On a family of AR-sets

by

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K. Borsuk has recently constructed [3] a family of \(\mathbb{R}^n\) 2-dimensional compact AR-sets such that none of them contains a 2-dimensional closed subset homeomorphic to a subset of another set. As an application of this family it has been shown that there is no universal 2-dimensional AR-set, and that a 3-dimensional cube \(Q\) has no \(r\)-neighbour on the left.

In the present note we shall show that these results can be extended to every finite dimension, and, with a slight modification, to an infinite dimension. The constructions and the proofs are suitably adapted constructions and proofs of [3].

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1. Zone. Let \(E^n\) denote the \(n\)-dimensional Euclidean space and \(E^m\) the Hilbert space. Let \(\Delta\) be an \(n\)-dimensional simplex in \(E^{n+k}\), where \(k\) is a natural number or the infinity. Let us denote by \(E^{\Delta}\) the orthogonal complementary space to the hyperspace \(E^\Delta\) of \(\Delta\) in \(E^{n+k}\), and by \(l_1, l_2, \ldots, l_n\) the orthogonal basis of the space \(E^{\Delta}\). Given a sequence \((\varepsilon_i)\) of positive numbers such that \(\varepsilon_i < 1/2^i, i = 1, 2, \ldots, k\), let us denote by \(L_i\) the segment with length \(2\varepsilon_i\) with the centre in the barycentric centre \(b_i\) of \(\Delta\), and in the direction of vector \(l_i\). By the \(k\)-dimensional zone of the simplex \(\Delta\) we understand the minimal convex subset of \(E^{n+k}\) containing the set \(\Delta\) and all segments \(L_i\). It will be denoted by \(Z(\Delta, (\varepsilon_i))\).

In this paper we use only the case where \(k = 1\) or \(k = \infty\).

Let us consider a homogeneously \(n\)-dimensional \((n > 2)\) polytope \(P \subset E^{n+k}\) with a triangulation \(T\). By the boundary of \(P\) we understand the union \(\partial P\) of \((n-1)\)-dimensional simplexes of \(T\) which are incident exactly with one \(n\)-dimensional simplex of \(T\), and by the edge of \(P\) the closure \(\partial P^*\) of the set of all points of \(P\) having a neighbourhood in \(P\) which cannot be disconnected by a simple arc. Obviously the notions of boundary and edge are independent of the choice of triangulation \(T\).

One can easily see that for \(\varepsilon_i\) sufficiently small the common part of the zones of different simplexes of \(T\) coincides with the common part
of the boundaries of those simplices. The sequence \( \{a_i\} \) satisfying this condition is said to be suitable for the triangulation \( T \). Let \( \{a_i\} \) be the sequence suitable for the triangulation \( T \). By the \( k \)-dimensional zone \( (h \text{ a natural number or the infinity}) \) of the triangulation \( T \) we shall understand the polytope

\[
Z^k(T, \{a_i\}) = \bigcup_{a \in T} Z^k(a, \{a_i\})
\]

Evidently the polytope \( P \) is a deformation retract of the zone \( Z^k(T, \{a_i\}) \).

2. Construction of a finite-dimensional membrane. Given a sequence \( \{a_k\} \) of natural numbers \( >1 \), let us assign to each natural number \( k \geq 1 \) a polytope \( P_k \), its triangulation \( T_k \) and a positive number \( e^{(k)} \) satisfying the following conditions:

1. \( P_k \) is a homogeneously \( n \)-dimensional polytope \( (n > 2) \) in \( E^{k+1} \) which is an \( AR \)-set with the boundary \( T_k = P_k \).

2. The edge \( P_k^e \) of \( P_k \) is a subset of \( P_k - P_k^e \) and its components are rectilinear segments.

3. All simplices of triangulation \( T_k \) of \( P_k \) have the diameter \( < 1/k \).

4. \( e^{(k)} \) is suitable for the triangulation \( T_k \), \( e^{(k)} < 1/k \); for \( k \geq 2 \) if \( n < k \) and \( n \) is such that all \( a \in T_k \) is such that \( \Delta \) is a \( k \)-simplex, then

\[
Z(\Delta, e^{(k)} = Z(\Delta, e^{(k-1)}).
\]

Let \( P \) denote a polytope in \( E^{k+1} \) homeomorphic to the \( n \)-dimensional simplex \( n \geq 2 \), \( T \) its triangulation with diameters of simplices \( < 1 \), and \( e^{(k)} \) a number \( < 1 \) suitable for the triangulation \( T \).

