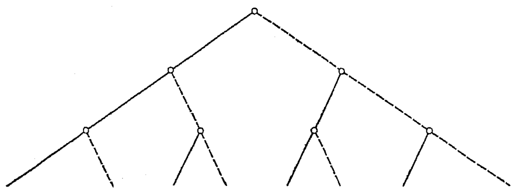
each of the n modal operators M we are dealing with is stronger than its predecessors. This suggests that no finitude result is possible without this assumption of nesting (linear order). In particular, since the proof of (3) already turns on this assumption, it may be expected that without the assumption of nesting we could already have an infinity of irreducible formulae of the form ... $N_{a_2}N_{a_1}p$. This expectation turns out to be justified. It is well known that of each partly ordered set we can obtain a topology by taking for the closure of each set S the set of all the elements e for which $e \leqslant s$ for at least one $s \in S$. The following infinite double tree will then serve as an example which shows the justifiability of our expectation:



Two partly ordering relations are defined on it whose covering relations are indicated by solid and dotted lines, respectively. From the unit set of the highest element one can obviously form an infinity of different sets by repeatedly using the two closure operations.

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Spaces in which sequences suffice*

by

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0. Introduction. Venkataraman [5] poses the following problem.

0.1. Problem. Characterize "the class of topological spaces which can be specified completely by the knowledge of their convergent sequences".

It is a well known and useful fact that every first-countable space falls into this class. Indeed, this is so by virtue of either of two properties of first-countable spaces:

(a) A point lies in the closure of a set iff there is a sequence in the set converging to the point.

(b) A set is open iff every sequence converging to a point in the set is, itself, eventually in the set.

But these properties are not equivalent (see Example 2.2 below) and each is of independent interest (see Arhangel'skiĭ [1], Dudley [3], Franklin and Sorgenfrey [4], Hukuhara and Sibuyo [6], Kelley and Namioka [8], Mazur [10]). Hence problem 0.1 becomes by mitosis the two problems (0.1 (a) and 0.1 (b)) of characterizing the class of spaces satisfying (a) and the class satisfying (b).

The first of these (0.1 (a)) has two known solutions. Kowalsky [9] has given a characterization in terms of the neighborhoods of a point as follows: *A space satisfies* (a) *iff the filter of neighborhoods of each of its points is a union of Fréchet filters*. Since little is known of unions of Fréchet filters, this solution is not completely satisfactory.

A more penetrating solution is given by Arhangel'skii who calls spaces satisfying (a) *Fréchet spaces*. In [1] he asserts, without proof, that *among Hausdorff spaces, Fréchet spaces, and only these, are pseudo-open images of metric spaces*: (Pseudo-open maps form a class between the open maps and the quotient maps. See section 2 below for the definition.) An analogous result, due to Ponomarev [3], characterizes first-countable

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spaces among T_0 spaces as follows: *first countable spaces, and only these, are continuous open images of metric spaces.*

Thus there are two important classes of spaces which are characterized as the images of metric spaces under a certain class of maps. From this fact and the relation between open, pseudo-open and quotient maps, the following problem is completely natural.

0.2. PROBLEM. Characterize the quotients of metric spaces.

Since first-countable spaces are Fréchet spaces and Fréchet spaces are, in turn, *sequential* (i.e., satisfy (b)), a very attractive conjecture would read: *sequential spaces, and only these, are quotients of metric spaces.* That this is indeed the case is shown in section 1 below, yielding simultaneously solutions to problems 0.1 (b) and 0.2. As is clear from Proposition 1.1 below, problem 0.1 (b) is what Venkataramen intended.

Section 1 is devoted to an investigation of sequential spaces, their properties, their behavior under the usual topological constructions, their relations to other topological properties, and lastly, the solution to 0.1 (b) and 0.2. Also, as a by-product, something is added to the question of when the product of two countably compact spaces is countably compact (Corollary 1.10.1).

Section 2 is concerned with Fréchet spaces and their properties and their relation to sequential spaces. The characterization of sequential spaces is used to prove Arhangel'skiĭ's characterization of Fréchet spaces.

Section 3 is a brief coda in which another attractive conjecture is put to rest.

As will become apparent through individual acknowledgments, the author is indebted to Ernest Michael for a number of helpful conversations.

