Coincidences of real-valued maps from the $n$-torus

by

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1. Introduction. Let $E_n$ denote euclidean $n$-space and suppose that $f: E_n \to E_n$ is a continuous periodic function of period 1. In the number interval $[0, 1]$ there is a point $p$ where $f$ attains its maximum. Suppose that $\lambda$ is a real number and let $g$ denote the function from $E_1$ such that $g(z + \lambda) = f_{\lambda}$. Then $g(p) \leq 0$ and $g(p - \lambda) \geq 0$. By the intermediate value theorem there is a point $x^*$ such that $g(x^*) = 0$; $f(x^* + \lambda) = f_{\lambda}$. If we take $\lambda = \frac{1}{2}$ and identify points in the domain of $f$ with coordinates congruent mod $1$, our result can be restated in the following form:

\[(1.1) \text{If } f \text{ is a real-valued mapping from a circle } C, \text{ there is a pair } (x^*, y^*) \text{ of diametrically opposite points of } C \text{ such that } f(x^*) = f(y^*).\]

A generalization of (1.1) to higher dimensional spheres is commonly known as the Borsuk-Ulam theorem. If $f$ is a continuous function from the $n$-sphere $S^n$ into $E_n$, there is a pair $(x^*, y^*)$ of diametrically opposite points of $S^n$ such that $f(x^*) = f(y^*)$ ([11], p. 178, Satz II (?)). A different generalization of (1.1), applying to topological products of circles, has been devised by W. Schmidt ([4], p. 86, Satz 1). The 2-dimensional case of Schmidt's theorem runs as follows:

Suppose that each of $f_1$ and $f_2$ is a real-valued mapping from $E_2$ and for each number pair $(x_1, x_2)$

\[
f_1(x_1, x_2) = f_2(x_1 + 1, x_2),
\]

\[
f_2(x_1, x_2) = f_1(x_1, x_2 + 1) = f_1(x_1 + 1, x_2).\]

Then there is a number pair $(x_1^*, x_2^*)$ such that $f_1(x_1^*, x_2^*) = f_1(x_1^* + 1/2, x_2^*)$ and $f_2(x_1^*, x_2^*) = f_2(x_1^*, x_2^* + 1/2)$. Thus there is a square in $E_2$, with sides of length $1/2$, on the vertices of which each of $f_1$ and $f_2$ is constant.

We proved (1.1) by particularizing the parameter $\lambda$ that appears in a more general theorem. This suggests a direction in which Schmidt's

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?) The numbers which appear in brackets in this paper correspond to the numbers in the bibliography of this paper.
A theorem might be extended. This paper is devoted to a proof of such an extension. A special case of the theorem proved here (17.9) below reads:

(1.2) If $p$ is a prime and each of $f_1$ and $f_2$ is a real-valued mapping from $E_2$ such that for any number pair $(x_1, x_2)$,

\[ f_1(x_1, x_2) = f_1(x_1 + 1, x_2) = f_1(x_1, x_2 + 1) \]

and

\[ f_2(x_1, x_2) = f_2(x_1 + 1, x_2) = f_2(x_1, x_2 + 1) \]

then there is a number pair $(x_1, x_2)$ such that $f_1(x_1, x_2) = f_1(x_1 + 1, x_2)$ and $f_2(x_1, x_2) = f_2(x_1 + 1, x_2)$. Thus there is a square in $E_2$ with sides of length $1/p$ on the interiors of which each of $f_1$ and $f_2$ is constant.

The compact space obtained from $E_2$ by identifying points with coordinates congruent (mod 1) will be called the $n$-torus $T^n$. $T^2$ is a simple closed curve; $T^3$ is an ordinary torus. Our principal theorem can be formulated as a statement about coincidences of real-valued maps from $T^n$ and can be derived from a theorem about the incidence relations in certain finite closed covers of $T^n$. The latter is deducible by a continuity argument from a theorem about intersections of sub complexes of a complex $K$ whose polyhedron is homeomorphic to $T^n$. We construct a homology theory on $K$ and a homomorphism $\rho_K$ from the homology groups on $K$ into a ring $Z_p$. To know that the intersection of two sub complexes of $K$ is nonempty, it suffices to show that their intersection carries a nonzero element of a homology group; to know that an element of a homology group is nonzero, it suffices to show that its image under $\rho_K$ is nonzero. We take advantage of the fact that $T^n$ is a topological product by expressing $K$ as a product of complexes $H_1, ..., H_n$, each with a polyhedron homeomorphic to the circle $T^1$, by defining a product $c_1, ..., c_n$ of the chains on $H_1, ..., H_n$ into the chains on $K$, and by showing that if $x$ is a cycle in the homology theory for $H_1, ..., H_n$, then

\[ (\rho_K c_1)(\rho_K c_2) ... (\rho_K c_n) = \rho_K(c_1 \times c_2 \times ... \times c_n) \]

where juxtaposition indicates the ring product in $Z_p$. To know that $\rho_K(c_1 \times ... \times c_n) \neq 0$ it suffices to prove that $\rho_K e_i = 1$ for $1, ..., m$. Such $e_i$ exist.

The general method, then, is the same as that used by Schmidt; the proof that follows may be regarded as a refinement and completeness of Schmidt's argument ([4], pp. 88-91).

2. Some definitions. Definitions of technical terms not defined in this paper may be found in [2] or [3].

If $n$ is a nonnegative integer, the statement that $S$ is an $n$-simplex means that there is a set $\{A^1, ..., A^n\}$ of $n + 1$ objects such that $S$ is the set of all functions $\alpha$ from $\{A^1, ..., A^n\}$ into the positive real numbers such that $\sum_{\alpha} \alpha(A^i) = 1$. Each of $A^1, ..., A^n$ is called a vertex of $S$. Each member of $S$ is called a point of $S$.

A complex $K$ is a finite collection of one or more simplexes such that, if $S$ is a simplex of $K$ with vertex set $V$ and $L$ is a nonempty subset of $V$ then, the simplex with vertex set $L$ is in $K$.

If $H$ is a finite collection of one or more nonempty sets, the nerve of $H$ is defined to be the simplex collection to which $S$ belongs if each vertex of $S$ is in $H$ and, if $S$ has more than one vertex, the vertices of $S$ have an element in common. The nerve of a finite collection of sets is a complex.

If $n$ is a nonnegative integer and $S$ is an $n$-simplex with vertex set $\{A^1, ..., A^n\}$, an orientation of $S$ is a function $f$ into $(-1, 1)$ from the set of simple orders for $\{A^1, ..., A^n\}$ such that, if $(A^n, ..., A^1)$ is an odd permutation of $\{A^1, ..., A^n\}$, then $f(A^n, ..., A^1) = -f(A^1, ..., A^n)$. A simplex has two orientations, one the negative of the other. The orientation of $S$ whose value at $(A^n, ..., A^1)$ is 1 will be denoted by $<A^1, ..., A^n>$. If $w$ is a simplicial map which is 1-1 on $\{A^1, ..., A^n\}$, then $<\alpha(A^1), ..., \alpha(A^n)>$ will denote $w <A^1, ..., A^n>$.

If $n$ is a nonnegative integer, $X$ is a complex, and $G$ is an abelian group, then a $G$-valued $n$-chain on $X$ is a function $c$ from the orientations of $n$-simplexes of $X$ into $G$ such that, if $E$ is an orientation of an $n$-simplex of $X$, $c(-E) = -cE$. The $G$-valued $n$-chains on $X$ form, under functional addition, an abelian group which will be denoted by $C_n(X, G)$.

Suppose that each of $X$ and $Y$ is a complex and $f: X \to Y$ is a simplicial map. Let $f$ be the function from $C_n(X, G)$ into $C_n(Y, G)$ such that, if $c \in C_n(X, G)$ and $F$ is an orientation of an $n$-simplex of $X$, $f(cF) = \sum F \cdot E$ where $G$ is the set of orientations of simplexes of $X$ to which $E$ belongs iff $fE = F$, $f$ is a chain map, i.e., is a homomorphism that commutes with the boundary operator $\partial$. Hereafter $f$ will be denoted by $f$. A similar convention holds for any other letter of the alphabet.

(2.3) Notice that, if $w$ is a 1-1 simplicial map from $X$ onto $X$ and $c$ is an $n$-chain on $X$, then $(wc)E = c(w^{-1}E)$ for each orientation $E$ of an $n$-simplex of $X$.

Suppose that $\{A^1, ..., A^n\}$ is an orientation of an $n$-simplex of $X$, $g \in G$ and $c$ is an $n$-chain on $X$ such that

(a) $c(A^1, ..., A^n) = g$ and

(b) if $E$ is an orientation of a $n$-simplex of $X$ different from $S$ then $cE = 0$.

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If \( G = \mathbb{Z} \), the additive group of the integers, and \( g = 1 \) then \( e \) is called an elementary chain and is denoted by \( A^0 \ldots A^n \). Whether \( G = \mathbb{Z} \) or not, \( e \) is denoted by \( gA^0 \ldots A^n \). Every member of \( C_d(X, G) \) may be represented in the form \( \sum J[\alpha_1A^0 \ldots A^n] \).

