *Let f be a Borel function. A necessary and sufficient condition that f be bimeasurable is that there are at most countably many values v in the range of f such that $\{x \in \text{domain } f \mid f(x) = v\}$ is uncountable.*

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Sufficiency is an immediate consequence of a theorem of Lusin ([2], p. 406). In the proof of necessity given here, it is helpful to have a relation of "similarity". Borel functions f, g are said to be *similar* if there is a 1-1 Borel function T, whose domain is the domain of f and whose range is exactly the domain of g, such that for all x, y in the domain of f, $f(x) = f(y)$ if and only if $g(T(x)) = g(T(y))$.

Here are two examples of how similar functions behave alike:

(i) Let f be a bimeasurable Borel function. If f is similar to g, where g is a Borel function, g is bimeasurable.

Why is this? The similarity of f, g induces a 1-1 Borel function T^* defined on the range of f and onto the range of g as follows: if v is in the range of f, $T^*(v)$ is $g(T(x))$, where x satisfies $f(x) = v$. The inverse image under T^* of a borel set E is

$$f\left(T^{-1}\left(g^{-1}(E)\right)\right)$$

and so analytic. The same is true of the complement of E, and by the famous theorem of Souslin ([2], p. 395), T^* is a Borel function. Since the range of f is a Borel set and T^* is 1-1, the range of g is also Borel ([2], p. 397). If B is a Borel set in the domain of g, f restricted to $T^{-1}(B)$ is similar to g restricted to B. Then $g(B)$ is a Borel set, by the above argument applied to the two restricted functions.

In the next instance, $U(f)$ is the set of those values v in the range of f such that $f^{-1}(v)$ $(= \{x \in \text{domain} f \mid f(x) = v\})$ is uncountable.

(ii) Let f be a Borel function such that $U(f)$ is uncountable. If f is similar to g, where g is a Borel function, $U(g)$ is uncountable.

For in terms of the function T^*, $v \in U(f)$ implies $T^*(v) \in U(g)$, and $U(g)$ contains an uncountable subset.

Two further properties of similarity which will be used implicitly throughout the exposition are:

(iii) Similarity is an equivalence relation.

The only difficult part, that it is a symmetric relation, depends on one of the first successes of the theory of analytic sets: the inverse of a 1-1 Borel function is Borel ([2], p. 398).

(iv) If f, g are similar Borel functions, the restriction of g to a Borel subset of its domain is similar to the restriction of f to a suitable Borel subset of its domain.

The following three sections are the three steps in the proof of the "necessity" half of the theorem. In the next section this is shown for a particular class of continuous functions, and in the final two sections a reduction is made to this special case. All three sections begin with a proposition and the purpose of the section is to establish that proposition. With the exception of any lemmas in a section, *the hypotheses of the proposition are assumed to be in force throughout the section.* That the three propositions constitute a proof of the theorem depends on a simple argument by contradiction beginning with the assumption that there is a bimeasurable Borel function f for which $U(f)$ is uncountable.

3. Proposition. *Let f be a continuous function defined on a Borel subset G of the Cantor set which satisfies*

(i) *$f^{-1}(v)$ is a perfect subset of the Cantor set for all v in the range of f,*

(ii) *f is bimeasurable.*

Then the range of f is countable.

Proof. In this and the next section, the Cantor set is represented as the space C of all sequences $\{x_i\}$ of 0's and 1's with the customary product topology.

Suppose for this paragraph and the next, that for each v in the range of f we have a 1-1 map φ_v from $f^{-1}(v)$ onto C such that the overall map s,

$$s(x) = \varphi_v(x) \quad \text{when} \quad f(x) = v$$

is a Borel function. Let R be the set of all those $x \in G$ which satisfy

$$\left(s(x)\right)_i = 0$$

for every positive integer i. Then R meets every non-empty $f^{-1}(v)$ in exactly one point. More than that, the map r which assigns to each $x \in G$ the unique member of R in $\{y \in G \mid f(y) = f(x)\}$ is a Borel function. As is easily seen,

$$\{x \in G \mid r(x) \in B\} = \{x \in G \mid f(x) = f(y) \quad \text{for some} \quad y \in B \cap R\}.$$

If B is Borel, the right side is analytic. This implies that r is Borel.