1. Sequential spaces. Let X be a topological space. A subset U of X is *sequentially open* iff each sequence in X converging to a point in U is eventually in U. A subset F of X is *sequentially closed* iff no sequence in F converges to a point not in F. Let τ be the topology of X, and Y be some topological space. Following Venkataramen [5] we say that Y *divides* X iff no topology τ' on X which is strictly finer than τ leaves every τ-continuous function from Y to X τ'-continuous. ω is the first infinite ordinal.

1.1. PROPOSITION. *For any topological space X the properties* 1.1.1 *through* 1.1.3 *are equivalent. If X is Hausdorff they are also equivalent to* 1.1.4 *and* 1.1.5.

1.1.1. $\omega+1$, *provided with its order topology, divides X.*

1.1.2. *Each sequentially open subset of X is open.*

1.1.3. *Each sequentially closed subset of X is closed.*

1.1.4. *Each subset of X which intersects every convergent sequence* (¹) *in a closed set is closed.*

1.1.5. *Each subset of X which intersects every compact metric subspace of X in a closed set is closed* (²).

Proof. 1.1.1 can be restated as follows: every subset of X whose inverse image under each continuous $f\colon \omega+1\to X$ is open in $\omega+1$, is open in X. Now $f\colon \omega+1\to X$ is continuous iff the sequence $\{f(n)\}$ converges to $f(\omega)$ in X. On the other hand each convergent sequence (¹) is the range of a continuous function on $\omega+1$. The correspondence is clearly one-to-one. Hence the inverse image of a subset U of X under each continuous $f\colon \omega+1\to X$ is open iff U is sequentially open. Thus 1.1.1 is equivalent to 1.1.2.

The equivalence of 1.1.2 and 1.1.3 is obvious. That of 1.1.3 and 1.1.4 follows immediately since for Hausdorff X the sequentially closed subsets are precisely those intersecting each convergent sequence in a closed set. For Hausdorff X, each convergent sequence is compact metric and hence 1.1.4 is equivalent to 1.1.5.

A space satisfying any of 1.1.1 thru 1.1.4, and hence all of them, will be called *sequential*, or a *sequential space*. Clearly each first-countable space, and hence each metric space and each discrete space, is sequential. On the other hand, there are countable normal spaces which are not sequential (see [7], p. 77). Such a space must fail to be locally compact. $\Omega+1$, where Ω is the first uncountable ordinal, with its usual order topology is a compact Hausdorff space which is not sequential, since $\{\Omega\}$ is sequentially open but not open. If X is any set, δ the discrete topology and τ a non-sequential topology on X, then the identity map $\mathrm{id}_X\colon \langle X,\delta\rangle\to\langle X,\tau\rangle$ shows that the continuous image of a sequential space need not be sequential.

1.2. PROPOSITION. *Every quotient of a sequential space is sequential.*

Proof. Let f be a quotient map of a sequential space X onto a space Y. If $U\subseteq Y$ is sequentially open and $\{x_n\}\to x_0\in f^{-1}(U)$, then $\{f(x_n)\}\to f(x_0)\in U$ and $\{f(x_n)\}$ is eventually in U. Hence $\{x_n\}$ is eventually in $f^{-1}(U)$ which is therefore open. Hence U is open.

1.3. COROLLARY. *There are sequential spaces which are not first-countable.*

Proof. There are quotients of first-countable spaces which are not first-countable (for example, shrink $\boldsymbol{Z}$ to a point in $\boldsymbol{R}$), but which, by Proposition 1.2, must be sequential.

(¹) By a convergent sequence we mean the union of the sequence and its limit points.

(²) The equivalence of this property to the others was pointed out by E. Michael.

1.4. Corollary. *The continuous open or closed image of a sequential space is sequential.*

1.5. Corollary. *If a product space is sequential, so is each of its factors.*

1.6. Proposition. *The disjoint topological sum of any family of sequential spaces is sequential.*

Proof. Let X be the disjoint sum of the family $\{X_i\}_{i\in A}$ of sequential spaces. If U is not open in X, then for some $i \in A$, $U \cap X_i$ is not sequentially open. Thus there is a point $p \in U \cap X_i$ and a sequence $\{p_n\} \subseteq X_i \setminus U$ converging to p in X_i and therefore in X. Hence U is not sequentially open and the contrapositive of 1.1.2 is established.