If \( n \) is an integer and \( e \) is a chain, then \( ne \) is uniquely defined by the equations

\[
0 = 0 \quad \text{and} \quad (n-1)e = ne + e.
\]

Suppose that \( g \in G \), \( \{A^0 \ldots A^n\} \cap \epsilon \) is a finite collection of elementary chains and each of \( u_1, \ldots, u_m \) is an integer. Then \( \sum u_igA^0 \ldots A^n \) will sometimes be denoted by \( \sum u_igA^0 \ldots A^n \).

The support of an \( n \)-chain \( e \) is the set of \( n \)-simplices on whose orientations \( e \) does not assume the value 0.

3. The cartesian product of chains. Definitions. For the simplicial product \( K \times K \) of complexes \( K \) and \( K \), an order for a complex and the cartesian product of ordered complexes are given in [2], pp. 66-67. (If \( \leq \) is an order for the complex, the pair \( \{K, \leq\} \) will be called an ordered complex.)

Remark. Any complex can be ordered by assigning a simple order to the vertices of the complex and then deleting from that order each vertex pair not connected by a 1-simplex. On the other hand there is a complex for which there is an order that cannot be imbedded in a simple order.

If each of \( G, H \) and \( J \) is an abelian group, a binary composition \( \varphi \colon (G, H) \to J \) is called a multiplication, if

\[
\varphi(g, h) = \varphi(g, h) + \varphi(h, g)
\]

and

\[
\varphi(g_1, g_2, h) = \varphi(g_1, h) + \varphi(g_2, h)
\]

whenever \( g, g_1, g_2 \in G \) and \( h, h_1, h_2 \in H \).

(3.1) Suppose that each of \( G, H \) and \( J \) is an abelian group, \( \varphi \colon (G, H) \to J \) is a multiplication, \( g \in G \) and \( h \in H \). Then \( \varphi(g, 0) = \varphi(0, h) = 0 \) and \( \varphi(-g, h) = -\varphi(g, h) = \varphi(g, -h) \). (Proof omitted.)

Suppose that \( X \) is a complex, \( A^0 \ldots A^n \) is an \( n \)-dimensional elementary chain on \( X \), and \( \{A^0, \ldots, A^n, B^0 \ldots B^m \} \) is the vertex set of a \((n+m)-1\)-simplex of \( X \). Then the elementary chain \( A^0 \ldots A^nB^0 \ldots B^m \) is called the join of \( A^0 \ldots A^n \) and \( B^0 \ldots B^m \) and is denoted by \( A^0 \ldots A^nB^0 \ldots B^m \). If each of \( \{A^0 \ldots A^n, B^0 \ldots B^m, C^0 \ldots C^t \} \) is a finite collection of elementary chains, each of \( u_1, \ldots, u_r, v_1, \ldots, v_s \) is an integer and \( A^0 \ldots A^n \), \( B^0 \ldots B^m \), \( C^0 \ldots C^t \) is defined when \( i \in \{1, \ldots, r\} \) and \( j \in \{1, \ldots, s\} \), then

\[
\left( \sum_{i=1}^r \sum_{j=1}^s u_i A^0 \ldots A^n + B^0 \ldots B^m \right) \cdot \left( \sum_{i=1}^t v_i C^0 \ldots C^t \right)
\]

will sometimes be denoted by

\[
\left( \sum_{i=1}^r u_i A^0 \ldots A^n \right) \cdot \left( \sum_{i=1}^t v_i C^0 \ldots C^t \right).
\]

(3.2) If each of \( \{K_0, \leq\} \) and \( \{K_1, \leq\} \) is an ordered complex, there is a semi-linear homeomorphism from \( X_1 \times X_2 \) onto the topological product of \( X_1 \) and \( X_2 \) ([2], p. 68). We wish to define a "cartesian" product of chains on \( X_1 \) and \( X_2 \) into those on \( X_1 \times X_2 \) which is the algebraic counterpart of the cartesian product of complexes and the topological product of spaces. For \( i \in \{1, 2\} \), suppose that \( S_i \) is a \( d_i \)-dimensional simplex in the complex \( X_i \), \( H_i \) is the subcomplex of \( K_i \) consisting of \( S_i \) and all its faces and \( c \) is an elementary \( d_i \)-chain with support \( S_i \). The cartesian chain product of \( c \) and \( c \) should be a chain whose dimension is \( d_1 + d_2 \), whose support is the set of \((d_1 + d_2)\)-simplices in \( H_1 \times H_2 \) whose boundary is 0 on any \((d_1 + d_2)\)-simplex in \( H_1 \times H_2 \) which is a common face of two \((d_1 + d_2)\)-simplices in \( H_1 \times H_2 \). We now give a definition having these properties.

Suppose that each of \( \{K_1, \leq\} \) and \( \{K_1, \leq\} \) is an ordered complex. If \( A \) is a vertex of \( K_1 \) and \( B \) is a vertex of \( K_2 \), the cartesian chain product of the elementary 0-chains \( A \) and \( B \) is defined to be the elementary 0-chain \( (A, B) \). For products of elementary chains the definition proceeds on the dimensions. Suppose \( A^0 \ldots A^n \) is an elementary \( a \)-chain of \( K_1 \), \( A^0 \leq \ldots \leq A^a \leq \ldots \leq A^n \leq B^0 \ldots B^m \) is an elementary \( b \)-chain of \( K_2 \). If \( a > 0 \), \( A^0 \ldots A^n \times B^0 \ldots B^m \) means \( (A^0, B^0) \ldots (A^n, B^m) \). (Thus \( A^a \times B^b = (A^a, B^b)(A^a, B^b) \ldots (A^a, B^b) \).)

If \( b > 0 \), \( A^0 \ldots A^n \times B^0 \ldots B^m \) means \( (A^0, B^0, B^1) \ldots (A^n, B^m, B^m) \). If \( a, b > 0 \), \( A^a \times B^b \) means

\[
\sum A^0 \ldots A^n \times B^0 \ldots B^m = (A^a, B^b)(A^a, B^b) \ldots (A^a, B^b)
\]

(3.3) Suppose \( \{G_1, G_2, G_3\} \) is an abelian group and \( \varphi \colon (G_1, G_2) \to G_3 \) is a multiplication. Then we define

\[
\times_i [G_1, G_2, G_3] \to C_{d+1}(X, X, G_3)
\]

by the equation

\[
\sum_i \varphi(g_i, g_1)(A^0 \ldots A^n \times B^0 \ldots B^m)
\]

The definition of cartesian chain product given here, unlike the one given in [3], p. 138, is suitable for use in Čech homology theory. The proofs of (3.2), (3.3), and (3.6) below are lengthy but routine and are omitted.

(3.2) The cartesian chain product is a multiplication.

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Page 67 of a paper by P. Bacon. The text discusses the definitions and properties of the cartesian product of chains in algebraic topology.
(3.3) Suppose that each of \((X, \leq_X)\) and \((Y, \leq_Y)\) is an ordered complex, \(e_i\) is an \(a\)-chain on \(X\), and \(e_i\) is a \(b\)-chain on \(Y\),
\[
\varepsilon(x \times y) = \begin{cases} 
\delta_X \times \delta_Y & \text{if } a > 0 \text{ and } b = 0, \\
\delta_X \times \delta_Y & \text{if } a = 0 \text{ and } b > 0, \\
\delta_X \times \delta_Y + \langle -1 \rangle \delta_X \times \delta_Y & \text{if } a > 0 \text{ and } b > 0.
\end{cases}
\]

(3.4) Corollary. If \((a \neq 0 \text{ or } \delta_X = 0)\) and \((b = 0 \text{ or } \delta_Y = 0)\), then \((a + b = 0 \text{ or } \varepsilon(x \times y) = 0)\).

Suppose that each of \((X, \leq_X)\) and \((Y, \leq_Y)\) is an ordered complex and \(f\) is a simplicial map from \(X\) to \(Y\), \(\mathcal{I}\) will be said to be order preserving if \(\mathcal{I}^a \leq \mathcal{I}^b\) implies \(f^{-1} \mathcal{I}^a \leq f^{-1} \mathcal{I}^b\) whenever \(\mathcal{I}^a, \mathcal{I}^b\) are vertices of a 1-simplex of \(X\).

(3.5) If \((X, \leq_X)\) is an ordered complex and \(f\) is a simplicial map from \(K\) to \(X\), then there is an order \(\leq\) for \(K\) such that \(f\) is order preserving. (Such an assertion was true if \(f\) is well defined as a partial order.)

Construction. For each vertex \(e\) of \(X\) let \(\leq\) be an order on the subcomplex of \(X\) whose points are \(e\)'s. Define the relation \(\leq\) as follows. Suppose that each of \(A\) and \(B\) is a vertex of \(K\). If \(fA = fB\), \(A \leq B\) iff \(A = fB\). If \(fA \neq fB\), \(A \leq B\) iff \(A\) is a vertex of \(K\) and \(fA \leq fB\).

