On open mappings and certain spaces satisfying the first countability axiom

by

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A. H. Stone [16] showed that a regular space E is metrizable and locally separable if E is the image of a locally separable metric space under an open mapping f such that, for each $p \in E$, $f^{-1}(p)$ is separable. In [5], S. Hanai showed that a T_1-space satisfies the first countability axiom if and only if it is the open continuous image of a metric space. Similar or related theorems are to be found in [2], [3], [6], [7], [8], [14], and [15]. In particular, A. Arhangel'skiĭ [3] showed that a T_1-space Y has a uniform basis (see [1] or [2]) if and only if there is an open mapping f from some metric space onto Y such that, for each $p \in Y$, $f^{-1}(p)$ is a bicompactum. Note that, by Theorem 4 of [11], a space Y has a uniform basis if and only if it is a pointwise paracompact developable space.

In this paper necessary and sufficient conditions are given for the open continuous image of a metric space to be (1) a semimetric space, (2) a developable space, (3) a Nagata space or (4) metrizable. Besides characterizing these four main classes of spaces which satisfy the first countability axiom, the four theorems point out a surprising relation that exists among those classes of spaces. Also a characterization of Nagata spaces similar to that for semi-metric, developable and metric spaces in [9] is given.

Terms not defined are used as in [17] or [2]. A *mapping* will be a continuous function. If X is a metric space, $a \in X$ and $e > 0$, $S(a, e)$ will denote the open ball, $\{x: |x-a| < e\}$, of radius e with center a.

DEFINITION 1. Let f be a mapping from the metric space X onto the topological space Y. Then f is a *P-mapping* provided that, for every $p \in Y$ and every open set R containing p, there is an $e > 0$ such that $f[S(f^{-1}[p], e)] \subset R$ (i.e., f is uniformly continuous on $f^{-1}(p)$); and f is a *C-mapping* provided that, for every compact subset K of Y and every open set R, $R \supset K$, there is an $e > 0$ such that $f[S(f^{-1}[K], e)] \subset R$.

DEFINITION 2. A topological space Y is *developable* provided that there is a sequence $G_1, G_2, G_3, \ldots$ of open coverings of Y such that (1) for each i, $G_i \supset G_{i+1}$ and (2) for any open set R containing any point p of Y there is an n such that $p \in g \in G_n$ implies $g \subset R$.

Note that a developable T_3-space is a Moore space (satisfies Axiom 1_3 of [13]). From Theorem 1 below one sees which part of the conclusion in Theorem 1 of [3] still holds when one removes the pointwise paracompactness (implicit in a uniform basis) from the hypothesis.

THEOREM 1. *A necessary and sufficient condition that a T_1-space Y be developable is that there exist an open P-mapping f from some metric space X onto Y.*

Proof. The condition is necessary. For suppose that $G_1, G_2, G_3, \ldots$ is a development for Y. For each n let A_n be a discrete topological space such that there is a one-to-one function g_n from A_n onto G_n. Let X then be that subspace of the product space $\prod_{n=1}^{\infty} A_n$ consisting of all points $x = x_1x_2x_3 \ldots$ such that, for $p \in Y$, $\bigcap_{n=1}^{\infty} g_n(x_n) = \{p\}$, and for each $x \in X$, let $f(x) = \bigcap_{n=1}^{\infty} g_n(x_n)$. Then X is a 0-dimensional metric space with a metric d defined as follows: if $x, y \in X$ and n is the smallest natural number k such that $x_k \neq y_k$ then $d(x, y) = 1/n$. It is easily seen that f is an open mapping of X onto Y (or see the proof of Theorem 1 in [5], noting that, for $x \in X$, $\{g_n(x_n)\colon n = 1, 2, \ldots\}$ is an open neighbourhood basis for $f(x)$). Also f is a P-mapping. For suppose that $p \in S$ and R is an open set containing p. By definition of development, there is a natural number n such that if $h \in G_n$ and $p \in h$ then $h \subset R$. But if $x, y \in X$, $f(x) = p$, and $d(x, y) < 1/n$, then $x_i = y_i$ for $i \leqslant n$, so that $p \in g_n(x_n) = g_n(y_n)$ and $\{f(y)\} = \bigcap_{n=1}^{\infty} g(y_n) \subset R$. Thus $f\big[S(f^{-1}[p], 1/n)\big] \subset R$. It follows then that f is a P-mapping.

