On a class of universal algebras

by

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1. Introduction. E. Marczewski introduced some classes of universal algebras in which the notion of independence (see [3]) has fundamental properties of linear independence. These classes are: v-algebras (called also Marczewski's algebras; see [2] and [6]), v^*-algebras (see [1], [4], [5], [8] and [9]) and v^*_*-algebras (called also v^{**}-algebras; see [6]). A full description of all v-algebras and v^*-algebras is contained in papers [1], [7], [8], [9] and [10]. The representation problem for v^*_*-algebras is not solved yet. A partial solution as well as examples are given in [6] and [10].

For the terminology and notation used here, see [3]. In particular, for a given universal algebra $(A; \boldsymbol{F})$, where A is a set and $\boldsymbol{F}$ is the class of fundamental operations, by $\boldsymbol{A}^{(n)}$ $(n \geqslant 1)$ we shall denote the class of all n-ary algebraic operations. Further, by $\boldsymbol{A}^{(0)}$ we shall denote the class of all algebraic constants. If $0 \leqslant k \leqslant n$, then $\boldsymbol{A}^{(n,k)}$ will denote the subclass of $\boldsymbol{A}^{(n)}$ consisting of all operations depending on at most k variables.

Let $f, g \in \boldsymbol{A}^{(n)}$ $(n \geqslant 1)$. We say that the equation

$$f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n)$$

depends on a certain variable if there exist an integer j $(1 \leqslant j \leqslant n)$ and a system $a_j', a_1, a_2, \ldots, a_n$ of elements of A such that

$$f(a_1, a_2, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n) = g(a_1, a_2, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n)$$

and

$$f(a_1, a_2, \ldots, a_{j-1}, a_j', a_{j+1}, \ldots, a_n) \neq g(a_1, a_2, \ldots, a_{j-1}, a_j', a_{j+1}, \ldots, a_n).$$

An algebra $(A; \boldsymbol{F})$ is called a *v_*-algebra* if for every pair $f, g \in \boldsymbol{A}^{(n)}$ $(n \geqslant 1)$ for which the equation

$$f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) \tag{1.1}$$

depends on a certain variable there exist an index k $(1 \leqslant k \leqslant n)$ and an operation $h \in \boldsymbol{A}^{(n-1)}$ such that equation (1.1) is equivalent to the equation

$$x_k = h(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n).$$

This concept is due to W. Narkiewicz. It is very easy to see that the class of v_*-algebras contains the class of v-algebras and is contained in the class of v_*^*-algebras. The aim of the present paper is to prove a representation theorem for v_*-algebras. The idea of the proof is similar to that in [7].

2. Admissible sets. In this section $\mathfrak{R}$ will denote an associative ring with the unit element, without divisors of zero, such that for every pair α, β of elements of $\mathfrak{R}$ there exists an element $\gamma \in \mathfrak{R}$ satisfying the equation $\alpha = \beta\gamma$ or the equation $\beta = \alpha\gamma$. Let A be a unital left-module over $\mathfrak{R}$ satisfying the cancellation law, i.e. a left-module satisfying the condition $1x = x$ for every $x \in A$ and such that for any $\alpha \in R$ and $y \in A$ the relation $\alpha y = 0$ implies $y = 0$ whenever $\alpha \neq 0$.

A subset B of the Cartesian product $\mathfrak{R} \times A$ is said to be *admissible* if it satisfies the following conditions:

(i) $\langle 1, 0\rangle \in B$.

(ii) If $\langle \lambda, a\rangle \in B$, then the element λ is invertible in $\mathfrak{R}$.

(iii) If $\mu_1, \mu_2, \ldots, \mu_n \in \mathfrak{R}$, $\sum_{j=1}^{n} \mu_j = 1$ and $\langle \lambda_j, a_j\rangle \in B$ $(j = 1, 2, \ldots, n)$, then

$$\Big\langle \sum_{j=1}^{n} \mu_j\lambda_j, \sum_{j=1}^{n} \mu_j a_j \Big\rangle \in B .$$

(iv) If $\alpha, \lambda \in \mathfrak{R}$, $\alpha \neq 0$, $a \in A$ and $\langle 1 + \alpha\lambda - \alpha, \alpha a\rangle \in B$, then $\langle \lambda, a\rangle \in B$.

Now we shall give some examples of admissible sets. A submodule A_1 of a unital left-module A over the ring $\mathfrak{R}$ is said to be *divisible* if for any $\alpha \in \mathfrak{R}$ $(\alpha \neq 0)$ and any $a \in A$ the relation $\alpha a \in A_1$ implies the relation $a \in A_1$.