(3.6) Suppose that each of \((X_i, \leq_X)\), \((Y_i, \leq_Y)\), \((X_i, \leq_X)\) and \((Y_i, \leq_Y)\) is an ordered complex, \(f_i\) is an order preserving simplicial map of \(X_i\) into \(X_i\), \(i \in \{1, 2, \ldots, n\}\), \(G_i\) is an abelian group, \(i \in \{1, 2, 3\}\), and \(\psi_i\) : \((G_i, G_i) \rightarrow G_i\) is a multiplication. Let the chains multiplications
\[
x' \times y' = \begin{cases} 
C_{X_1} \times C_{X_2} & \text{if } i = 2, \text{ and } i \neq 3, \\
C_{X_1} \times C_{X_2} & \text{if } i = 3, \text{ and } i \neq 2, \\
C_{X_1} \times C_{X_2} + C_{X_1} \times C_{X_2} & \text{if } i \neq 2, \text{ and } i \neq 3.
\end{cases}
\]
be defined using \(\psi_i\) as the underlying group multiplications. (Note that each of \(x\) and \(x'\) is used in two ways here, to indicate a product of complexes and to indicate a product of chains.) Let \((f_1, f_2)\) denote the vertex map of \(X_i \times X_j\) into \(X_i \times X_j\) such that \((f_1, f_2)\) \((A, B) = (f_1 A, f_2 B)\) whenever \(A\) is a vertex of \(X_i\) and \(B\) is a vertex of \(X_j\). Then \((f_1, f_2)\) defines a simplicial map which will also be denoted by \((f_1, f_2)\) \((A, B) = (f_1 A, f_2 B)\) whenever \(A\) is a vertex of \(X_i\) and \(B\) is a vertex of \(X_j\). Then \((f_1, f_2)\) defines a simplicial map which will also be denoted by \((f_1, f_2)\) \((A, B) = (f_1 A, f_2 B)\) whenever \(A\) is a vertex of \(X_i\) and \(B\) is a vertex of \(X_j\). Then \((f_1, f_2)\) defines a simplicial map which will also be denoted by \((f_1, f_2)\) \((A, B) = (f_1 A, f_2 B)\) whenever \(A\) is a vertex of \(X_i\) and \(B\) is a vertex of \(X_j\), respectively.

4. Homology groups for actions. An action is an ordered triple \((W, X, \leq)\) such that \(W\) is a group under multiplication, \(X\) is a topological space and \(\leq\) is a function from \(W\) such that

(1) if \(x \in W\), then \(x\) is a homeomorphism of \(X\) onto \(X\) and

(2) if \(x, y \in W\) and \(x \leq y\), then \((xy) \leq (yx)\).

Note that \(1 \leq 1\) and \(x^{-1} = x^{-1}\).

Throughout the remainder of this paper the conventions listed in the paragraphs (4.1) and (4.2) below will be observed except when explicitly suspended.

(4.1) If \(W\) is a group and \(X\) is a space, then there will be under consideration at most one function \(\ast\) such that \((W, X, \ast)\) is an action. In particular if \(Y\) is a subspace of \(V\) and \((W, Y, \ast)\) is an action, \(w^* = w^*y\) for each \(w\) in \(W\). Accordingly an action \((W, X, \ast)\) will be denoted by \((W, X)\), \((w^* : w \in W)\) will be denoted by \(W\) and, for each \(w\) in \(W\), \(w^*\) will be denoted by \(w\).

(4.2) If \((W, X)\) is an action, \(W\) is finite, \(X\) is a bicompact Hausdorff space and for each two different members \(w_1\) and \(w_2\) of \(W\) and each point \(x \in X\), \(w_1 x \neq w_2 x\).

An example of an action satisfying our conditions is provided by \((W, S^n)\) where \(S^n\) is the \(n\)-dimensional sphere, \(W\) is a group whose only elements are 1 and \(T\), 1 is the identity map on \(S^n\) and \(T\) maps each point of \(S^n\) onto its antipode.

An action \((W, X)\) is simplicial if \(X\) is a complex and each map in \(W\) is simplicial. We require in addition that, if \(x\) is a vertex of \(X\) and \(w \in W\), \(w \neq 1\), then no simplex has both \(x\) and \(w x\) as faces. Consequently, \(S\) and \(S\) are disjoint and (4.2) holds. If \(S\) is a simplex and \(w \in W\), \(w \neq 1\), then \((S) \neq wS\). Any complex \(X\) may be regarded as a simplicial action \((\{1\}, X)\), where \(1\) is the degenerate group.

(4.3) Suppose that \((W, X)\) is a simplicial action, \(F\) is a subcomplex of \(X\) such that \(\bigcup wF = X\) and \(n\) is a nonnegative integer. There is a subset \(Y\) of the set of \(n\)-simplices of \(F\) such that \(\bigcup wY = X\) is the set \(X_0\) of \(n\)-simplices of \(X\) and \(w_1 Y \cap w_2 Y = \emptyset\) for \(w_1, w_2 \in W\), \(w_1 \neq w_2\).

Construction. If each of \(H\) and \(K\) is in \(X_0\), write \(H \sim K\) if there is a \(w\) in \(W\) such that \(H = wK\). Corresponding to the equivalence relation \(\sim\) there is a partition of \(X_0\) into equivalence classes, each of which intersects \(F\). Let \(Y\) be a subset of the \(n\)-simplices of \(F\) consisting of just one simplex from each equivalence class.

An ordered simplicial action is a triple \((W, X, \leq)\) such that \((W, X)\) is a simplicial action, \((X_0, \leq)\) is an ordered complex and each \(w\) in \(W\) is order preserving.

(4.4) If \((W, X)\) is a simplicial action there is an order \(\leq\) for \(X\) such that \((W, X, \leq)\) is an ordered simplicial action.

Proof. Let \(X/W\) denote the simplicial complex obtained from \(X\) by identifying each point of the polyhedron \(X\) of \(X\) with all its images under \(W\). Assign an order \(\leq\) to \(X/W\). By (3.5) there is an order \(\leq\) for \(X\) such that the natural map of \(X\) onto \(X/W\) is order preserving. \(\leq\) has the desired properties.
If each of \((W_1, X_1, \ast)\) and \((W_2, X_2, \ast)\) is an action, their product \((W_1 \times W_2, X_1 \times X_2, \ast)\) is the action \((W_1 \times W_2, X_1 \times X_2, \ast)\), where \(W_1 \times W_2\) is the direct product, \(X_1 \times X_2\) is the topological product and \(\ast\) is defined by
\[
(w_1, w_2) \ast (x_1, x_2) = (w_1 w_2, w_1^* x_2).
\]
It can be shown that if each of \((W_1, X_1)\) and \((W_2, X_2)\) is an action satisfying (4.2) then their product is an action satisfying (4.2).

If each of \((W_1, X_1, \ast)\) and \((W_2, X_2, \ast)\) is a simplicial action their simplicial product \((W_1 \times W_2, X_1 \times X_2, \ast)\) is the simplicial action \((W_1 \times W_2, X_1 \times X_2, \ast)\), where \(W_1 \times W_2\) is the direct product, \(X_1 \times X_2\) is the simplicial product and, for \(w_1 \in W_1\) and \(w_2 \in W_2\),
\[
(w_1, w_2) \ast : X_1 \times X_2 \to X_1 \times X_2
\]
is the unique simplicial map such that
\[
(w_1, w_2) \ast (A, B) = (w_1^\ast A, w_2^\ast B)
\]
whenever \(A\) is a vertex of \(X_1\) and \(B\) is a vertex of \(X_2\).

If each of \((W_1, X_1, \leq_1)\) and \((W_2, X_2, \leq_2)\) is an ordered simplicial action, their cartesian product \((W_1 \times W_2, X_1 \times X_2, \leq)\), where \(X_1 \times X_2\) is the subcomplex of \(X_1 \times X_2\) associated with the order \(\leq\), is the simplicial maps in \(W_1 \times W_2\) are the restrictions to \(X_1 \times X_2\) of the corresponding simplicial maps for \((W_1, X_1)\) and \((W_2, X_2)\).

The elements \((w_1, 1)\) and \((1, w_2)\) of the direct product \(W_1 \times W_2\) of groups \(W_1\) and \(W_2\) will sometimes be denoted by \(w_1\) and \(w_2\), respectively.

Suppose that \((W, X)\) is a simplicial action, \(G\) is an abelian group and \(R : C_\delta(X, G) \to C_\delta(X, G)\) is a map. A \(G\)-chain \(e\) on \(X\) is called an \((R, g)\)-chain if there is a chain \(d\) such that \(e = Rd\). The group of \(G\)-chains \(R G\) is denoted by \(R G(X, R, G)\). Thus \(R G(X, R, G) = R C_\delta(X, G)\). If \(g > 0\), the boundary operator \(\partial\) is a homomorphism from \(C_\delta(X, R, G)\) into \(C_{\delta - 1}(X, R, G)\). Define
\[
Z_\delta(X, R, G) = C_\delta(X, R, G),
\]
\[
Z_\delta(X, R, G) = \{ e : e \inId(X, R, G), \partial e = 0 \},
\]
\[
B_\delta(X, R, G) = \partial C_{\delta + 1}(X, R, G),
\]
\[
H_\delta(X, R, G) = Z_\delta(X, R, G) / B_\delta(X, R, G).
\]
If each of \((W, X)\) and \((W, Y)\) is a simplicial action and \(f : X \to Y\) is a simplicial map, \(f\) is said to be equivariant if \(fw = f w\) for each \(w\) in \(W\) and each point \(p\) in \(X\).