Let us assume that we have defined the polytope \( P_k \), its triangulation \( T_k \) and the number \( e^{(k)} \) in such a manner that conditions (1a), (1b), (4a), (4b) are satisfied. For each \( n \)-dimensional simplex \( \Delta \) of \( T_k \), let us consider a system consisting of \( n \) \( n \)-simplices \( \Delta_1, \Delta_2, ..., \Delta_n \) lying in the interior of the simplex \( \Delta \) and such that \( b \) is their common vertex and that \( \Delta_i \cap \Delta_j = b \) for \( i \neq j \). Let \( a_k \) be a point lying on the axis \( L(\Delta, e^{(k)}) \) at a distance \( e^{(k)/2} \) from \( b \). Consider the system of \( (n+1)\) \( n \)-dimensional simplices \( \Delta_1, \Delta_2, ..., \Delta_{(n+1)n} \), their vertices are: \( a_k \) and \( n \) vertices of the simplex \( \Delta_k \). \( i = 1, 2, ..., n \).

One can easily see that the polytope

\[
R_k = (e^{(k)} \setminus (\bigcup_{k=1} a_k \setminus (\Delta_1)) \setminus (\bigcup_{k=1} a_k \setminus (\Delta_2)) \setminus \ldots \setminus (\bigcup_{k=1} a_k \setminus (\Delta_n))
\]

is homogeneously \( n \)-dimensional and is a deformation retract of the zone \( Z(\Delta, e^{(k)}) \). We set \( P_{k+1} = \bigcup_{e^{(k)}} R_k \). The set \( P_{k+1} \) is said to be a modification set of the polytope \( P_k \) corresponding to the triangulation \( T_k \) and to the number \( n_k \).

As \( T_{k+1} \) we choose an arbitrary triangulation of \( P_{k+1} \) with simplices of diameter \( < 1/k+1 \) and as \( e^{(k+1)} \) the number satisfying \( (4a) \). It is easily seen by the same argument as in [1] that \( P_{k+1} \) and its triangulation \( T_{k+1} \) satisfy conditions (1a), (2a) and (3a).

The construction implies that the edge \( P_k^e \) coincides with the segments \( a_k b_k \), where \( \Delta \times T_k \), \( k < m \). Since \( a_k b_k \) is a common part of \( n_k \) simplices, we shall say that it is a segment of ramification of order \( n_k \).

It follows from (4a) that

\[
Z(T_{k+1}, e^{(k+1)}) \subset Z(T_k, e^{(k)}), \quad k = 1, 2, ...
\]

i.e. the sequence of the polytopes \( Z(T_k, e^{(k)}) \) is decreasing.

Every space \( X \) homeomorphic to the set

\[
P(\{a_k\}) = \bigcap_{k=1}^{\infty} Z(T_k, e^{(k)})
\]

will be called a membrane corresponding to the sequence \( \{a_k\} \). The polytope \( P \) will be called the base of membrane \( X \), the boundary \( P_1 \) of base \( P_1 \) will be called the boundary of membrane \( X \) and will be denoted by \( X \).

As in [1] we can prove that every membrane with a base homeomorphic to an \( n \)-dimensional simplex is an \( n \)-dimensional AR-set.