(a) Let A_1 be a divisible submodule of a unital left-module A over the ring $\mathfrak{R}$. The set $\{1\} \times A_1$ is an admissible subset of $\mathfrak{R} \times A$. In fact, conditions (i), (ii) and (iii) are obvious. The condition (iv) is simply the divisibility condition for A_1.

(b) Let p be a prime and let $\mathfrak{R}_p$ be the ring of all rationals $n/(pm+1)$, where n and m are arbitrary integers, under usual addition and multiplication. It is very easy to prove that for any pair $\alpha, \beta \in \mathfrak{R}_p$ at least one of the elements α and β is left-divisible by the other one. Let A be an arbitrary unitary left-module over $\mathfrak{R}_p$. Further, let A_1 be a divisible submodule of A and let c be an element of A such that for any $\alpha, \beta \in \mathfrak{R}_p$ and $a \in A_1$ the left-divisibility of $\alpha c + a$ by β implies the left-divisibility of α and a by β. For instance, if A is a product of n copies of the ring $\mathfrak{R}_p$: $A = \mathfrak{R}_p \times \mathfrak{R}_p \times \ldots \times \mathfrak{R}_p$ and $A_1 = \{0\} \times \mathfrak{R}_p \times \ldots \times \mathfrak{R}_p$, then as an element c each element $\langle \gamma, 0, 0, \ldots, 0\rangle$ with invertible $\gamma \in \mathfrak{R}_p$ can be taken.

Let B be the set of all elements

$$\Big\langle \frac{pn+1}{pm+1}, \frac{n-m}{pm+1}c + a \Big\rangle, \tag{2.1}$$

where n, m are arbitrary integers and $a \in A_1$. We shall prove that the set B is an admissible subset of $\mathfrak{R}_p \times A$.

We note that the equation

$$\frac{pn+1}{pm+1} = \frac{pq+1}{pr+1}$$

implies the equation

$$\frac{n-m}{pm+1} = \frac{q-r}{pr+1}$$

and, consequently, the coefficient of c in (2.1) is uniquely determined. Since $\langle 1, 0\rangle \in B$ and all elements of the form $(pn+1)/(pm+1)$ are invertible in $\mathfrak{R}_p$, conditions (i) and (ii) for admissible sets are satisfied. In order to prove condition (iii) consider a system $\mu_1, \mu_2, \ldots, \mu_n \in \mathfrak{R}_p$, with $\sum_{j+1}^{n} \mu_j = 1$ and a system $\langle \lambda_j, a_j\rangle \in B$ $(j = 1, 2, \ldots, n)$. It is very easy to prove that the elements μ_j, λ_j $(j = 1, 2, \ldots, n)$ can be written in the form

$$\mu_j = \frac{m_j}{pm+1}, \qquad \lambda_j = \frac{pk_j+1}{pm+1},$$

where m, m_j, k_j $(j = 1, 2, \ldots, n)$ are integers. Moreover,

$$a_j = \frac{k_j - m}{pm+1} c + b_j \qquad (j = 1, 2, \ldots, n),$$

where $b_j \in A_1$. Hence, setting

$$q = m + \sum_{j=1}^{n} m_j k_j, \qquad r = m^2 p + 2m, \qquad b = \sum_{j=1}^{n} \mu_j b_j,$$

we get the equations

$$\sum_{j=1}^{n} \mu_j \lambda_j = \frac{pq+1}{pr+1}, \qquad \sum_{j=1}^{n} \mu_j a_j = \frac{q-r}{pr+1} c + b,$$

which, according to the relation $b \in A_1$ and (2.1), imply

$$\Big\langle \sum_{j=1}^{n} \mu_j\lambda_j, \sum_{j=1}^{n} \mu_j a_j \Big\rangle \in B .$$

Condition (iii) is thus proved.

Suppose that $\alpha, \lambda \in \mathfrak{R}_p$, $\alpha \neq 0$, $a \in A$ and $\langle 1+\alpha\lambda-\alpha, \alpha a\rangle \in B$. Consequently,

$$(2.2)\qquad \alpha = \frac{k}{pm+1}, \quad 1+\alpha\lambda-\alpha = \frac{pn+1}{pm+1}, \quad \alpha a = \frac{n-m}{pm+1}c+b,$$

where k, n, m are integers, $k \neq 0$ and $b \in A_1$. From the definition of the element c it follows that b and $(n-m)/(pm+1)$ are left-divisible by α. Thus $b = \alpha b_0$ and, by (2.2), $n-m = ks/(pq+1)$, where s and q are integers. Consequently,

$$a = \frac{s}{pq+1}c+b_0$$

and

$$\lambda = \frac{p(s+q)+1}{pq+1}.$$

Hence it follows that the element $\langle\lambda, a\rangle$ is of the form (2.1) and, consequently, belongs to B. Condition (iv) is thus proved. The set B in question is an admissible subset of $\mathfrak{R}_p\times A$ and is not of the form $\{1\}\times A_1$, where $A_1 \subset A$.