The map

$$T\colon\ x \to \left(r(x), s(x)\right), \qquad x \in G,$$

shows f and the projection to the first coordinate on $R \times C$ to be similar.

If R is uncountable, there is a 1-1 Borel function defined on R and onto the unit interval ([2], p. 538). The same is true of C and the transformation obtained by taking the ordered pair of these two maps and composing the result with T establishes the similarity of f and the projection to the first coordinate on the unit interval. As is well known, the latter is not bimeasurable and by contradiction, R is countable.

It remains to construct the map s. The next two definitions and lemmas are directed toward setting up a tractable function φ_D, which for any perfect non-empty subset D of C, is a homeomorphism between D and C. The remaining discussion shows that putting the maps together in the case that the D are the sets $f^{-1}(v)$ yields an over-all map s, which is a Borel function.

DEFINITION. Let D be a subset of the Cantor set C and $x \in D$. A positive integer k is said to be a *free coordinate* of x (with respect to D) if there is a $y \in D$ such that $y_i = x_i$, $1 \leqslant i < k$, and $y_k \neq x_k$.

Our candidate for φ_D is the 'projection'

$$(x_1, x_2, x_3, \ldots) \to (x_{k_1}, x_{k_2}, x_{k_3}, \ldots)$$

where $k_1 = k_1(x), k_2 = k_2(x), \ldots$ are the free coordinates of x, written in increasing order. For this to be well-defined we need the

LEMMA 1. *Let D be dense-in-itself. Given any $x \in D$ and n a positive integer, there is an $m > n$ such that m is a free coordinate of x.*

DEFINITION. Let D be dense-in-itself and $x \in D$. Then

$$k_1(x) = \text{the least free coordinate of } x;$$
$$k_{j+1}(x) = \text{the least free coordinate of } x \text{ greater than } k_j(x);$$

and

$\varphi_D(x)$ *is the sequence* $y \in C$ *satisfying*

$$y_i = x_{k_i(x)}, \qquad i = 1, 2, \ldots$$

LEMMA 2. *If $D \neq 0$ is a perfect subset of the Cantor set, φ_D is a homeomorphism from D onto the Cantor set.*

Proof of Lemma 2. Let I be the set of all finite sequences $(a_1, \ldots, a_j)$ of 0's and 1's. The number j, of terms in the sequence will be called the *length* of the sequence. If $a \in I$, define $N(a)$ to be the neighbourhood

$$\{x \in C|\ x_i = a_i,\quad 1 \leqslant i \leqslant j\}$$

where j is the length of a. We are going to prove that if $d \in I$ of length n is given, there is an $a \in I$ of length $m \geqslant n$ such that

$$D \cap N(a) \neq 0,$$
$$(*) \qquad \textit{If } x \in D \cap N(a),\ k_n(x) = m,$$
$$x \in D \cap N(a) \text{ if and only if } \varphi_D(x) \in N(d).$$

The proof is essentially an inductive rule for determining $\varphi_D^{-1}(N(d))$. For $n = 1$, let m be the least integer i for which there are x, y in D such that $x_i \neq y_i$. Then there is a $u \in D$ such that $u_m = d_1$ and every $x \in D$ satisfies $x_i = u_i$, $1 \leqslant i < m$. Set $\hat{u} \in I$ to be the sequence of length m composed of the first m terms of u. Now if $x \in D$, then x has m as its first free coordinate, and if, further, $x \in N(\hat{u})$, then $\varphi_D(x)$ is in $N(d)$. Conversely, if $\varphi_D(x) \in N(d)$, then

$$d_1 = x_{k_1(x)} = x_m$$

and $d_1 = u_m$ imply $x_m = u_m$. We have already shown why $x_i = u_i$, $1 \leqslant i < m$, so $x \in N(\hat{u})$. Finally, $u \in D \cap N(\hat{u})$ so $D \cap N(\hat{u})$ is not empty.