3. Construction of an infinite-dimensional membrane. Let \( W \) be a polytope and \( T \) a triangulation of \( W \). We shall say that the polytope \( W \) is strongly connected in the dimension \( m \) if, given two simplices \( \Delta \) and \( \Delta' \) of dimension \( \geq m \) of \( T \), there exists a sequence of simplices \( \Delta = \Delta_1, \Delta_2, ..., \Delta_m = \Delta' \) of \( T \) such that for \( \Delta_i, \Delta_{i+1} \) one of them is the face of the other and \( \dim \Delta_i \geq m \), \( i = 2, ..., m-1 \). Obviously this property is independent of the choice of the triangulation \( T \). One can easily see that if an \( n \)-dimensional polytope with a triangulation \( T \) is strongly connected in the dimension \( m \), then its \((n-1)\)-skeleton, that is the union of all \((n-1)\)-simplices of \( T \), is strongly connected in the dimension \( (m-1) \).

Now, given a sequence \( \{a_k\} \) of natural numbers \( >1 \), let us assign to each natural number \( k \geq 1 \) a polytope \( P_k \), its triangulation \( T_k \) and a sequence \( e^{(k)} \) of positive numbers satisfying the following conditions:

1. \( P_k \) is a homogeneously \( (q+1) \)-dimensional \( (q \geq 3) \) polytope in \( E^{q+1} \) and is an \( AR \)-set strongly connected in a dimension \( \geq 3 \).

2. The edge \( P_k^e \) of \( P_k \) is a union of disjoint rectilinear segments.
(3) The simplex of the triangulation $T_k$ has diameters $<1/k$. For each point $x \in P_k \cap P_k'$ the union of all simplices of $T_k$ containing $x$ is a homogeneously $(k+q)$-dimensional $\mathbb{R}$-set strongly connected in a dimension $\geq 3$.

(4) The sequence $(\delta_k)$ is suitable for the triangulation $T_k$, $\delta_k \geq 1/k$, $\delta_k < 1/2k$, $\delta_k < 1/8$, $\delta_k > 1/4$, $k = 1, 2, ...$, if $\delta' > 0$, and $\delta' \in T_k$ is a $(q+k)$-dimensional simplex of $T_k$ such that $\delta' \in T_k(\delta_k)$, then

$$Z^{(q)}(\delta', (\delta_k)) \subseteq Z^{(q)}(\delta', (\delta_k)).$$

As $P_k$ we take a polytope in $\mathbb{R}^{\delta_k+1}$ homogeneously $(q+1)$-dimensional ($q \geq 3$) which is an $\mathbb{R}$-set strongly connected in a dimension $\geq 3$ satisfying (3), so $P_k$, its triangulation with simplices of diameter $<1$, as $(\delta_k)$ we set a sequence satisfying (4).

Let us assume that we have defined the polytope $P_k$, its triangulation $T_k$ and the sequence $(\delta_k)$ in such a manner that conditions (1), ..., (4) are satisfied. Let $A_1, A_2, ..., A_{k+1}$ denote two systems of $(q+k)$-simplexes defined in the same manner as in the construction of polytope $P_{k+1}$ in the finite-dimensional case. We set

$$E_k = (\delta - \sum_{i=1}^{k+1} A_i) \cup \left( \bigcup_{j=1}^{k+1} A_j \right),$$

and $E = \bigcup_{k=1}^{\infty} E_k$. Let $T_k$ be a triangulation of the polytope $E$ with the simplices of diameter $<1/k+1$, and let $\delta_{k+1}$ be a number suitable for the triangulation $T_k$ and such that $\delta_{k+1} < 1/(k+1)$.

We set $P_{k+1} = Z(T_k(\delta_{k+1}))$ and let $T_{k+1}$ be its triangulation. $Z(T_k(\delta_{k+1}))$ is an $\mathbb{R}$-set and since $E$ is a deformation retract of it, $E$ is also an $\mathbb{R}$-set and consequently $P_{k+1}$ is an $\mathbb{R}$-set. One can easily see that we can choose the triangulations $T_k$ and $T_{k+1}$ in such a manner that the $(q+k-1)$-dimensional skeleton of $T_k$ is included in the triangulation $T_{k+1}$. The edge $P_k^{1+k}$ is the union of the edge $P_k'$ and all segments of the form $\alpha_{k+1}k$, where $\alpha$ is a $(q+k)$-dimensional simplex of $T_k$.

From the construction it follows that the condition (4) is satisfied. We can also choose a sequence $(\delta_{k+1})$ so that $\delta_{k+1}$ is decreasing.