Now we shall prove some Lemmas for arbitrary admissible subsets B of $\mathfrak{R}\times A$.

LEMMA 2.1. *If $\langle\lambda, a\rangle \in B$, then $\langle\lambda^{-1}, -\lambda^{-1}a\rangle \in B$.*

Proof. We note that, by condition (ii), the element λ is invertible. Put $\mu_1 = -\lambda^{-1}$, $\mu_2 = 1+\lambda^{-1}$ and $\langle\lambda_1, a_1\rangle = \langle\lambda, a\rangle$, $\langle\lambda_2, a_2\rangle = \langle 1, 0\rangle$. Since $\mu_1+\mu_2 = 1$, we have, according to condition (iii),

$$\langle\mu_1\lambda_1+\mu_2\lambda_2, \mu_1a_1+\mu_2a_2\rangle \in B.$$

Taking into account the equations $\mu_1\lambda_1+\mu_2\lambda_2 = \lambda^{-1}$, $\mu_1a_1+\mu_2a_2 = -\lambda^{-1}a$, we obtain the assertion of the Lemma.

LEMMA 2.2. *If $\langle\lambda_j, a_j\rangle \in B$ $(j = 1, 2, \ldots, n)$ and*

$$\Big\langle\sum_{j=1}^{n}\nu_j, a\Big\rangle \in B,$$

then

$$\Big\langle\sum_{j=1}^{n}\nu_j\lambda_j, \sum_{j=1}^{n}\nu_ja_j+a\Big\rangle \in B.$$

Proof. Set $\mu_j = \nu_j$ $(j = 1, 2, \ldots, n)$, $\mu_{n+1} = 1$,

$$\mu_{n+2} = -\sum_{j=1}^{n}\nu_j, \qquad \langle\lambda_{n+1}, a_{n+1}\rangle = \Big\langle\sum_{j=1}^{n}\nu_j, a\Big\rangle$$

and

$$\langle\lambda_{n+2}, a_{n+2}\rangle = \langle 1, 0\rangle.$$

Since $\sum_{j=1}^{n+2}\mu_j = 1$, we have, according to condition (iii) of the definition of admissible sets, the relation

$$\Big\langle\sum_{j=1}^{n+2}\mu_j\lambda_j, \sum_{j=1}^{n+2}\mu_ja_j\Big\rangle \in B.$$

Hence and from the equations

$$\sum_{j=1}^{n+2}\mu_j\lambda_j = \sum_{j=j}^{n}\nu_j\lambda_j, \quad \sum_{j=1}^{n+2}\mu_ja_j = \sum_{j=1}^{n}\nu_ja_j+a$$

the assertion of the lemma follows.

LEMMA 2.3. *If $\langle\lambda, a\rangle \in B$, then the element $1-\lambda$ is not invertible.*

Proof. Contrary to this let us suppose that the element $1-\lambda$ is invertible in $\mathfrak{R}$. Put $\alpha = 1-\lambda$ and $b = (1-\lambda)^{-1}a$. Since $\langle\lambda, a\rangle = \langle 1+\alpha\times\times 0-\alpha, \alpha b\rangle$, we infer, in view of condition (iv), that $\langle 0, b\rangle \in B$. But this contradicts condition (ii), which completes the proof.

In the sequel $C+c$, where $c \in A$ and $C \subset A$, will denote the set $\{a+c\colon a \in C\}$. Further, for any $\lambda \in \mathfrak{R}$ by A_λ we shall denote the set of all elements $a \in A$ such that $\langle\lambda, a\rangle \in B$.

THEOREM 2.1. *The set A_1 is a divisible submodule of A. Moreover, for any $\lambda \in \mathfrak{R}$ either the set A_λ is empty or $A_\lambda = A_1+c$, where $c \in A_\lambda$.*

Proof. Let $a_1, a_2 \in A_1$ and $\mu_1, \mu_2 \in \mathfrak{R}$. Setting $a_3 = 0$, $\mu_3 = 1-\mu_1-\mu_2$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$, we have, by condition (iii) of the definition of admissible sets, the relation $\langle\sum_{j=1}^{3}\mu_j\lambda_j, \sum_{j=1}^{3}\mu_ja_j\rangle \in B$. Since $\sum_{j=1}^{3}\mu_j\lambda_j = 1$ and $\sum_{j=1}^{3}\mu_ja_j = \mu_1a_1+\mu_2a_2$, we infer that $\mu_1a_1+\mu_2a_2 \in A_1$. Consequently, A_1 is a submodule of A.