The following is immediate from Propositions 1.2 and 1.6.

1.7. Corollary. *The inductive limit of any family of sequential spaces is sequential.*

A negative answer to a question posed by L. C. Robertson is given by the following example.

1.8. Example. To see that *a subspace of a sequential space need not be sequential* (3), even in "nice spaces", let X be the real numbers provided with the topology generated by the usual topology and all sets of the form $\{0\} \cup U$ where U is a usual open neighborhood of the sequence $\{1/n\}$. Thus the topology of the line is altered only at 0 and in such a way that a sequence converging to 0 is either eventually constant a subset of $\{1/n\}$. X is σ-compact (4) and regular but not locally compact.

Next define a subset $Y = \{(x,0) \mid 0 \neq x \in \boldsymbol{R}\} \cup \{(0,1)\} \cup \{(1/n, 1) \mid n \in \boldsymbol{N}\}$ of the plane. Y is the topological sum of a convergent sequence (1) and the punctured real line and is therefore sequential by Proposition 1.6. But the first projection is a quotient map of Y onto X and hence, by Proposition 1.2, X is sequential.

But deleting the sequence $\{1/n\}$ from X leaves a subspace $X\setminus\{1/n\}$ which is not sequential since $\{0\}$ is sequentially open in $X\setminus\{1/n\}$ but not open.

The proof of the next proposition is routine and will be omitted.

1.9. Proposition. *Each open or closed subspace of a sequential space is sequential.*

(3) This was known to Dudley [3]. Example 1.8 is included because it is specific and, more importantly, for later reference. It was known to the author prior to Dudley's paper.

(4) A space which is the countable union of compact sets is called σ-compact. A σ-compact regular space is also Lindelöf, paracompact, normal, etc.

As is well known, countable compactness and sequential compactness are equivalent in the class of first-countable spaces. Since sequential compactness always implies countable compactness, the following proposition establishes their equivalence on the larger class of sequential spaces.

1.10. Proposition. *Every countably compact sequential Hausdorff space is sequentially compact.*

Proof. Let X be sequential and countably compact, and suppose that $\{x_n\} \subseteq X$ has no convergent subsequence. Then $\{x_n\}$ is sequentially closed and hence closed. Thus $\{x_n\}$ is a countable compact Hausdorff space and hence first-countable in the relative topology. But this contradicts $\{x_n\}$ having no convergent subsequence.

Novak [11], answering a question of Čech, showed that the product of two countably compact spaces need not be countably compact.

But Ryll-Nardzewski [4] has shown that the first countability of one of the spaces is sufficient. Hence it is natural to ask if one of the spaces being sequential is enough. But this follows at once from Proposition 1.10 and a result of Mrówka [1] which says that one of the spaces being sequentially compact is enough.

1.10.1. Corollary. *The product of two countably compact spaces, one of which is sequential, is countably compact.*

This proposition is a generalization of Ryll-Nardzewski's result since first countable spaces are sequential.

L. C. Robertson also asked about the preservation of products of sequential spaces. Since for uncountable X, 2^X is compact but not sequentially compact, Proposition 1.10 shows that uncountable products of sequential spaces can fail to be sequential. Although this is not at all surprising, it is remarkable indeed that the *product of two sequential spaces need not be sequential.* An example using distribution spaces of Schwarz is given by Dudley [3]. The following example is somewhat more accessible (5).

1.11. Example. Let Q' be the rationals, Q, with the integers identified, and let $X = Q \times Q'$. X is the product of two sequential spaces but contains a sequentially open set W which is not open.

To describe W let $\{x_n\} \subseteq \boldsymbol{R}$ be a sequence of irrational numbers less than one converging monotonically downward to 0. For $n = 0, 1, \ldots$ let T_n be the interior of the plane triangle determined by the points (x_n, n), $(1, n+\frac{1}{2})$, $(1, n-\frac{1}{2})$. Let T'_n be the reflection of T_n on the y-axis and let R_n be the interior of the rhombus determined by the points $(-x_n, n)$, $(0, n+\frac{1}{2})$, (x_n, n) and $(0, n-\frac{1}{2})$. Then $W_n = T_n \cup R_n \cup T'_n$ is an open

(5) This example was known to the author prior to Dudley's paper.

subset of the plane. Thinking of X as a subset of the plane with the horizontal integer lines identified, let $W = X \cap \bigcup_{0}^{\infty} W_n$.