Every space $X$ homeomorphic to the set

$$P'((\delta_{k+1})) = \bigcap_{j=1}^{\infty} Z(T_k(\delta_{k+1}))$$

is said to be an infinite-dimensional membrane corresponding to the sequence $((\delta_{k+1}))$. The polytope $P_k'$ will be called the base of the membrane $X$, and its boundary $P_1'$ the boundary of the membrane $X$ and will be denoted by $X'$. From the construction of $T_{k+1}$ it follows that the triangulation $T_{k+1}$ contains the $(q+k-1)$-dimensional skeleton of $T_k$. Thus $P'((\delta_{k+1}))$ contains the $(q+k-1)$-dimensional skeleton of $T_k$ for $i = 1, 2, ...$, and consequently the set $P'((\delta_{k+1}))$ has an infinite dimension.

We can say more. Namely, every point of $P'((\delta_{k+1}))$ has arbitrary small neighborhoods whose boundaries have finite dimension. Thus $P'((\delta_{k+1}))$ has transfinite dimension (6).

The $(q+k-1)$-dimensional simplices of $T_{k+1}$ contained in $Z(T_k(\delta_{k+1})))$, where $\delta$ is a $(q+k)$-dimensional simplex of $T_k$ form the triangulation $T_{k+1}$ of the polytope $Z(T_k(\delta_{k+1})))$, which is an $\mathbb{R}$-set. It follows that $Z(T_k(\delta_{k+1})))$ is also an $\mathbb{R}$-set. Thus there exists a retraction $r_k$ of the set $Z(T_k(\delta_{k+1})))$ to the set $Z(T_k(\delta_{k+1})))$. Since $Fr(\delta, (\delta_{k+1})))$ (the boundary $Fr(\delta)$ is taken relatively to the polytope $P_k$), $r_k(x)$ is for $x \in Fr(\delta)$, $r_k(x) = r_k(x)$ for $x \in Z(T_k(\delta_{k+1})))$, $\delta \subset T_k$, we infer that the mapping $r_k$ is a retraction of $Z(T_k(\delta_{k+1})))$ to $Z(T_k(\delta_{k+1})))$ such that for every $\delta \subset T_k$

$$r_k(Z(T_k(\delta_{k+1}))) = Z(T_k(\delta_{k+1}))) \quad (\delta_{k+1}).$$

Let us set $r_k(x) = r_k(x) \quad r_k(x)$ for $x \in Z(T_k(\delta_{k+1})))$. The mapping $r_k$ is a retraction of $Z(T_k(\delta_{k+1})))$ to $Z(T_k(\delta_{k+1})))$, and if $x \in Z(T_k(\delta_{k+1})))$ and $\delta \subset T_k$, $\delta \subset T_k$, then every point $r_k(x)$ for $i = 1, 2, ...$, belongs to $Z(T_k(\delta_{k+1})))$. Since the diameter of the zone $Z(T_k(\delta_{k+1}))) < 1/(k+1)$, then the sequence $r_k(x)$ converges uniformly to a map $r$ of $Z(T_k(\delta_{k+1}))) \to P'((\delta_{k+1})))$. For every $x \in P'((\delta_{k+1})))$ we have $r(x) = Z(T_k(\delta_{k+1}))) \to Z(T_k(\delta_{k+1})))$, and $r_k(x) = x$ for every $k = 1, 2, ...$, consequently $r$ is a retraction of $Z(T_k(\delta_{k+1}))) \to P'((\delta_{k+1})))$. Since the zone $Z(T_k(\delta_{k+1})))$ is a compact set, we conclude that every infinite-dimensional membrane is a compact $\mathbb{R}$-set.