Further, suppose that $\alpha \in \mathfrak{R}$, $\alpha \neq 0$, $a \in A$ and $\alpha a \in A_1$. Since $\langle 1, \alpha a\rangle = \langle 1+\alpha-\alpha, \alpha a\rangle$, we infer, by condition (iv), that $\langle 1, a\rangle \in B$ and, consequently, $a \in A_1$. Thus the submodule A_1 is divisible.

If $c \in A$ and $a \in A_1$, then, by Lemma 2.2, $\langle\lambda, a+c\rangle \in B$. Consequently, $A_1+c \subset A_\lambda$. Further, by Lemma 2.1, $\langle\lambda^{-1}, -\lambda^{-1}c\rangle \in B$ and, consequently, for every pair $\langle\lambda, c_1\rangle \in B$ we have, in view of Lemma 2.2, the relation $\langle\lambda\lambda^{-1}, c_1-c\rangle \in B$. Thus $c_1-c \in A_1$, which implies the inclusion $A_\lambda \subset A_1+c$. The theorem is thus proved.

The following theorem is a direct consequence of Theorem 2.1 and Lemma 2.3.

THEOREM 2.2. *If $\mathfrak{R}$ is a field and A a linear space over $\mathfrak{R}$, then each admissible subset of $\mathfrak{R}\times A$ is of the form $\{1\}\times A_1$, where A_1 is a linear subspace of A.*

3. Examples of v_*-algebras.

3.1. Let $\mathcal{R}$ be an associative ring with the unit element, without divisors of zero and such that for any pair $\alpha, \beta \in \mathcal{R}$ the element α is left-divisible by β or the element β is left-divisible by α. Let A be a unital left-module over $\mathcal{R}$ satisfying the cancellation law and A_0 a divisible submodule of A. Further, let F be the class of all operations

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \lambda_j x_j + a, \tag{3.1}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{R}$ and $a \in A_0$. It is very easy to prove that the algebra $(A; F)$ is a v^*-algebra. This algebra is a modification of the algebra presented in [6] Section 2.

3.2. Let B be an admissible subset of $\mathcal{R} \times A$ and F the class of all operations (3.1) where

$$\Big\langle \sum_{j=1}^{n} \lambda_j, a \Big\rangle \in B. \tag{3.2}$$

We shall prove that the algebra $(A; F)$ is a v^*-algebra.

First of all we shall prove that each algebraic operation in the algebra in question is of the form (3.1) with coefficients satisfying condition (3.2). To prove this it suffices to show that the composition of operations of the form (3.1) with condition (3.2) is of the same form. Suppose that

$$g(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \nu_j x_j + a, \qquad g_j(x_1, x_2, \ldots, x_k) = \sum_{i=1}^{k} \lambda_{ij} x_i + a_j$$

and

$$\Big\langle \sum_{j=1}^{n} \nu_j, a \Big\rangle \in B, \qquad \Big\langle \sum_{i=1}^{k} \lambda_{ij}, a_j \Big\rangle \in B \qquad (j = 1, 2, \ldots, n).$$

Since, by Lemma 2.2, the pair $\langle \sum_{j=1}^{n} \sum_{i=1}^{k} \nu_j \lambda_{ij}, \sum_{j=1}^{n} \nu_j a_j + a \rangle$ belongs to B, the composition

$$g\big(g_1(x_1, x_2, \ldots, x_k), g_2(x_1, x_2, \ldots, x_k), \ldots, g_n(x_1, x_2, \ldots, x_k)\big)$$

is of the form (3.1) and satisfies condition (3.2).

Let $f_1 \in A^{(1)}$, i.e. $f_1(x) = \lambda x + a$, where $\langle \lambda, a \rangle \in B$. Since $f_1^{-1}(x) = \lambda^{-1}x - \lambda^{-1}a$ and, by Lemma 2.1, $\langle \lambda^{-1}, -\lambda^{-1}a \rangle \in B$, we infer that the inverse operation f_1^{-1} belongs to $A^{(1)}$.