For the inductive step, let $c \in I$ of length $n+1$ be given and d be the first n terms of c. By the inductive hypothesis, there is an $a \in I$ of length $m \geqslant n$ such that $(*)$ holds. By the preceding lemma there are integers i such that, for some x, y in $D \cap N(a)$, $x_i \neq y_i$. If j is the least of these, then $j \geqslant m+1 \geqslant n+1$ and there is a $u \in D \cap N(a)$ such that $u_j = c_{n+1}$ and every $x \in D \cap N(a)$ satisfies $x_i = u_i$, $1 \leqslant i < j$. Setting $\hat{u} \in I$ to be the first j terms of u, we know that $D \cap N(\hat{u})$ is not empty.

If $x \in D \cap N(a)$, $k_n(x) = m$. But j is a free coordinate of x greater than m and is the least such. Thus $k_{n+1}(x) = j$. Further, $x \in D \cap N(a)$ implies that the first n coordinates of $\varphi_D(x)$ are $d_1, \ldots, d_n$. When $x \in D \cap N(\hat{u})$,

$$(\varphi_D(x))_{n+1} = x_j = u_j = c_{n+1}.$$

If $\varphi_D(x) \in N(c)$, $\varphi_D(x) \in N(d)$ and $x \in N(a) \cap D$. Therefore $x_i = u_i$, $1 \leqslant i < j$, and $k_{n+1}(x) = j$. Now

$$c_{n+1} = (\varphi_D(x))_{n+1} = x_j \quad \text{and} \quad u_j = c_{n+1}$$

imply $x_j = u_j$. This completes the inductive step.

The last clause of $(*)$ establishes that φ_D is continuous. Suppose that $v = \varphi_D(x) = \varphi_D(y)$. Then if $\hat{v} \in I$ is composed of the first n terms of v, by $(*)$ applied to $d = \hat{v}$, there is an $a \in I$ of length at least n such that both x, y belong to $D \cap N(a)$. This holds for every n so $x = y$ and φ_D is 1-1. By the first and third clauses of $(*)$, the range of φ_D is dense in C, and by the compactness of D, φ_D is a homeomorphism of D onto C. This completes the proof of the lemma.

We now return to proving the main proposition. The aim is to show that

$$s(x) = \varphi_D(x) \quad \text{when} \quad f(x) = v \quad \text{and} \quad D = f^{-1}(v)$$

is a Borel function.

If we can show that

$$\{x \in G|\ (s(x))_i = 1\}$$

is a Borel set for every i, we will be done. This set can be expressed as the countable union

$$\bigcup_j \{x \in G|\ x_j = 1\} \cap \{x \in G|\ k_i(x) = j\}$$

and we may concentrate on the measurability of

$$E_{ij} = \{x \in G \mid k_i(x) = j\}$$

where $k_i(x)$ is the ith free coordinate of x with respect to $\{y \in G \mid f(y) = f(x)\}$. Set

$$F(i) = \{x \in G \mid i \text{ is a free coordinate of } x\}.$$

Then each E_{ij} is a Boolean combination of sets $F(1), \ldots, F(j)$ and it suffices to check that the $F(i)$'s are Borel sets. That is, the set of $x \in G$ such that for some $y \in G$,

$$f(x) = f(y); \quad \text{and} \quad x_j = y_j,\ 1 \leqslant j < i; \quad \text{and} \quad x_i \neq y_i;$$

is Borel. If $a \in I$ is a finite sequence of length i

$$F(i) \cap N(a) = \{x \in G \mid \text{for some } y \in G \cap N(\bar{a}),\ f(y) = f(x)\} \cap N(a)$$

where $\bar{a} = (a_1, a_2, \ldots, a_{i-1}, 1-a_i)$. The right side is

$$f^{-1}\big(f(G \cap N(\bar{a}))\big) \cap N(a)$$

which is Borel by the assumed bimeasurability of f. It follows by taking the union over all a of length i that $F(i)$ is a Borel set.