Conversely, suppose that f is an open P-mapping from the metric space X onto the topological space Y. For each natural number n, let $G_n = \{f[S(x, 1/m)]\colon x \in X,\ m = n, n+1, \ldots\}$. Then $G_1, G_2, \ldots$ is a development. For suppose that p is a point of the open subset R of Y. Since f is a P-mapping, there is a natural number n such that $f\big[S(f^{-1}[p], 1/n)\big] \subset R$. Note that, if $m \geqslant 2n$ and $p \in f[S(x, 1/m)]$, then $S(x, 1/m) \cap f^{-1}(p) \neq \emptyset$ so that $S(x, 1/m) \subset S(f^{-1}[p], 1/n)$, and hence $f[S(x, 1/m)] \subset R$. Thus, if $p \in h \in G_{2n}$, then $h \in R$.

The condition therefore is sufficient.

Note that Theorem 9 of [3] is an immediate corollary of the next theorem (but apparently not conversely).

THEOREM 2. *A (necessary and) sufficient condition that a T_2-space Y be metrizable is that there exist a metric space X and an open C-mapping from X onto Y.*

Proof. The condition is sufficient. For suppose that f is an open C-mapping from the metric space X onto the T_2-space Y. For each natural number n, let $G_n = \{f[S(x, 1/m)]\colon x \in X,\ m = n, n+1, \ldots\}$. Since f is a C-mapping, then, for any two nonintersecting closed subsets H and K of Y one of which is compact, there is an n such that no member of G_n intersects both H and K. Therefore, by a theorem of F. B. Jones [12], Y is metrizable.

A topological space Y is a *semi-metric space* provided that there is a distance function d defined on Y such that, for $x, y \in Y$, (1) $d(x, y) = d(y, x)$, (2) $d(x, y) \geqslant 0$ and $d(x, y) = 0$ only in case $x = y$, and (3) the topology of Y is invariant with respect to d.

The following theorem from [9] is useful in Theorem 4.

THEOREM 3. *The topological space Y is semi-metric if and only if there exists a collection of open sets, $\{g_n(x)\colon x \in Y,\ n = 1, 2, \ldots\}$, such that (1) for each $x \in Y$, $\{g_n(x)\colon n = 1, 2, \ldots\}$ is a local neighborhood base for x and (2) if $y \in Y$ and x is a point sequence in Y such that, for each m, $y \in g_m(x_m)$, then x converges to y.*

By Theorem 4, for the image of a metric space X under an open mapping f to be semi-metric it is necessary only that f be a P-mapping relative to a certain subset of X.

THEOREM 4. *A T_1-space Y is semi-metric if and only if there exist an open mapping f from some metric space X onto Y and a subset X' of X such that (1) $f(X') = Y$ and (2) if $p \in Y$ and R is a neighborhood of p, then, for some $\varepsilon > 0$, $f\big[S(f^{-1}[p], \varepsilon) \cap X'\big] \subset R$.*

Proof. The condition is necessary. For suppose that Y is a semi-metric space and $\{g_n(x)\colon x \in Y,\ n = 1, 2, \ldots\}$ is an open basis satisfying the conditions in Theorem 3. For each n let A_n be the discrete topological space whose points are the points of Y; let X be the subspace of $\prod_{n=1}^{\infty} A_n$ consisting of all points $x = x_1x_2x_3 \ldots$ such that, for some $p \in Y$, $\{g_n(x_n)\colon n = 1, 2, \ldots\}$ is a local base for p; and for each $x \in X$ let $f(x) = \bigcap_{n=1}^{\infty} g_n(x_n)$. That X is metric and f is an open mapping from X onto Y again follows as in the proof of Theorem 1 in [5]. Also, if X' consists of all points x of X such that $x_1 = x_2 = x_3 = \ldots$, then it is easily seen that X' satisfies (1) and (2).

Conversely, suppose that there is an open mapping f from some metric space X onto the T_1-space Y and that there is a subset X' of X satisfying (1) and (2). Then, for each $p \in Y$, let x be a point of X' such that $f(x) = p$ and let $g_n(p) = f[S(x, 1/n)]$ for $n = 1, 2, \ldots$ It is easily seen that $\{g_n(p)\colon p \in Y,\ n = 1, 2, \ldots\}$ satisfies the conditions of Theorem 3, so that Y is a semi-metric space.