(4.5) Suppose that each of \((W, X)\) and \((W, Y)\) is a simplicial action, \(G\) is an abelian group, for each \(w\) in \(W\), \(w_0\) is an integer, and \(R\) is used to denote both the chain map from \(C_\delta(X, G)\) into \(C_\delta(Y, G)\) such that
\[
Rc = \sum_{w \in W} n_w w_0 c
\]
and the chain map from \(C_\delta(X, G)\) into \(C_\delta(Y, G)\) defined by the same formula. Then, if \(f : X \to Y\) is an equivariant simplicial map, the chain map \(f^*\) maps \(C_\delta(X, R, G)\) into \(C_\delta(Y, R, G)\), \(Z_\delta(X, R, G)\) into \(Z_\delta(Y, R, G)\) and \(B_\delta(X, R, G)\) into \(B_\delta(Y, R, G)\) and induces a homomorphism
\[
f_* : H_\delta(X, R, G) \to H_\delta(Y, R, G).
\]

Proof. Since \(f\) is equivariant, it commutes with \(R\). Suppose that \(e \in C_\delta(X, R, G)\). There is a \(d\) such that \(e = Rd\). Let \(fe = f(Rd)\). Suppose \(g > 0\) and \(z \in Z_\delta(Y, R, G)\). Since \(f\) is equivariant, \(e = 0\). Let \(z = f e\). Suppose \(z \in Z_\delta(Y, R, G)\). There is a \(w\) in \(C_{\delta - 1}(X, R, G)\) such that \(z = f w\). Since \(f\) is equivariant, \(e = f w\). Since \(f\) is equivariant, \(w\) is an element of \(B_\delta(Y, R, G)\).

The word "cover" will be used in its usual sense: a collection of sets is a cover of any subset of the union of its members. The noun "covering" will be used in a more restricted sense. A covering of an action \((W, X)\) is a finite open cover \(\lambda\) of \(X\) such that

(a) if \(U \in \lambda\) and \(w \in W\), \(wU \in \lambda\), and

(b) if \(U \in \lambda\), \(w_1 \in W\) and \(w_1 \neq w_2\), then \(w_1 U \cap w_2 U = \emptyset\).

If \(\lambda\) is a covering of \((W, X)\), its nerve will be denoted by \(X_{\lambda}((W, X))\). \(\lambda\) is a simplicial action, where the simplicial maps in \(W\) are defined by the vertex maps \(w : \lambda \to \lambda\), \(w \in W\).

(4.6) If \(\alpha\) is an open cover of \(X\), there is a covering of \((W, X)\) that refines \(\alpha\).

Proof. For each \(p\) in \(X\) there is an open set \(U_p\) containing \(p\) whose closure \(\overline{U_p}\) does not intersect the finite point set \(\{w : w \in W, w \neq \lambda\}\) and is a subset of a member of \(\alpha\). Let \(F_p\) denote \(F_p = \bigcup_{w \in W} U_{\overline{w}}\). \(V_p\) is an open set containing \(p\) that is a subset of a member of \(\alpha\) and does not intersect \(w_{\overline{p}}\), if \(w \in W\) and \(w \neq \lambda\). Let \(S_p\) denote \(\bigcup_{w \in W} U_{\overline{p}}\).

\(S_p\) is an open set containing \(p\) such that

(a) \(w_1 S_p \cap w_2 S_p = \emptyset\) if \(w_1, w_2 \in W\) and \(w_1 \neq w_2\)

(b) \(w S_p\) is a subset of a member of \(\alpha\), if \(w \in W\).

Since \(X\) is bicomplex, some finite subcollection
\[\left[ \bigcup_{w \in W} w S_p : i \in \{1, \ldots, n\} \right] \bigcup \left[ \bigcup_{w \in W} w S_p : p \in X \right]\]
covers \(X\). \(w S_p : w \in W, i \in \{1, \ldots, n\}\) is a covering of \((W, X)\) that refines \(\alpha\).

Suppose that each of \(\alpha\) and \(\beta\) is a finite cover of the space \(X\), \(\beta\) refines \(\alpha\) and \(\beta = \alpha\) is a function such that, for all \(U \in \beta\), \(U \subseteq \pi U\). The unique extension of \(\pi\) to a simplicial maps from \(X_\lambda\) into \(X\) is called a projection and will be denoted by \(\pi\).
For the purposes of this paper it is not necessary to know whether the homology group of a complex is isomorphic to the corresponding Čech-Smith homology group of its polyhedron.

5. \textbf{L-systems.} To say that \((A, B, C)\) is an \textit{L-triple} for \((X, G)\) means that \(X\) is a complex, \(G\) is an abelian group, each of \(A, B\) and \(C\) is a chain map from the groups \(C_q(X, G)\) into the groups \(C_q(X, G)\) and, if \(c\) is a chain such that \(Co = 0\), there is a chain \(d\) such that \(Cd = Ad\).

\textbf{EXAMPLES.} If \(X\) is a complex, \(T: X \to X\) is a fixed-point free simplicial involution and \(Z\) is the group of order 2, then \((1+T, 1+T)\) is an \textit{L-triple} for \((X, Z_2)\) which plays a fundamental role in [6], [7], and [8]. If \(o\) and \(\delta\) are defined as in [5], p. 355, each of \((o, 1, o)\) and \((\delta, 1, o)\) is an \textit{L-triple}. See also (6.3) below.

If \(X\) is a complex and \(G\) is an abelian group, we define a homomorphism \(In: C_q(X, G) \to G\) by the rule \(\ln(x) = g\), where the \(V\)'s are the vertices of \(X\) and the \(g(V)\) are in \(G\). The properties of \(In\) used in the sequel are:

(a) \(In\) is a homomorphism,

(b) \(In(o) = 0\) if \(o \in C_0(X, G)\),

(c) \(In(f) = In(x)\) if \(Y\) is a complex and \(f: X \to Y\) is a simplicial map,

(d) \(In(g \times c) = g(\ln(1), In(1))\) if each of \(G_1, G_2\) and \(G_3\) is an abelian group, \(g: (G_1, G_2) \to G_3\) is a multiplication, each of \((X_1, \leq_1)\) and \((X_2, \leq_2)\) is an ordered complex, \(X\) is the cartesian chain product using \(c\) as the underlying multiplication, and \(a \in C_q(X_1, G_1), b \in C_q(X_2, G_2)\), \(i \in (1, 2)\).

To say that \(a\) is an \textit{L-system} of depth \(n\) for \((X, G)\) means that \(X\) is a complex, \(G\) is an abelian group, \(a\) is a nonnegative integer and \(c\) is a sequence \((A_0, S_1, A_1, \ldots, S_n, A_n)\),

of chain maps from \(C_q(X, G)\) into \(C_q(X, G)\) such that

(a) \(c \in C_q(X, G)\) and \(A_0c = 0\) then \(In(c) = 0\), and

(b) if \(0 < c < n\) then \(A_{c-1}, S_c, A_c\) is an \textit{L-triple} for \((X, G)\).

(5.1) Suppose that \((A_0)\) is an \textit{L-system} of depth \(0\) for \((X, G)\). Let \(\nu_0\) be the relation to which \((x, g)\) belongs if it is \(a\) chain \(c\) such that \(x = A_0c \in C_q(X, A_0, G)\) and \(g = In(c)\). Then \(\nu_0\) is a homomorphism from \(Z_2(X, A_0, G)\) into \(G\) and \(\nu_0Rg(x, A_0, G) = 0\).

\textbf{Proof.} (a) If \((x, g) \in Z_2(X, A_0, G)\), there is a \(g\) such that \((x, g) \in \nu_0\).

(b) \(\nu_0\) is a function.
Suppose \((z, g_1), (z, g_2) \in \eta_2\). For \(k \in \{1, 2\}\), there is an \(a_0\) such that 
\[ z = A_k a_0 \] and 
\[ g_k = \text{In}_{a_0}. \] \(A_k (a_0 - e) = A_k a_0 - A_k = z - e = 0.\) \(\text{In}(a_0 - e) = 0.\) \(\text{In}_{a_0} = \text{In}_{a_0} = 0.\) \(g_k = 0.\)

(c) \(\eta_2\) is a homomorphism.

Suppose \(A_k \in A_k \in \mathbb{Z}(X, A, G), \eta_2(A_k e + A_k d) = \text{In}(e + d)\) and \(\text{In} + \text{In} d = \eta_2(A_k e) + \eta_2(A_k d).\)

(d) \(\eta_2(A_k e, A_k d, G) = 0.