4. Hits of a membrane. As in [3], by a hit of a membrane $X$ (of finite or infinite dimension) we understand a membrane $Y$ (corresponding to an arbitrary sequence $((\delta_{k+1})))$ of naturals $\geq 2$ such that $Y \subset X$ and that $X \cap \overline{X} \subset \overline{Y}$. One can easily see that if a set $Q$ is a union of simplices of the triangulation $T_k$ homeomorphic to the $(q+k)$-dimensional ball $(n > 2)$ in the finite-dimensional case, and to a homogeneously $(q+k)$-dimensional $\mathbb{R}$-set strongly connected in the dimension $\geq 3$ in the infinite-dimensional case, and if $T$ denotes the triangulation of $Q$ which consists of simplices included in $T_k$, then the constructions of § 2 and § 3 applied only to the simplices of the triangulations $T_{k+1}$ ($k = 1, 2, ...$, lying in $Z(T_k(\delta_{k+1})))$ ($j = 1, \infty$) define a set $X_Q = P'((\delta_{k+1}))) \cap Z(T_k(\delta_{k+1})))$ where $j = 1, \infty$. ?
which is a bit of membrane \( X \) corresponding to the sequence \((m_{k+1})\), and with base \( Q \).

By an \( m \)-membrane we shall understand a set \( Y \) which is a union of \( m \) membranes \( X_1, \ldots, X_m \) (of finite or infinite dimensions) such that there exists a simple arc \( L \) satisfying the condition \( X_i \cap X_j = X_i \cap X_j = L \), \( i \neq j \). The arc \( L \) will be called the edge of the \( m \)-membrane \( Y \) and will be denoted by \( Y^* \). By \( Y \) we shall denote the interior of \( Y^* \). The membranes \( X_i, i = 1, 2, \ldots, m, \) will be called the strings of the \( m \)-membrane \( Y \). By the boundary \( \partial \) of the \( m \)-membrane \( Y \) we shall understand the set \( Y = \bigcup_{i=1}^m X_i \).

By the \( m \)-bit we shall understand a subset \( Y \) of a membrane \( X \) which is an \( m \)-membrane and \( Y \cap X - \{y \} \subset Y^* \).

We omit the proof of the following lemma because it is completely analogous to the proof which (in the case \( m = 2 \)) is included in [3].

**Lemma 1.** A closed subset \( Y \) of an \( m \)-dimensional membrane \( X \) is \( m \)-dimensional if and only if it contains at least one bit of \( X \).

Obviously every open subset of an infinite-dimensional membrane contains a bit of this membrane.

5. Topological classification of points of a membrane. Let us consider the following subsets of the membrane \( X \) (of dimension \( m \) or \( \infty \)):

- \( X_1 \) consists of all points \( x \in X \) such that for every \( e > 0 \) there exists a neighbourhood of \( x \) in \( X \) which is a bit with diameter \( < e \). The points of \( X_1 \) are said to be regular points of \( X \).
- \( X_{m} \) consists of all points \( x \in \partial X \) such that for every \( e > 0 \) there exists a neighbourhood of \( x \) in \( X \) which is an \( m \)-bit with diameter \( < e \). The points of \( X_{m} \) are said to be points of the order \( m \) of the membrane \( X \).

\[ X_{m} = X - X_{\infty} \cup \bigcup_{m=1}^\infty X_{m} \]

The points of \( X_{m} \) are said to be singular points of \( X \).

These definitions imply the topological invariance of the sets \( X_1 \), \( X_{m} \) and \( X_{m} \).

**Example.** Let \( X \) be an infinite-dimensional membrane (in the finite-dimensional case the argument is analogous) and let \( x \in X \). There occurs one of three cases:

(i) There exists a natural \( l \) such that \( x \) belongs to the \((l+q-1)\)-skeleton of triangulation \( T_l \) and \( x \) does not belong to \( P' \). It follows that \( x \) belongs to the \((j+q-1)\)-skeleton of triangulation \( T_j \), and that \( x \) does not belong to \( P' \) for any \( j \geq l \). From (3) we infer that \( x \in X_1 \).

(ii) For every \( l = 1, 2, \ldots, \), the point \( x \) belongs to the set \( \bigcup_{j=1}^\infty (Z_{(q)}(d_j, \{c_{d_j}\}) - \delta) \),

thus \( x \in X_1 \).

(iii) There exists a natural \( l \) such that \( x \in P \). Only in this case \( x \) can belong to \( \infty \cup_{m=1} X_{m} \).