If $\pi_1\colon X \to Q$ and $\pi_2\colon X \to Q'$ are the canonical projections, it is easy to see that for any neighbourhoods U and U' of 0 in Q and Q' respectively, $\pi_1^{-1}(U) \cap \pi_1^{-1}(U')$ cannot be contained in W. Hence $(0, 0)$ is not an interior point of W which, therefore, cannot be open.

Now suppose $\{y_n\} \subseteq X \setminus W$ and $y_n \to y \in W$. If $\pi_2(y) \neq 0$, convergence in X is simply convergence in $Q \times Q$, and this contradicts $y_n \to y$. Hence $\pi_2(y) = 0$. If $\pi_1(y) \neq 0$, then W can be replaced by a scaled down version of itself with y at the symmetric position. Hence without loss of generality assume that $y = (0, 0)$. But $y_n \to (0, 0)$ implies $\pi_2(y_n) \to 0$ in Q', which can occur iff some subsequence converges in Q to some integer k. But this would restrict $\{y_n\}$ eventually to arbitrarily small strips $\pi_2^{-1}(k-\varepsilon, k+\varepsilon)$ and, since $y_n \to (0, 0)$ in X, would eventually put y_n in W. Hence W is sequentially open.

T. Seidman points out that a similar construction can be carried out in $Q' \times Q'$ so that *the square of a sequential space need not be sequential.*

The principal result of this section is a consequence of the following fact.

1.12. Proposition. *Every sequential space is a quotient of a topological sum of convergent sequences* (1).

Proof. Let X be a sequential space. For each $x \in X$ and for each sequence $\{s_n\}$ in X converging to x, let $S(s, x) = \{s_n \mid n = 1, 2, \ldots\} \cup \{x\}$ be a topological space where each s_n is a discrete point and $s_n \to x$ in $S(s, x)$. Let T be the disjoint topological sum of all possible $S(s, x)$.

Each point of T arose from some point of X yielding a surjection $f\colon T \to X$ which is continuous since it is continuous on each summand (see the proof of Proposition 1.1). To see that f is a quotient map suppose $U \subseteq X$ and $f^{-1}(U)$ is open in T. If $x_0 \in U$ and $s_n \to x_0$ in X, then $x_0 \in f^{-1}(U) \cap S(s, x_0)$ which is open in $S(s, x_0)$. Thus $\{s_n\}$ as a subset of $S(s, x_0)$ is eventually in $f^{-1}(U)$ and thus $\{s_n\}$ as a subset of X is eventually in U. Hence U is sequentially open and thus open. f, therefore, is a quotient map and the proof is complete.

1.13. Corollary. *Every sequential space is the quotient of a zero-dimensional, locally compact, complete metric space.*

1.14. Corollary. *The following are equivalent*

(a) *X is sequential;*

(b) *X is the quotient of a first countable space;*

(c) *X is the quotient of a metric space.*

Proof. (a) implies (b) implies (c) by Proposition 1.12. (c) implies (a) by Proposition 1.2.

Thus the promised complete solution to the problem posed by Venkataramen is established.

E. Michael posed and answered negatively the question of whether each separable sequential space is the quotient of a separable metric space, his example being the real line in the half-open interval topology.

2. Fréchet spaces. A topological space X is called a *Fréchet space* iff the closure of any subset A of X is the set of limits of sequences in A. Again, each first-countable space (and so each metric space and each discrete space) is a Fréchet space, and every Fréchet space is sequential.

The proof of the following proposition is routine and will be omitted.