Suppose \(z \in B_k(X, A, G)\). There is an \(a \in C_{\zeta}(X, A, G)\) such that 
\[ z = \text{In}_a. \] \(\text{In} | C_{\zeta}(X, G)\) such that \(u = A_k e.\) \(\eta_2(z) = \eta_2(A_k e) = \eta_2(A_k e) = \eta_2(A_k e) = \eta_2(A_k e) = \eta_2(A_k e) = 0.\)

(c) \(\eta_2\) is a homomorphism.

Suppose \(A_k \in A_k \in \mathbb{Z}(X, A, G), \eta_2(A_k e + A_k d) = \text{In}(e + d)\) and \(\text{In} + \text{In} d = \eta_2(A_k e) + \eta_2(A_k d).\)

(d) \(\eta_2(A_k e, A_k d, G) = 0.\)

Suppose \(z \in B_k(X, A, G)\). There is an \(a \in C_{\zeta}(X, A, G)\) such that 
\[ z = \text{In}_a. \] \(\text{In} | C_{\zeta}(X, G)\) such that \(u = A_k e.\) \(\eta_2(z) = \eta_2(A_k e) = \eta_2(A_k e) = \eta_2(A_k e) = \eta_2(A_k e) = \eta_2(A_k e) = 0.\)

(c) \(\eta_2\) is a homomorphism.

Suppose \(A_k \in A_k \in \mathbb{Z}(X, A, G), \eta_2(A_k e + A_k d) = \text{In}(e + d)\) and \(\text{In} + \text{In} d = \eta_2(A_k e) + \eta_2(A_k d).\)

(d) \(\eta_2(A_k e, A_k d, G) = 0.\)

Suppose \(z \in B_k(X, A, G)\). There is an \(a \in C_{\zeta}(X, A, G)\) such that 
\[ z = \text{In}_a. \] \(\text{In} | C_{\zeta}(X, G)\) such that \(u = A_k e.\) \(\eta_2(z) = \eta_2(A_k e) = \eta_2(A_k e) = \eta_2(A_k e) = \eta_2(A_k e) = \eta_2(A_k e) = 0.\)

Consider the real-valued maps:
\[ v: H_d(X, A, G) \to \mathbb{R}, \quad g \in \{0, \ldots, n\} \]
by the rule \(v[z] = \text{In}_a.\) (Here \([z]\) denotes the homology class containing \(z.\)

Remark. Up to this point the discussion of \(d\)-triples and their associated homomorphisms can be interpreted even in the abstract context of chain complexes ([2], p. 124). But in concrete applications an \(L\)-system \(a\) will be defined just for simplicial actions \((W, X)\) and the terms of \(a\) will be of the form \(\sum w^w N_w\), where the \(N_w\)'s are integers.

(5.3) Suppose that each of \((W, X)\) and \((W, Y)\) is a simplicial action, 
\[ a = (R_0, S_1, R_1, \ldots, S_n, R_n) \]
and \((X, G)\) is an \(L\)-system of depth \(n\) for each of \((X, G)\) and \((Y, G)\), each term of \(a\) is of the form \(\sum w^w N_w\) where each \(N_w\) is an \(L\)-system, \(X \to Y\) is an equivariant simplicial map and \(v\) is an \(a\)'s homomorphism.

Then for each \(g \in \{0, \ldots, n\}\) and \(z \in Z_d(X, A, G)\), \(v(z) = v(z)\) (and, for \(z \in H_d(X, A, G)\), \(v(z) = v(z)\)).

Proof. Induction on \(g\). Suppose \(R_0 e \in Z_d(X, R_0, G)\). \(v[R_0 e] = v[R_0 e] = \text{In} e = \text{v}[R_0 e].\) Suppose \(0 < g \leq n\) and \(R_0 e \in Z_d(X, R_0, G)\).

\[ v[R_0 e] = v[R_0 e] = v[R_0 e] = v[R_0 e] = v[R_0 e] = v[R_0 e]. \]

Remark. It can be shown that if \(G\) is an abelian group, each of \((W, X)\) and \((W, Y)\) is a simplicial action, each of \(a, b, c\) and \(u\) is a function from \(W\) into the integers and 
\[ \sum a w^w, \quad \sum b w^w, \quad \sum c w^w \]
is an \(L\)-triple for \((X, G)\) then it is also an \(L\)-triple for \((Y, G)\). This fact does not contribute to the proof of our principal theorem and so is not proved here. I mention it only in order to render more palatable the following curious definition.

If \((W, X)\) is an action or a simplicial action and, for every simplicial action \((W, Y)\), \(a = (R_0, S_1, R_1, \ldots, S_n, R_n)\) is an \(a\)-system of depth \(n\) for \((Y, G)\) and each term of \(a\) is of the form \(\sum w^w N_w\), where each \(N_w\) is an \(L\)-system, then we shall say that \(a\) is an \(L\)-system of depth \(n\) for \((W, X)\).

Suppose each of \((W_1, X_1)\) and \((W_2, X_2)\) is an ordered simplicial action, each of \(G_1, G_2\) and \(G_3\) is an abelian group,
\[ A: C_d(X_1, G_1) \to C_d(X_2, G_1) \]
is a chain map of the form \(\sum w^w N_w\) and 
\[ B: C_d(X_2, G_2) \to C_d(X_3, G_2) \]
Thus $A(c \times B_3 d) = Z_{c+3}(X_1 \times X_2, A_1 B_1, G_0)$. To show that $v(A(c \times B_3 d) = \varphi(v_1 A_1 c, v_2 B_3 d)$ we use induction on $j$.

(I) Suppose $i = 0$. Induction on $j$.

(II) Suppose $j = 0$.

\begin{align*}
\varphi(A(c \times B_3 d) &= v_1 A_1 c_0 B_3 d) \quad (3.2) \text{ and } (3.6) \\
&= \text{In}(c \times d) \quad \text{definition of } v_1 \\
&= \varphi(\text{In}, c \times d) \quad \text{definition of } v_1 \\
&= \varphi(v_1 A_1 c, v_2 B_3 d) \quad \text{definitions of } v_1, v_2.
\end{align*}

(II) Suppose $0 < j < b$. The proof consists of a calculation similar to, but simpler than, the one given for case (II) below.

(II) Suppose $0 < i < a$, $j = b$. Suppose $b > 0$.

\begin{align*}
\varphi(A(c \times B_3 d) &= v_2 A_1 B_3 d) \quad (3.2) \text{ and } (3.6), \\
&= v_2 A_1 B_3 d(c \times d) \quad \text{definition of } v_2 \\
&= \text{In}(c \times d) \quad \text{definition of } v_2 \\
&= v_2 A_1 B_3 d(c \times d) \quad \text{inductive hypothesis} \\
&= \varphi(v_1 A_1 c, v_2 B_3 d) \quad \text{definition of } v_1.
\end{align*}

In case $b = 0$ the calculation is similar but simpler.

Suppose that $(W, X)$ is an action and $\alpha = (E_0, S_1, E_2, \ldots, S_n, E_n)$ is an $L$-system of depth $n$ for $(W, X, G)$. By (5.3) all the coordinates of an element of $H_d(X, E_n, G)$, $q \in \{0, \ldots, n\}$, have the same value under $\alpha$'s homomorphism $\nu$. A homomorphism

\[\nu: H_d(X, E_n, G) \to G\]

is defined by taking the value of an element of $H_d(X, E_n, G)$ under $\nu$ to be the common value of its coordinates under $\nu$.

If $(W, X)$ is a simplicial action (action) and $\alpha = (E_0, S_1, A_1, \ldots, S_n, A_n)$ is an $L$-system of depth $n$ for $(W, X, G)$, $(W, X)$ will be said to be $\alpha$-admissible if there is an element $\nu$ of $H_d(X, E_n, G)$ such that $\nu \neq 0$, where $\nu$ is $\alpha$'s homomorphism.

Suppose that $K$ is a complex, $SdK$ is its first barycentric subdivision and $G$ is an abelian group. Define a homomorphism

\[Sd: C_d(K, G) \to C_d(SdK, G)\]
6. The admissibility of the n-torus.

(6.1) Suppose that \((W, X)\) is a simplicial action, \(G\) is an abelian group and \(c\) a chain in \(C_q(X, G)\) such that \(\sum w_c = 0\). Then \(\text{Inc} = 0\).

Proof. By an argument similar to the one used for (4.3) there is a subset \(Y\) of the set \(V\) of 0-simplices of \(X\) such that \(\sum w_{\alpha} Y = V\) and \(w_1 Y \cap w_2 Y = \emptyset\) whenever \(w_1, w_2 \in W\) and \(w_1 \neq w_2\). With the aid of (2.1) we have:

\[
\text{Inc} = \sum_{\alpha_1} \sum_{w_{\alpha_1}} c_{\alpha_1} = \sum_{\alpha_1} \sum_{w_{\alpha_1}} w_{\alpha_1} c_{\alpha_1} = \sum_{\alpha_1} \sum_{w_{\alpha_1}} w_{\alpha_1} c_{\alpha_1} = 0\, < c_{\alpha} = 0.
\]

The statements in (6.2) below will be hypotheses in (6.3) and (6.4).

(6.2) \((W, X)\) is a simplicial action, \(p\) is an integer \(\geq 1\), \(m\) is a positive integer, \(\varepsilon = mp\).