Definition 3. A *Nagata space* X is a T_1-space such that, for each $x \in X$, there exist sequences of neighborhoods of x, $\{U_n(x)\colon n = 1, 2, \ldots\}$ and $\{S_n(x)\colon n = 1, 2, \ldots\}$ such that (1) for each $x \in X$, $\{U_n(x)\colon n = 1, 2, \ldots\}$ is a local base for x, (2) for all $x, y \in X$, $S_n(x) \cap S_n(y) \neq \emptyset$ implies that $x \in U_n(y)$.

Remark. A Nagata space is a paracompact semi-metric space [4], but it is not known whether the converse is true. By Theorem 3.1 of [4], a T_1-space is a Nagata space if and only if it is first countable and has a σ-cushioned pair-base (defined below).

Definition 4. A *pair-base* P for a space X is a collection of ordered pairs of subsets of X such that (1) for every $(p_1, p_2) \in P$, p_1 is open and $p_1 \subset p_2$ and (2) for every $x \in X$ and every neighborhood U of x, there exists a $(p_1, p_2) \in P$ such that $x \in p_1 \subset p_2 \subset U$. Also P is called *cushioned* if, for every subcollection P' of P, $\mathrm{Cl}(\bigcup\{p_1\colon (p_1, p_2) \in P'\}) \subset \bigcup\{p_2\colon (p_1, p_2) \in P\}$ ($\mathrm{Cl} =$ closure); and P is *σ-cushioned* if it is the union of countably many cushioned subcollections.

A characterization of Nagata spaces which is somewhat simpler and is useful in Theorem 6 is given by the next theorem.

Theorem 5. *The T_1-space Y is a Nagata space if and only if there exists a collection $\{g_n(x)\colon x \in Y,\ n = 1, 2, \ldots\}$ of open sets such that, for each $x \in Y$, (1) $\{g_n(x)\colon n = 1, 2, \ldots\}$ is a local base for x and (2) for every neighborhood R of x there is a natural number n such that $g_n(x) \cap g_n(y) \neq \emptyset$ implies that $y \in R$.*

Proof. If Y is a Nagata space, such a collection is obtained as follows: for each $x \in Y$, let $g_n(x)$ be the interior of $S_n(x)$ for $n = 1, 2, \ldots$ (see Definition 3).

Conversely, suppose that there exists such a collection $\{g_n(x)\colon x \in Y,\ n = 1, 2, \ldots\}$ of open subsets of the T_1-space Y. It will be shown that Y has a σ-cushioned pair-base. For each natural number m and each $n > m$, let $M(m, n)$ be the set of all points x of Y such that if $g_n(x) \cap g_n(y) \neq \emptyset$ then $y \in g_m(x)$. Clearly, for each m and n and each subset $M'(m, n)$ of $M(m, n)$,

$$\mathrm{Cl}\big(\bigcup\{g_n(x)\colon x \in M'(m, n)\}\big) \subset \bigcup\{g_m(x)\colon x \in M'(m, n)\}.$$

Thus $\{(g_n(x), g_m(x))\colon x \in M(m, n),\ m, n = 1, 2, \ldots\}$ is a σ-cushioned pair-base for Y and since Y is also first countable, it is a Nagata space (see above remark).

By Theorem 6, together with Theorems 1, 2 and 4, Nagata spaces would to be related to semi-metric spaces in the same way as metric spaces are to developable spaces. This suggests, for example, that, since a paracompact developable space is metrizable, a paracompact semi-metric space should be a Nagata space.

Theorem 6. *A T_2-space Y is a Nagata space if and only if there is an open mapping f from some metric space X onto Y and a subset X' of X such that (1) $f(X') = Y$ and (2) for every compact subset K of Y and every neighborhood R of K, there is an $\varepsilon > 0$ such that $f[S(f^{-1}(K), \varepsilon) \cap X'] \subset R$.*