It follows that every point of ramification and every singular point of \( X \) belongs to one of the segments of ramification \( a_k b_k \), and one can easily see that if \( x \) belongs to the interior of the segment \( a_k b_k \), then \( x \) is not a singular point. Thus only the end-points of the segments of ramification \( a_k b_k \) can be singular and consequently the set of singular points is countable.

6. Points of ramification.

**Lemma 2.** There are only two possibilities: either the simple arc disconnects the \( m \)-membrane into \( m \) components or it does not disconnect it at all.

**Proof.** Let us suppose that there exists a simple arc \( L \) which disconnects the \( m \)-membrane \( Y \). Let us assume at first that the arc \( L \) is an irreducible cutting, that is that no subset \( L \subsetneq L \) disconnects \( Y \). Let \( G_1, G_2, \ldots, G_6 \) denote the components of the set \( Y - L \). Then \( L \) is a common boundary of \( G_1, G_2, \ldots, G_6 \) ([5], p. 175). Let us show that no regular point of \( Y \) belongs to \( L \). Suppose to the contrary that \( x \in L \cap X_{m} \). Then there exists an arbitrary small neighbourhood which is a bit of \( Y \). We can assume that this neighborhood is of the form \( X_{m} \), where \( Q \) is an \( r \)-dimensional polytope connected in a dimension \( \geq 3 \) if \( r > 3 \), and \( Q \) is homeomorphic to an \( r \)-dimensional ball if \( r = 3 \). Since \( L \) is the common boundary of the components of \( Y - L \), we infer that the set \( L \cap X_{m} \) disconnects \( X_{m} \) and since \( Q = (Q^{\infty} - \partial Q^{\infty}) \) denotes the \((r-1)\)-skeleton of the polytope \( Q \), the set \( L \cap (Q^{\infty} - \partial Q^{\infty}) \) disconnects \( Q^{\infty} - \partial Q^{\infty} \). But this is impossible because, if \( r > 3 \), then \( Q^{\infty} - \partial Q^{\infty} \) is connected in a dimension \( > 2 \) and \( \dim (L \cap Q^{\infty} - \partial Q^{\infty}) < 2 \). If \( r = 3 \) it is impossible because then \( L \) disconnects the set \( Q^{\infty} \), homeomorphic to a 2-dimensional sphere.

Thus the arc \( L \) contains only the points of ramification or the singular points of \( Y \) and, since these points belong to the segments of ramification which are disjoint, we infer that \( L \) is included in one of...
them. If $m = 1$, that is if $Y$ is a membrane, one can see at once that none of the segments of ramification disconnects $Y$. If $m > 1$, then by the definition of an $m$-membrane there exists an edge $Y^*$ which disconnects $Y$ into $m$ components $X_1, X_2, \ldots, X_m$. If there exists in $Y$ an other simple arc $L$ which disconnects $Y$ into $p$ components and $p \neq m$, then $L \nsubseteq Y^*$. However, no arc is included in the edge disconnects $Y$.

Now let $L$ be any simple arc in $Y$. Since $Y$ is an AR-set, $L$ contains an irreducible cutting of $Y$ ([7], pp. 176, 287, 335). If $Y$ is a membrane, that is, if $m = 1$, then $L$ does not disconnect $Y$, since no irreducible cutting does. If $m > 1$, and if $L$ disconnects $Y$, then $L \supsetneq Y$ and therefore $L$ disconnects $Y$ into $k$ components where $k \geq m$. But if $k > m$ then for some $i$ the arc $L \cap Y_i$ disconnects the membrane $Y_i$, which is impossible. Thus $k = m$ and the proof is finished.

Let us put $X_i = X_i^1$.