\(W\) is the (internal) direct product of a subgroup \(H\) and a cyclic subgroup \(W_s\) of order \(e\) with generator \(T_s\), \(m\) is a nonnegative integer.

\(Y'\) is a subset of the set \(X_n\) of \(n\)-simplices of \(X\) such that

(i) \(hY' = Y'\) for each \(h\) in \(H\),

(ii) \(\bigcup_{i=0}^{m-1} T_i Y' = X_n\),

(iii) \(T_i Y' \cap T_j Y' \neq \emptyset\) if \(i \neq j \pmod{e}\).

\(X_n\) is an integer for each \(h\) in \(H\),

\(M\) denotes the chain map \(\sum_{\lambda} N_\lambda h: C_q(X, Z_p) \rightarrow C_q(X, Z_p)\),

\(Z_p\) is the cyclic group of order \(p\),

\(A\) denotes the chain map \(\sum_{\lambda} jT_i M: C_q(X, Z_p) \rightarrow C_q(X, Z_p)\),

\(S\) denotes the chain map \(\sum_{\lambda} jT_i M: C_q(X, Z_p) \rightarrow C_q(X, Z_p)\).

(6.3) Suppose (6.2), \(c \in C_q(X, Z_p)\) and \(\Delta e = 0\). Let \(d\) be the unique chain such that, if \(E\) is an orientation of an \(n\)-simplex in \(Y'\),

\[
d(T_i E) = \begin{cases} \sum_{\lambda} jT_i c \, E & \text{if } a = 0, \\ 0 & \text{if } a \in \{1, \ldots, e-1\}. \end{cases}
\]

Then \(S \circ d = A d\). (Thus, \((A, S, A)\) is an \(L\)-triple for \((X, Z_p)\).)
Proof. Suppose that $E$ is an orientation of an $n$-simplex in $Y$ and $b \in \{0, \ldots, e-1\}$. Then

\[
\left( \sum_{s=0}^{e-1} T^s M a \right) (T^b E)
\]

(a) \[= \left( \sum_{s=0}^{e-1} T^s \sum_{u \in R} N_u u a \right) (T^b E) \]

(b) \[= \sum_{s+b \in R} \sum_{u \in R} N_u (T^s u a) (T^{b-s} E) \]

(c) \[= \sum_{s \in R} N_u (\sum_{b \in R} a (T^{s-b} u^{-1} E)) \]

(d) \[= \sum_{u \in R} N_u (\hat{a} (u^{-1} E)) \]

(e) \[= \sum_{u \in R} N_u (\sum_{j \in J} j^s (T^{e-j} E)) \]

(f) \[= \sum_{u \in R} N_u \left( \sum_{j \in J} j^s \hat{a} a E \right) \]

(g) \[= \left( \sum_{j \in J} j^s \hat{a} a E \right) \]

(h) \[= \left( \sum_{j \in J} j^s \hat{a} a E \right) \]

(i) \[= \left( \sum_{j \in J} j^s \hat{a} a E \right) \]

(j) \[= \left( \sum_{j \in J} j^s \hat{a} a E \right) \]

Equations (b), (f) and (j) are justified by (2.1); (a) and (g), by the definition of $M_1$ (c) and (g), by the fact that $T$ commutes with each member of $H_1$ (d) and (e), by the definition of $d_1$ (h), by the fact that the order of $Z_p$ divides the period of $T$; and (i), by the fact that $\alpha e = 0$.

(6.4) Suppose (6.2) and $b \in C_0(X, Z_p)$. There is a chain $d$ with support in $Y'$ such that $Ab = Ad$.

Proof. Let $e$ denote the chain $(1 - T)b$. Since $A(1 - T)$ is the 0 operator, $Az = 0$. Let $d$ be defined as in the hypothesis of (6.3). The support of $d$ is a subset of $Y'$. By (6.3), $\delta c = Ad$. Since $(1 - T) - A$, we have $Ab = Ad$.

(6.5) Suppose that:

$a$ is a positive integer, $p$ is an integer $> 1$, $e = pm$.

$X$ is a 1-complex consisting of the 2e vertices $v_0, \ldots, v_{2e-1}$ and the 2e 1-simplexes $e_0, \ldots, e_{2e-1}$,

the 0-faces of $e_i$ are $v_i$ and $v_{i+1}$, if $i \in \{0, \ldots, 2e-1\}$ (addition in the subscripts is modulo 2e),

$k$ is an integer in $\{1, \ldots, e-1\}$ such that the greatest common divisor of $k$ and $e$ is 1,

$T: X \rightarrow X$ is the simplicial map such that

$T_{e_j} = v_{j+1}$, if $j \in \{0, \ldots, 2e-1\}$.

Let $W$ denote the group of homeomorphisms generated by $T$, $A$ denote the chain map $\sum_{j \in J} T^j: C_0(X, Z_p) \rightarrow C_0(X, Z_p)$, $S$ denote the chain map $\sum_{j \in J} T^j: C_0(X, Z_p) \rightarrow C_0(X, Z_p)$, and $a$ denote the triangle $(A, S, A)$. Then $T$ is periodic of period $e_i$, $(W, X)$ is a simplicial action, $a$ is an $L$-system of depth 1 for $(W, X, Z_p)$ and, if $v$ is $a$'s homomorphism and the members $Z_p$ are denoted by $0, 1, \ldots, p-1$, there is a $z$ in $Z_p(X, A, Z_p)$ such that $z = 1$.

Proof. That $(A, S, A)$ is an $L$-system follows from (6.1) and (6.3).

Let $c$ denote the 1-chain: $\sum_{i=0}^{eb} v_{i+1} v_i$ and let $A$ denote the 0-chain:

(6.6) Suppose that $p$ is an integer $> 1$, $n$ is a positive integer and, for $i \in \{1, \ldots, n\}$, $n_i$ is a positive integer, $e_i = pm_i$.

Proof. That $(A, S, A)$ is an $L$-system follows from (6.1) and (6.3).
Let $W$ denote the group of homomorphisms generated by $(T_i, \ldots, T_k)$, $A$ denote the chain map $\sum_{w \in A} c_i(x, z) \rightarrow c_i(x, z)$ and, for $i \in (1, \ldots, n)$, let $H_i$ denote the subgroup of $W$ generated by $(T_{i1}, \ldots, T_{in}, T_0)$ and $S_i$ denote the chain map

$$\sum_{k \in K_i} j(T_k) \rightarrow c_i(x, z) \rightarrow c_i(x, z).$$

Then $\alpha = (A, B, A, \ldots, S_n, A)$ is an $L$-system of depth $n$ for $(W, T^*, Z_P)$ and $(W, T^*)$ is $\alpha$-admissible.

Proof. That $(W, T^*)$ is an action satisfying (4.2) may be verified by the reader. That $\alpha$ is an $L$-system of depth $n$ for $(W, T^*, Z_P)$ is a consequence of (6.1) and (6.3).

Suppose $i \in (1, \ldots, n)$. The collection of points and segments in the real numbers

$$\{j(2a_1), j(2a_1, (j+1)(2a_1)) : j \in \{0, \ldots, 2a_1-1\}\}$$

defines a simplicial decomposition of $T^*$ which will be denoted by $K_i$. There is an order $\leq$ for $K_i$ such that $j(2a_1, (j+1)(2a_1)) = 0$ if $j \in \{0, \ldots, 2a_1-2\}$ and $(2a_1-1)(2a_1) \leq 0$. Let $T_{e_i} : K_i \rightarrow K_i$ denote the simplicial map such that $T_{e_i} = x + k_i e_i$ and let $W_i$ denote the group generated by $T_i$, $(W_i, K_i, \leq)$ is an ordered simplicial action. Let $K$ denote $K_1 \times \cdots \times K_n$, $(W, K) = (W_1, K_1) \times \cdots \times (W_n, K_n)$ and $(W, T^*) = (W_i, K_i)$. By (5.1), to prove the $\alpha$-admissibility of $(W, T^*)$ it will suffice to prove the $\alpha$-admissibility of $(W, K)$. An induction on $n$ will show that there is a $z$ in $Z_{\alpha}(K, A, Z_P)$ such that $z = 1$. In case $n = 1$, (6.3) disposes of the question. Suppose $n > 1$. By the inductive hypothesis there is a $z$ in $Z_{\alpha}(K_1 \times \cdots \times K_{n-1}, \sum_{k \in K_n} j x, z)$ such that $z = 1$. By (6.5) there is a $u$ in $Z_{\alpha}(K_n, \sum_{k \in K_n} j x, z)$ such that $uv = 1$. A chain product

$$\times : (\sum_{k \in K_n} j x, z) \rightarrow (\sum_{k \in K_n} j x, z)$$

can be defined using the usual ring product $(z_P, Z_P) \rightarrow Z_P$ as the underlying multiplication. Then by (5.4) $\nu(x \cdot u) - \nu(ux) = 1 - 1 = 1$.

7. The principal theorem. Throughout this section we will use the fact, that if $X$ is a set with subsets $A$ and $B$ and $T : X \rightarrow X$ is 3-1 and onto, then $T(A \cup B) = T(A) \cup T(B)$ and $T(A \cap B) = T(A) \cap T(B)$.