Proof. Suppose that Y is a Nagata space. Let $\{g_n(x)\colon x \in Y,\ n = 1, 2, \ldots\}$ be a basis for Y satisfying (1) and (2) of Theorem 5. For each n, let A_n be the discrete topological space whose points are the points of Y; let X be the subspace of $\prod_{n=1}^{\infty} A_n$ consisting of all points $x = x_1x_2x_3\ldots$ such that, for some $p \in Y$, $\{g_n(x_n)\colon n = 1, 2, \ldots\}$ is a local base for p; for each $x \in Y$, let $f(x) = \bigcap_{n=1}^{\infty} g_n(x_n)$; and, let X' consist of all points x of X such that $x_1 = x_2 = x_3 = \ldots$ Clearly f is an open mapping from X onto Y (see proof of Theorem 1 in [5]) and X' satisfies condition (1). Suppose that X' does not satisfy (2). Then there is a compact subset K of Y, a neighborhood R of K, and point sequences x and y such that, for every natural number n, $x_n \in X'$, $f(y_n) \in K$, $d(x_n, y_n) < 1/n$ (the metric d is defined in the proof of Theorem 1), but $f(x_n) \notin R$. Note that, for each n, $d(x_n, y_n) < 1/n$ and $x_n \in X'$ imply that $f(y_n) \in g_n(f(x_n))$. Since K is compact, the sequence $f(y_1), f(y_2), \ldots$ has a cluster point z in K, and, further, it may be assumed without loss of generality that $f(y_n) \in g_n(z)$ for each n. Thus $g_n(f(x_n)) \cap g_n(z) \neq \emptyset$ for each n, so that by Theorem 5 there is an n such that $f(x_n) \in R$ contrary to assumption. Thus X' satisfies (2).

Conversely suppose that there exist such an open mapping f from a metric space X onto the T_1-space Y and such a subset X' of X. For each n and each $y \in Y$, let $x \in f^{-1}(y) \cap X'$ and let $g_n(y) = f[S(x, 1/n)]$. Clearly $\{g_n(y)\colon y \in Y,\ n = 1, 2, \ldots\}$ satisfies condition (1) of Theorem 5. Suppose that it does not satisfy (2). Then there is a point $y \in Y$, a neighborhood R of y, and a point sequence x in Y such that, for each n, $x_n \notin R$ but $g_n(x_n) \cap g_n(y) \neq \emptyset$. If, for each n, $z_n \in g_n(x_n) \cap g_n(y)$, then, for some natural number N, $K = \{y\} \cup \{z_n\colon n \geqslant N\}$ is a compact set of which R is a neighborhood. Then, by the hypothesis on f and X and the definition of $g_n(x_n)$, there is a natural number M such that, for $n > M$, $x_n \in R$ contrary to assumption. Hence, by Theorem 5, Y is a Nagata space.

It is hoped that the above theorems might be useful for, among other things, determining whether a normal Moore space is metrizable and whether a paracompact semi-metric space is a Nagata space. An answer to the following question should help considerably to solve those problems.

QUESTION. *What is a necessary condition on an open mapping f from a metric space onto a T_2-space Y for Y to be normal (or paracompact)?*

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A closure and complement result for nested topologies

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It is well known that from a given set one can in a topological space construct at most 14 different sets by repeatedly using the operations of complementation and closure. The main purpose of this note is to establish a similar result for any finite number of nested topologies. Given any finite sequence of topologies each member of which is finer than its predecessor, from a fixed set one can construct only a finite number of different sets by repeatedly using the operations of complementation and of closure with respect to any topology of the sequence. It will be shown how this number is determined, and an upper bound will be given to it.

This will be accomplished by means of methods developed in modal logic. It is well known that the system of modal logic which is called S4 is interpretable as the closure algebra: If M and N are understood as the closure and the interior operator, respectively, and if $\sim$, &, and $\vee$ are understood in their usual Boolean sense, then a function formed by their means from set variables is identically the whole space in all topological spaces if and only if the same function is provable in S4 when the variables are interpreted as propositional variables, when $\sim$, &, and $\vee$ have their normal propositional senses, and when M and N are interpreted as the symbols for possibility and necessity, respectively [1]. This connection is extended to the case of a finite sequence of finer and finer topologies by considering a sequence of modal operators $M_0, N_0, M_1, N_1, \ldots, M_{n-i}, N_{n-i}$ where each pair M_i, N_i $(i = 0, 1, \ldots, n-1)$ is subject to the laws of S4 and where we have as an additional assumption the axiom schema

(1) $$M_i f \supset M_{i-1} f$$

for each $i = 1, \ldots, n-1$ (or, equivalently,

(2) $$N_{i-1} f \supset N_i f$$

for each $i = 1, \ldots, n-1$), where f is an arbitrary formula.

[1] For a lucid summary of many interesting results concerning the relation of modal logics to topology, see H. Rasiowa [6].