2.1. Proposition. (a) *Every subspace of a Fréchet space is a Fréchet space.*

(b) *The disjoint topological sum of any family of Fréchet spaces is a Fréchet space.*

2.2. Example. The space X of Example 1.8 is *a sequential space which is not a Fréchet space.* Let $A = X \setminus (\{1/n \mid n = 1, 2, \ldots\} \cup \{0\})$. For each n, there is a sequence $\{x_j^n\}$ in A converging to $1/n$. By the double limit theorem ([7], p. 69), $0 \in \mathrm{cl}A$. But every sequence in X converging to 0 is eventually constant or a subsequence of $\{1/n\}$. Hence X is not Fréchet. Since X is a quotient of the first-countable (and hence Fréchet) space Y, *quotients of Fréchet spaces need not be Fréchet spaces.*

Since the spaces Q and Q' of Example 1.11 are both Fréchet spaces, *the product of two Fréchet spaces need not be a Fréchet space.*

Arhangel'skiĭ [1] calls a map $f\colon X \to Y$ *pseudo-open* iff for any $y \in Y$ and for any open neighborhood U of $f^{-1}(y)$, $y \in \mathrm{int} f(U)$. It is easy to see that any open or closed map is pseudo-open and that each pseudo-open map is a quotient map. (See also the work of Din'N'ë T'ong [2].) The following proposition was asserted without proof by Arhangel'skiĭ ([1], Theorem 4). A proof will be given since none appears in the literature and the result will be used below.

2.3. Proposition. *If X and Y are Hausdorff, X is a Fréchet space and $f\colon X \to Y$ is a quotient map, then Y is a Fréchet space iff f is pseudo-open.*

Proof. Suppose that Y is a Fréchet space, $y \in Y$ and U is an open neighborhood of $f^{-1}(y)$. If $y \notin \mathrm{int} f(U)$, then $y \in \mathrm{cl}(Y \setminus f(U))$. Hence there is a sequence $\{y_n\} \subseteq Y \setminus f(U)$ converging to y. Then $\mathrm{cl}\{y_n\} = \{y_n\} \cup \{y\}$. If $F = f^{-1}(\{y_n\})$, then $\mathrm{cl}F \subseteq f^{-1}(\mathrm{cl}\{y_n\}) = F \cup f^{-1}(y)$. But $f^{-1}(y) \subseteq U$ and $U \cap F = \emptyset$. Hence $f^{-1}(y) \cap \mathrm{cl}F = \emptyset$ and, thus, F is closed. But F closed implies that $X \setminus F = f^{-1}(Y \setminus \{y_n\})$ is open and, therefore, that $Y \setminus \{y_n\}$ is open, contradicting $\{y_n\} \to y$. Hence $y \in \mathrm{int} f(U)$ and f is pseudo-open.

Conversely, if f is pseudo-open, let $y \in \mathrm{cl}M$ with $M \subset Y$. If $f^{-1}(y) \cap \mathrm{cl} f^{-1}(M) = \emptyset$, let $U = X \setminus \mathrm{cl} f^{-1}(M)$. Then $y \in \mathrm{int} f(U) \subseteq X \setminus M$ contradicting $y \in \mathrm{cl} M$. Thus there is some $x_0 \in f^{-1}(y) \cap \mathrm{cl} f^{-1}(M)$. Choose a sequence $\{x_n\} \subseteq f^{-1}(M)$ converging to x_0. Then $\{f(x_n)\} \subseteq M$ and $f(x_n) \to y$. Thus Y is a Fréchet space.

With the help of this proposition Arhangel'skiĭ's characterization of Fréchet spaces ([1], Theorem 2) can be easily deduced from Proposition 1.12.

2.4. Proposition. *Among Hausdorff spaces, Fréchet spaces are precisely the pseudo-open images of a topological sum of convergent sequences.*

Proof. Each Fréchet space is sequential and hence the quotient of such a sum by Proposition 1.12. But the sum is a Fréchet space and hence, by Proposition, 2.9 the quotient map must be pseudo-open. The converse follows immediately from Proposition 2.9.

A number of corollaries analogous to those following Proposition 1.12 can, of course, be stated.

3. First-countable spaces. In light of Propositions 1.12 and 2.4 it is only natural to ask whether Ponomarev's characterization of first-countable spaces (see section 0) can be strengthened to read "every first countable space is the open image of a topological sum of convergent sequences (1)"? But any such sum is a Baire space (see, for example, de Groot [5]) as are continuous open images of Baire spaces. But many spaces (for example Q) are first-countable but not Baire spaces. Hence the answer is no.

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