(7.1a) (7.1b)) Suppose that $p$ is an integer $> 1$, $Z_P$ is the cyclic group of order $p$, $(W, Z)$ is a simplicial action (b) an action), $m$ is a positive integer, $\epsilon = pm$.

$W$ is the (internal) direct product of a subgroup $H$ and a cyclic subgroup $Z_P$ of order $\epsilon$ with generator $T_e$. $F$ is a (a) subcomplex of (b) a closed subset of $X$ such that

(i) $X = \bigcup_{\epsilon=1}^{T \epsilon} F$,

(ii) $hF = T \epsilon F$, $h \in H$,

$K = \bigcup_{\epsilon=1}^{T \epsilon} (T \epsilon F \cap T \epsilon F)$,

$a = (A_0, S_1, A_1, \ldots, S_n, A_n)$ is an $L$-system of depth $n$ for $(W, T^*, Z_P)$, $A = A_0, S_1, A_1, \ldots, S_n, A_n$ is an $L$-system of depth $n$ for $(W, T^*, Z_P)$, $v$ is a $w$ homomorphism, there is a function $N$ from $H$ into the integers such that

$A_n = A_{n-1} = \sum_{k \in K_n} j \epsilon \sum_{k \in K_n} N_k h$, $S_n = \sum_{j \in J} j \epsilon \sum_{k \in K_n} \sum_{k \in K_n} N_k h$.

Then (a) if $A$ is the relation to which $(\alpha, \omega)$ belongs, if there is a chain $e$ such that

the support of $e$ is a subset of F,

$\alpha$ is a homology class in $H_2(X, A_0, Z_P)$ that contains $A_{n-1}$,

$\omega$ is a homology class in $H_{n-1}(K, A_n, Z_P)$ that contains the restrictions of $S_n$ to $H$,

then $A$ is a homomorphism from $H_0(X, A_0, Z_P)$ into $H_{n-1}(K, A_{n-1}, Z_P)$ such that $v^\epsilon = \nu^\epsilon$ for each $\epsilon$ in $H_0(X, A_0, Z_P)$;

(b) there is a homomorphism $A$ from $H_0(X, A_0, Z_P)$ into $H_{n-1}(K, A_{n-1}, Z_P)$ such that $v^\epsilon = \nu^\epsilon$ for each $\epsilon$ in $H_0(X, A_0, Z_P)$.

Proof. Notice first that $K$ is invariant under $W$, so that $(W, K)$ is a (a) simplicial action (b) an action). Let $A_0$ denote $A_{n-1} = A_0$ and $S_n$ denote $S_n$. We now restrict our attention to (7.1a).

(A) If $g$ is a nonnegative integer, $\epsilon = c_i(x, z)$, $A \epsilon = 0$ and the support of $\epsilon$ is a subset of $F$, then the support of $\epsilon$ is a subset of $K$ and $(S_0, K) \rightarrow (A, Z_P)$.

Let $b$ denote $\sum_{k \in K_n} j \epsilon \sum_{k \in K_n} N_k h$. Since the support of $e$ is a subset of $F$ and $F$ is invariant under both $H$ and $T_\epsilon$, the support of $b$ is a subset of $F$. Since $S_0 = \sum_{k \in K_n} j \epsilon b$, the support $J$ of $S_0$ is a subset of $\bigcup_{\epsilon=1}^{T \epsilon} F$.

Since $J$ is an L-triple for $(W, X, \epsilon Z_P)$ (see (6.3)), and $A_\epsilon = 0$, there is a chain $d$ such that $S_0 = A_\epsilon$. Since $A_\epsilon = \epsilon A$, $J \subseteq T \epsilon J$. Hence, for any integer $i, J \subseteq J_{T \epsilon} \subseteq T \epsilon (J_{T \epsilon}) \subseteq F$. Therefore $J \subseteq \bigcup_{\epsilon=1}^{T \epsilon} (T \epsilon J_{T \epsilon}) \subseteq K$. 


Since \( K \) is \( W \)-invariant, we have \((Ad)K = A(\mathcal{d}K)\). Thus \((Ad)K \in \mathcal{Q}(K, A, Z_p)\).

(B) If \( \zeta \in H_0(X, A, Z_p) \), then there is an \( \omega \) such that \( \langle \zeta, \omega \rangle \in \mathcal{A} \).

By (4.3) there is a subset \( Y \) of the \( n \)-simplices of \( F \) such that \( \bigcup_{w \in W} Y \) is the set of \( n \)-simplices of \( X \) and \( wY = \emptyset \) if \( w \neq w_1 \) and \( w_1, w_2 \in W \). Let \( Y' \) denote \( \bigcup_{w \in W} Y \). Suppose \( \zeta \in H_0(X, A, Z_p) \). Let \( \omega \) be in \( \mathcal{A} \). By (6.4) there is a chain \( e \) with support in \( Y' \) such that \( e = \omega \). Since \( F \) is closed and \( Y' \subseteq F \), the support of \( \omega \) is a subset of \( F \). Since \( \omega \in \mathcal{Q}(K, A, Z_p) \), the \( \omega \)-homology class in \( H_n(K, A, Z_p) \) that contains \( (\mathcal{S}e\mathcal{c})K \).

(C) A is a function.

Suppose that \( \langle \zeta, e_0 \rangle, \langle \zeta, e_1 \rangle \in \mathcal{A} \). By (B), for \( k \in \{1, 2\} \), there is an \( n \)-chain \( a \) with support in \( F \) such that \( \omega \in \mathcal{Q}(K, A, Z_p) \) and \( \omega \neq \emptyset \). Since \( \omega \in \mathcal{Q}(K, A, Z_p) \), there is a chain \( A \) in \( C_n(K, A, Z_p) \) such that \( A \subseteq \omega \). By (6.4) there is a \( (n+1) \)-chain \( e \) with support in \( F \) such that \( \omega = \omega \). The support of \( \omega \) is a subset of \( F \) and \( A \subseteq \omega \). By (K), contains the support of \( \omega \) and \( \omega \). Thus \( (\mathcal{S}e\mathcal{c})K = \mathcal{Q}(K, A, Z_p) \).

(D) A routine computation shows that \( \omega \) is a homomorphism.

(E) If \( \zeta \in H_0(X, A, Z_p) \), then \( \mathcal{A} = \mathcal{A} \).

Adopt the following convention: if \( x \) is a cycle, \( [x] \) denotes the homology class containing \( x \). Suppose \( \zeta \in H_0(X, A, Z_p) \) and let \( e \) be an \( n \)-chain with support in \( F \) such that \( \omega \in \mathcal{Q}(K, A, Z_p) \). Since the inclusion map \( i: K \rightarrow X \) is equivariant, (5.3) justifies the starred equation: \( \mathcal{X} = \omega \).

We now turn to the proof of (7.1b). For each covering \( \lambda \) of \( (W, X) \), let \( \lambda \) denote the nerve of \( \lambda \) and let \( \mathcal{F}(\lambda)X \) denote the subcomplex of \( X \) to which a simplex belongs iff the common part of its vertices intersects \( F(X) \). Let \( \lambda \) denote the collection of coverings of \( (W, X) \) to which \( \lambda \) belongs iff \( \lambda \in \mathcal{G}(K_1, A, Z_p) \).

It can be shown that \( D \) is cofinal in the collection of all coverings of \( (W, X) \). For each \( \lambda \) let \( D \) define

\[
A: H_0(X, A, Z_p) \rightarrow H_0(K_1, A, Z_p)
\]
Proof. Suppose \( F \) is a closed subset of \( X \) such that \( f = \chi F \) for all \( h \in H_s \), \( T^h F = F \) and \( \bigcup_{j=0}^{p-1} T^j F = X \). Let \( K \) denote \( \bigcup_{j=0}^{p-1} (T^j A \cap T^j B) \) and let \( \psi \) be \( \zeta \)'s homomorphism. Since \( (W, X) \) is \( a \)-admissible, there is a \( \xi \in H_s(X, A, Z_p) \) such that \( \psi \xi \neq 0 \). By (7.1b), \( v \delta \xi \neq 0 \), where \( \delta \in H_{a-1}(K, A, Z_p) \). The remainder of the argument is an induction on \( n \). If \( n = 1 \), \( v \delta \xi \neq 0 \Rightarrow \delta \xi \neq 0 \Rightarrow K \ni 0 \Rightarrow \delta \xi \). Suppose that \( n > 1 \) and let \( \beta \) denote the \( \delta \)-system \( (A, S_1, \ldots, S_{a-1}, A) \) of depth \( n - 1 \) for \( (W, K, Z_p) \). Since \( v \delta \xi \neq 0 \), \( (W, K) \) is \( \beta \)-admissible. By the inductive hypothesis \( A_{\beta}(K, A_{\beta}(X)) \).

(7.5) COROLLARY. If \( n > 0 \) \((W, T^n)\) is the action specified in the hypothesis of (6.5), then \( A_{\beta}(T^n) \).