4. Proposition. *Let g be a continuous function defined on the Cantor set whose range is uncountable and coincides with $U(g)$. There is a Borel subset G of the Cantor set such that f, the restriction of g to G, satisfies*

(i) *$f^{-1}(v)$ is a perfect subset of the Cantor set for all v in the range of f,*

(ii) *the range of f is uncountable.*

Proof. The proof of the proposition is given in six steps, the first being a summary of some known results which will be needed in the succeeding steps.

1. Let X be a compact metric space and 2^X the space of closed, non-empty subsets of X, endowed with the usual compact metric topology ([2], p. 106). Then we have ([3], Sections 38, 39):

(a) $\{(K, L) \in 2^X \times 2^X \mid L \text{ is a subset of } K\}$ *is closed in* $2^X \times 2^X$.

(b) $\{(x, K) \in X \times 2^X \mid x \in K\}$ *is closed in* $X \times 2^X$.

(c) *The set P, of all perfect non-empty subsets of X, is a G_δ in 2^X.*

(d) *If g is continuous, defined on X, the function*

$$v \to g^{-1}(v), \quad v \in \operatorname{range} g$$

is Borel.

2. We are going to use the results of (1) in the special case that $X = C$. Returning to the situation of the proposition, let V be the range of g, and S be the set of all pairs

$$(v, K) \in V \times 2^C$$

with K perfect in C and K a subset of $g^{-1}(v)$. By 1(d) and a familiar property of measurable functions, the function

$$(v, K) \to \big(g^{-1}(v), K\big), \quad (v, K) \in V \times 2^C$$

is Borel. The inverse image under it of the Borel set defined in 1(a) is therefore Borel. As S is the intersection of this inverse image and $V \times P$, S is Borel.

3. As each $g^{-1}(v)$ contains a set homeomorphic to C, it is clear that the projection of S to the v-axis is all of V. The lemma in (6) below shows that S contains a subset D, compact in $V \times 2^C$, which has an uncountable projection to the v-axis.

4. Applying a selection theorem ([1], p. 135) D contains a Borel subset B such that $\pi(B) = \pi(D)$ and

$$B_v = \{K \in 2^C \mid (v, K) \in B\}$$

contains exactly one point for all $v \in \pi(D)$. In other words, B is the graph of a function

$$v \to Q(v)$$

and since B is Borel, this function is a Borel function ([2], p. 398) defined on $\pi(D)$ to 2^C.

5. Set

$$H = \{x \in C \mid g(x) \in \pi(D)\},$$

$$G = \big\{x \in H \mid x \in Q\big(g(x)\big)\big\}.$$

It is clear that $G \cap g^{-1}(v)$ is empty for all $v \notin \pi(D)$. If $v \in \pi(D)$

$$G \cap g^{-1}(v) = Q(v)$$

which is a perfect, non-empty, subset of C. Once G is shown to be Borel, then it will follow that G meets all the requirements of the proposition. Now the function

$$x \to Q\big(g(x)\big), \quad x \in H,$$

is a Borel function, so that

$$x \to \Big(x, Q\big(g(x)\big)\Big), \quad x \in H,$$

is also a Borel function. The inverse image under this function of the Borel set defined in 1(b) is just the set G.