**Lemma 3.** The sets $X_i^1 \cap X_j^1$ are disjoint for $p \neq m$.

**Proof.** Obviously it suffices to consider the case $p < m$. Suppose to the contrary that $x \in X_i^1 \cap X_j^1$. Since $x \in X_i^1$, there exists a neighbourhood $Z_i$ which is a $p$-bit with the wings $Z_1, Z_2, \ldots, Z_p$ and since $x \in X_i^1$, there exists a neighbourhood $Z'$ which is a $p$-bit with the wings $Z_1', Z_2', \ldots, Z_p'$ and such that $Z_1 \cap Z_1' \neq \emptyset$ for each pair $i, j$, the arc $Z' \subseteq Z_i$ and $Z_i \subseteq X_i^1$ into $m$ components $Z_1, Z_2, \ldots, Z_m$ and since $Z_1 \cap Z_1' \neq \emptyset$ for each pair $i, j$, the arc $Z' \subseteq Z_i$ and $Z_i \subseteq X_i^1$ into at least $m$ components, which by Lemma 2 is impossible because $p < m$.

Lemma 3 implies that if $X = P((x_k), i)$, then every point lying in the interior of one of the segments $a_k b_k$, where $d$ is an $n$-simplex of the triangulation $T_k$ in the finite-dimensional case ($k + 1$) of $T_k$ in the infinite-dimensional case), belongs to the set $X_i^{1+n}$ (to the $X_i^{1+n}$ in the infinite-dimensional case). Consequently, for each subsequence $(n_k)$ of the sequence $(n_k)$ and for each open set $G$ of the membrane $X$, the set $G \cap X_i^{1+n_k}$ (the set $G \cap X_i^{1+n_k}$ for $k > k_0$) is of the power $2^m$. On the other hand, the set $\bigcup X_i^1 \cap X_i^1$,

where $N$ is the set of all natural numbers which do not belong to the sequence $(n_k)$ (to the sequence $(k + 1) \cdot n_k$), is at most countable because it contains only the end-points of the segments of ramification.

**7. Main theorem and corollaries.**

**Theorem.** For each $n$, where $n$ is a natural number or infinity, there exists a function $\Phi(t) \in \mathbb{R}^n$ in such a manner that, for $t \neq t'$, if $n$ is a finite number then no $n$-dimensional closed subset of $\Phi(t)$ is homeomorphic to any subset of $\Phi(t')$, and if $n$ is the infinity then no open subset of $\Phi(t)$ is homeomorphic to any subset of $\Phi(t')$ which contains an inner point.

**Proof.** If $n = 2$ the theorem was proved in [3]. The proof in the case of $n > 2$, $n$ finite, is completely analogous and will be merely outlined. In the same manner as in [3] we construct a function assigning to every real number $t$ an increasing sequence of natural numbers $(n_k)$ such that for $t < t'$ the sequence $(n_k)$ contains an infinite sequence $(m_k)$ whose terms do not belong to $(n_k)$. We set

$$\Phi(t) = P((n_k)).$$

Let us suppose that there exists a homomorphism $h$ of the subset $A$ of the membrane $\Phi(t)$ to the subset $h(A)$ of the membrane $\Phi(t')$, where $A$ is an $n$-dimensional closed set. The set $A$ contains a bit $Y$ of the membrane $\Phi(t)$. The points of ramification of order $n_k$ included in an arbitrary open set of $\Phi(t)$ form a set of the power $2^{n_k}$ while the points of ramification of $n_k$ included in the membrane $\Phi(t')$ form the set at most countable. Consequently, there exists in the open set $Y = Y$ a dense subset $R$ consisting of all points of ramification of order $n_k$ and such that any point of the set $h(R)$ is a singular point or a point of ramification of order $n_k$. Further there exists a point $a \in R$ such that $h(a)$ is an interior point of $h(Y)$. But this is impossible because $a$ is a point of ramification of order $n_k$ while $h(a)$ is neither a singular point nor a point of ramification of order $n_k$.

In the infinite-dimensional case, we construct a function assigning to every real number $t$ an increasing sequence of natural numbers $(n_k)$ in a little different manner. It is easy to construct an enumeration $(x_k)$ of all rational numbers $t \in [0, 1]$ such that the set $(x_k; n = 1, 2, \ldots)$ is dense in the segment $[0, 1]$. Let us define an increasing sequence of natural numbers $(n_k)$ by the formula

$$n_k(t) = \min \{n_t : n > n_t(t), |t - n_t + 2\delta_t| < 1/k \}, t \in [0, 1].$$

It is easy to see that if $t \neq t'$, then the sequence $(n_k(t))$ contains a subsequence $(m_k)$ such that the sequence $(q + k) \cdot m_k)$ does not belong to $(n_k(t))$. We set $\Phi(t) = P((n_k(t)))$.