(7.6) Suppose that
(a) \( p \) is a prime number,
(b) \( w \) is a positive integer and \( e = p^w \),
(c) \( m = c p^w \),
(d) the group \( W \) is the internal direct product of a subgroup \( H \) and a cyclic subgroup \( W' \) of order \( e \) with generator \( T \),
(e) \((W, X)\) is an action, not necessarily biocompact Hausdorff,
(f) \( T \) is a continuous function from \( X \) into the real numbers such that,
if \( x \in X \) and \( h \in H \), \( f x = f x h \).

Then there is a closed subset \( F \) of \( X \) such that
(h) \( F = T^h F \),
(i) \( F = \chi F \) for each \( h \in H \),
(j) \( \bigcup_{j=0}^{p-1} T^j F = X \),
(k) if \( x \in \bigcup_{j=0}^{p-1} (T^j F \cap T^j X) \),

then there is an integer \( a \) such that \( f T^a x = f T^{a+1} x \).

Proof. For each \( i \) in the integers \( Z \) define \( B(i, 1) \) to be \( \{ x \in X, f T^i x \leq f T^{i+1} x \} \) and \( B(i, -1) \) to be \( \{ x \in X, f T^i x > f T^{i+1} x \} \). Notice that

(m) \( B(i, -1) \cup B(i, 1) = X \), \( i \in Z \),
(n) \( \bigcup_{j=0}^{p-1} B(i, 1) \subseteq B(j, -1), j, e \in Z \), \( i \in \{ -1 \}, \)
(o) \( B(j, -1) \subseteq B(j-1, -1), j, e \in Z \), \( \delta \in \{ -1 \}, \)
(p) if \( x \in H \), \( h B(j, -1) = B(j, 1), j, e \in Z \), \( \delta \in \{ -1 \}, \)
(q) \( \chi X = \bigcup_{i \in \mathbb{Z}} B(i, -1), \)

Let \( C \) be the set of functions from the integers \( Z \) onto \((-1, 1)\) such that, if \( e \in C \) and \( n \in \mathbb{Z} \), \( e n = e(n + e) \). \( C \) has \( 2^p - 2 \) members. As a consequence of (m) and (n) we have

\[
(q) \quad \chi X = \bigcup_{i \in \mathbb{Z}} B(i, -1).
\]

Let \( t \) denote the function from \( C \) into \( C \) such that, if \( d \in C \),

\[
(t d)(i) = d(i + 1), \quad i \in Z.
\]

If \( d, d' \in C \) write \( d \sim d' \) if there is an integer \( i \) such that \( d_i = d_{i+1} \). Associated with the equivalence relation \( \sim \) is a partition of \( C \) into equivalence classes. Let \( M \) be a subset of \( C \) containing just one member of each equivalence class and let \( N \) denote \( \bigcup_{a \in M} \).

Suppose \( d \in C, a \in Z \) and \( d = \chi a \). Since \( a \) is a period of \( d \) and \( e \) is a period of \( d \), the greatest common divisor \( (a, e) \) of \( a \) and \( e \) is a period of \( d \). \( (a, e) \) divides \( e \) and is \( >1 \). Since \( e = p^w \), \( a, e \) is a power of \( p \). \( p \) divides \( a \), so if for any \( d \) in \( C \) and \( a \) in \( Z \), \( d = \chi a \) implies that \( p \) divides \( a \). This enables us to show that

\[(r) \quad f^p N \) is a partition of \( C \) into \( p \) disjoint sets.

Let \( \beta \) be the function from \( (\mathbb{Z}^n, \ldots, \mathbb{Z}) \) such that, if \( i < 0, \ldots, p-1 \),

\[
\beta(i) N = \bigcup_{a \in N} B(i, a) N = \bigcup_{a \in N} B(i, a) \bigcup_{a \in N} B(i, a) N.
\]

By using (o) above it can be shown that

\[(s) \quad T B(i) N = \beta T B(i) N, \quad i < 0, \ldots, p-1.
\]

Let \( F \) denote \( \beta N \). By (q), (r), and (s),

\[
\bigcup_{i=0}^{p-1} T F = \bigcup_{i=0}^{p-1} \beta F = \bigcup_{i=0}^{p-1} \beta F = \bigcup_{i=0}^{p-1} B(i, a) N = X,
\]

which proves (j). Suppose \( x \in \bigcup_{i=0}^{p-1} (T F \cap T F) \). There are two integers \( i \) and \( k \) in \( \{ 0, \ldots, p-1 \} \) such that

\[
(\text{for } \beta N \cap T F N = (\beta F N) \cap (T F N) = X.
\]

There is a \( d \) in \( T F N \) such that \( x \in \bigcup_{i=0}^{p-1} B(i, a) N \) and there is a \( c \) in \( T F N \) such that \( x \in \bigcup_{i=0}^{p-1} B(i, c) N \). Since \( T F N \) and \( T F N \) are disjoint, \( e \neq c \). There is an \( e \) in \( \{ 1, \ldots, e \} \) such that \( d e = c e \), in fact \( d e = c e \). \( x \in B(a, e) N \cap B(a, e) N = B(a, e) N \cap B(a, e) N \). Hence \( f T^a x \leq f T^{a+1} x \) and \( f T^a x \geq f T^{a+1} x \), which proves (k); (h) and (i) follow from the definition of \( F \) and from (p).
(7.7) DEFINITION. Suppose (7.2) and that \((W, X)\) is an action, not necessarily bicom pact Hausdorff. We define sentences \(Q_i(X), k \in \{1, \ldots, n\}\), as follows:

\[ Q_i(X) \text{ if each of } f_1, \ldots, f_k \text{ is a continuous function from } X \text{ into the real numbers and } \]

\[ f_jT_a = f_j, \quad a \in X, \quad i \in \{1, \ldots, n\}, \quad j \in \{1, \ldots, k\}, \]

then there is an \(a^* \) in \(X\) such that \(f_i(a^*) = f_iT_a(a^*) \) for \(i \in \{1, \ldots, k\}\).

(7.8) Suppose (7.2), \(p\) is prime, \(u_1\) is a positive integer and \(e_i = p^u\), \(i \in \{1, \ldots, u_1\}\), and \((W, X)\) is an action, not necessarily bicom pact Hausdorff. For each \(i\) in \(\{1, \ldots, n\}\), \(A_i(X)\) implies \(Q_i(X)\).

Proof. Induction on \(i\). Suppose \(i = 1\). By (7.6) there is a closed subset \(F\) of \(X\) such that \(F = T_1F, F = \emptyset\) for each \(h \in H_1\), \(\bigcup_{j=0}^{n-1} T_j^*F = X\) and if \(a \in K = \bigcup_{j=1}^{m} \bigcup_{b=1}^{m} (T_j^*F \cap T_b^* F)\) then there is an integer \(a\) such that \(f_iT_a = f_iT_a\) for \(i \in \{1, \ldots, k\}\). Since \(A_1(X), K \neq \emptyset\). Let \(a \in \emptyset\) a point of \(K\), let \(a\) be an integer such that \(f_iT_a = f_iT_a\) for \(i \in \{1, \ldots, k\}\) and \(a^* = T_a^*\).

Suppose \(1 < i \leq n\). By (7.6) there is a closed subset \(F\) of \(X\) such that \(F = T_1F, F = \emptyset\) for each \(h \in H_1\), \(\bigcup_{j=0}^{n-1} T_j^*F = X\) and if \(a \in K = \bigcup_{j=1}^{m} \bigcup_{b=1}^{m} (T_j^*F \cap T_b^* F)\) then there is an integer \(a\) such that \(f_iT_a = f_iT_a\) for \(i \in \{1, \ldots, k\}\). Since \(A_i(X), A_{i-1}(K)\) and, by the inductive hypothesis, \(Q_{i-1}(K)\).

Let \(a^*\) be a point of \(K\) such that \(f_iT_a = f_iT_a\) for \(i \in \{1, \ldots, i-1\}\). By (7.6) there is an integer \(a\) such a that \(f_iT_a = f_iT_a\). Let \(a^*\) denote \(T_a^*\).

From (7.5) and (7.8) we have our principal theorem:

(7.9) Suppose that \(n\) is a positive integer, \(p\) is a prime and, for each \(i\) in \(\{1, \ldots, n\}\), \(u_i\) is a positive integer and \(b_i\) is a positive integer not divisible by \(p\). Let \(T_i, i \in \{1, \ldots, n\}\), denote the function from \(T^n\) into \(T^n\) such that

\[ T_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_i + (K/p^u), x_{i+1}, \ldots, x_n). \]

If each of \(f_1, \ldots, f_k\) is a continuous function from \(T^n\) into the real numbers and

\[ f_iT_a = f_i, \quad a \in T^n, \quad i, j \in \{1, \ldots, n\}, \quad j \neq i, \]

then there is an \(a^* \) in \(T^n\) such that \(f_i(a^*) = f_iT_a(a^*) \) for \(i \in \{1, \ldots, n\}\).

Remark. Taking \(p = 2\) and \(u_1 = \ldots = u_n = 1\) yields Schmidt's Satz 1 ([4], p. 86).