6. Lemma. *Let V, Y be complete separable metric spaces and $S \subset V \times Y$. If S is analytic and $\pi(S)$ is uncountable, there is a $D \subset S$, compact in $V \times Y$ such that $\pi(D)$ is uncountable.*

Proof of Lemma. S is analytic and so is the continuous image under h (say) of the Baire space J. As $\pi(S)$ is uncountable and analytic it contains a subset homeomorphic to the Cantor set ([2], p. 352) and we can define a probability measure on the Borel subsets of $\pi(S)$ which give singletons measure zero. The reasoning of Sion (see the proof of Theorem 4.2 of [6]) leads to a compact set D_1 in J with the property that its image under the composition $\pi(h)$ has measure at least one-half, say. Now take $D = h(D_1)$.

5. Proposition. *Let f be a Borel function for which $U(f)$ is uncountable. There is a Borel subset F, of the domain of f, such that $f|F$ is similar to a continuous function g, defined on the Cantor set, whose range is uncountable and coincides with $U(g)$.*

Proof. 1. The function f is similar to a continuous function defined on a closed subset of the Baire space J. To show this, let $Y(V)$ be a complete separable metric space which includes (resp.) the domain (range) of f. The graph of f is a Borel subset Γ of $Y \times V$ and therefore can be represented as the range of a 1-1 continuous function defined on a closed subset E of J ([2], p. 354). Composition of this function with the projection

$$(y, v) \to v, \qquad (y, v) \in Y \times V,$$

gives us a continuous function defined on E similar to f. The similarity is provided by the function

$$x \to \big(x, f(x)\big), \qquad x \in \operatorname{domain} f,$$

composed with the inverse ([2], p. 398) of the function whose domain is E and range is Γ.

By this argument, and (2) of section 2, we may proceed as if the function f in the proposition is continuous and its domain is a closed subset E of J.

2. Now to find a compact set K in J such that $U(f|K)$ is uncountable. The method turns on the simple fact that a closed set K in J is compact if and only if there is a $b \in J$ so that for all $d \in K$

$$d_i \leqslant b_i, \qquad i = 1, 2, 3, \ldots$$

We abbreviate the displayed relation by $\leqslant$.

Let S be the set of all pairs (v, b) in $V \times J$ such that $f^{-1}(v)$ and $\{y \in J \mid y \leqslant b\}$ have an uncountable intersection. The projection of S to the v-axis is $U(f)$. For if $v \in U(f)$, $f^{-1}(v)$ is an uncountable closed set. As such it contains an uncountable compact set and there is $b \in J$ such that b dominates (in the sense of $\leqslant$) all the members of this compact set. Then $(v, b) \in S$. The other direction is obvious.

As will be shown below, S is an analytic set. Applying the lemma of section 4, there is a compact subset D of S whose projection to the v-axis is uncountable. The projection of D to the other axis is compact and so dominated by some $b_0 \in J$. Claim: if f is restricted to the compact set

$$K = \{y \in J \mid y \leqslant b_0\} \cap E,$$

$U(f|K)$ is uncountable. For the section of S at b_0 includes $\pi(D)$ and is precisely the set of v such that $(f|K)^{-1}(v)$ is uncountable.

To shows that S is analytic, set

$$H = \{(v, b, x) \in V \times J \times E \mid x \leqslant b,\ f(x) = v\}.$$

Then H is closed and

$$S = \{(v, b) \in V \times J \mid H_{(v,b)} \text{ is uncountable}\}.$$

By a theorem of Sierpiński and Mazurkiewicz ([2], p. 405) S is analytic.

3. In order to end with a function which takes on all of its values uncountably often, we select a compact uncountable subset L, of $U(f|K)$ and restrict f to the set $K_1 = f^{-1}(L) \cap K$. This we can do since $U(f|K)$ is uncountable and analytic ([2], p. 405). By deleting at most a countable number of points from K_1, we get a perfect set F. If J is considered to be the subspace of irrational numbers in the unit interval, F becomes a compact, perfect, nowhere dense subset of the unit interval. All such linear sets being homeomorphic ([5], p. 146) we have a homeomorphism h defined on the Cantor set onto F. Further restriction of f to F gives a function similar to $f(h)$ defined on the Cantor set satisfying all the conditions of the proposition of this section.

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