Further the proof is the same as in the finite-dimensional case. It suffices only to replace the points of ramification of order $n_k$ by the points of ramification of order $(q + k) \cdot m_k$.

**Remark.** Let $D'_t$, $t = 1, 2, \ldots, f_k$, denotes the set of all $n$-simplexes (the $(q + k)$-simplexes in the infinite-dimensional case) of the triangulation $T_k$. Let us assign to each pair $(i, k)$, a number $r(i, k) = t + t_i + t_k + s = 0$, and let $(n_{(k)})$ be an increasing sequence of natural numbers $> 1$. It is easy to see that if we build the membrane by constructing the modification set on $D'_t T_k$ by means of $n_{(k)}$ simplexes $A_1, A_2, \ldots, A_{n_{(k)}}$ that is, if we cut off from every simplex in
every path of the construction another number of simplexes, then we obtain a compact AR-set \( P((n_w)A) \) with the following property: if \( \dim P((n_w)A) = n \), then no two homogeneously \( n \)-dimensional different closed subsets of \( P((n_w)A) \) will be homeomorphic, and if \( \dim P((n_w)A) = \infty \), then no two different open subsets of \( P((n_w)A) \) will be homeomorphic.

**Remark 2.** The polytopes \( P_s \) are smoothly connected ([6], p. 124) in the dimension \((n-1)\), and we can construct a polytope ([6], p. 128) subordinate to the polytope \( P_s \).

Let us denote it by \( P_s \). Now if \( P_s+1 \) denotes the modification set on \( P_s \), then we obtain a sequence of polytopes \( \{P_s\} \). If we use them for the construction of the membrane in the same manner as in § 2 and, moreover, if we modify in a suitable manner the definition of the subordinate polytope, then we can obtain a family \( \Phi(t) \) the elements of which are all irreducible \( n \)-dimensional AR-sets [6].

**Corollary 1.** Let \( Y \) be an arbitrary \( n \)-dimensional (infinite-dimensional) ANR-set. There exists a family \( \Psi \) consisting of \( 2^X \) \( n \)-dimensional (infinite-dimensional) ANR-sets such that \( Y \in \Psi \) and none of the elements of \( \Psi \) contains an \( n \)-dimensional closed subset (open subset) homeomorphic to a subset (containing an inner point) of the other element.

**Proof.** In the finite-dimensional case it is the consequence of the following theorem [2].

In an \( n \)-dimensional ANR-set every family of \( n \)-dimensional subsets which are ANR-sets with the common part of any two of them at most \((n-1)\)-dimensional is necessarily at most countable. From this theorem we infer that the subset of elements of \( \Phi(t) \) such that their \( n \)-dimensional closed subsets are homeomorphic with the subsets of \( Y \) is at most countable. Thus, if we remove these elements from the family \( \Phi(t) \), and add the set \( Y \), then we obtain a family \( \Psi \) which has the desired property. In the infinite-dimensional case the proof follows and once from the separability of \( Y \).

The theorem cited in the proof of corollary 1 can be used also in the proof of

**Corollary 2.** There is no universal \( n \)-dimensional AR-set, that is an AR-set which contains all the other \( n \)-dimensional AR-sets.

In the infinite-dimensional case we can formulate this corollary in the following manner:

**Corollary 2’.** For an arbitrary infinite-dimensional AR-set \( X \), there exists another infinite-dimensional AR-set \( Y \) such that for every injection \( \varphi \) of \( Y \) in \( X \) the set \( \varphi(Y) \) is a non-dense set in \( X \).

**Corollary 3.** The \( n \)-dimensional cube has no \( r \)-neighbours on the left ([1]).

The proof is the same as in [3] in the case of \( n = 2 \).