Suppose that $f, g \in A^{(n)}$ and the equation

$$f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) \tag{3.3}$$

depends on a certain variable. Put

$$f_1(x) = f(x, x, \ldots, x),$$
$$f_0(x_1, x_2, \ldots, x_n) = f_1^{-1}\big(f(x_1, x_2, \ldots, x_n)\big),$$
$$g_0(x_1, x_2, \ldots, x_n) = f_1^{-1}\big(g(x_1, x_2, \ldots, x_n)\big).$$

Of course, equation (3.3) is equivalent to the equation

$$f_0(x_1, x_2, \ldots, x_n) = g_0(x_1, x_2, \ldots, x_n). \tag{3.4}$$

Moreover, $f_0(x, x, \ldots, x) = x$ and, consequently,

$$f_0(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \lambda_j x_j,$$

where

$$\sum_{j=1}^{n} \lambda_j = 1. \tag{3.5}$$

Furthermore,

$$g_0(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \mu_j x_j + a,$$

where

$$\Big\langle \sum_{j=1}^{n} \mu_j, a \Big\rangle \in B. \tag{3.6}$$

Put

$$\alpha_j = \lambda_j - \mu_j \qquad (j = 1, 2, \ldots, n). \tag{3.7}$$

Then equation (3.4) is equivalent to the equation

$$\sum_{j=1}^{n} \alpha_j x_j = a. \tag{3.8}$$

Since this equation depends on a certain variable, at least one coefficient $\alpha_1, \alpha_2, \ldots, \alpha_n$ is different from zero.

Taking into account the divisibility properties of the ring $\mathcal{R}$, we can prove by induction with respect to n that there exists an index k $(1 \leqslant k \leqslant n)$ such that all elements α_j $(j = 1, 2, \ldots, n)$ are left-divisible by the element α_k. In fact for $n = 2$ this holds by the definition of the ring $\mathcal{R}$. Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are left-divisible by α_1. If α_n is not left-divisible by α_1, then α_1 and, consequently, all elements α_j $(j = 1, 2, \ldots, n)$ are left-divisible by α_n, which completes the proof.

Therefore without loss of generality we may assume that $\alpha_1 \neq 0$ and all elements $\alpha_2, \alpha_3, \ldots, \alpha_n$ are left-divisible by α_1, i.e.

$$\alpha_j = \alpha_1 \beta_j \qquad (j = 2, 3, \ldots, n), \tag{3.9}$$

where $\beta_j \in \mathfrak{R}$. Since equation (3.8) depends on a certain variable, there exists a system $a_1, a_2, \ldots, a_n$ of elements of A such that

$$\sum_{j=1}^{n} a_j a_j = a\,.$$

Consequently, by (3.9), setting $b = a_1 + \sum_{j=2}^{n} \beta_j a_j$ we have the equation

$$a = a_1 b\,. \tag{3.10}$$

Hence and from (3.9) in view of the cancellation law it follows that equation (3.8) is equivalent to the equation

$$x_1 = -\sum_{j=2}^{n} \beta_j x_j + b\,.$$

Put $\lambda = -\sum_{j=1}^{n} \beta_j$. It remains to prove that $\langle \lambda, b\rangle \in B$. From (3.5), (3.7), (3.9) and (3.10) we obtain the equations

$$\langle 1 + a_1\lambda - a_1, a_1 b\rangle = \Big\langle 1 - \sum_{j=1}^{n} a_j, a\Big\rangle = \Big\langle 1 - \sum_{j=1}^{n} \lambda_j + \sum_{j=1}^{n} \mu_j, a\Big\rangle = \Big\langle \sum_{j=1}^{n} \mu_j, a\Big\rangle,$$

which, by (3.6) and condition (iv) of the definition of admissible sets, imply the relation $\langle \lambda, b\rangle \in B$. Thus $(A; \boldsymbol{F})$ is a v^*-algebra.

3.3. Let $\mathcal{S}$ be a semigroup of one-to-one transformations of a non-empty set A into itself containing the identical transformation and satisfying the following conditions

(*) each transformation that is not the identical transformation has at most one fixed point in A,

(**) if $g_1, g_2 \in \mathcal{S}$ and $g_1(A) \cap g_2(A) \neq \emptyset$, then there exists a transformation $g \in \mathcal{S}$ such that $g_1 = g_2 g$ or $g_2 = g_1 g$.

Let A_0 be a subset of A containing all fixed points of transformations from $\mathcal{S}$ that are not the identical transformation and satisfying the conditions $g(A_0) \subset A_0$ and $g^{-1}(A_0) \subset A_0$ for all $g \in \mathcal{S}$. If $\boldsymbol{F}$ is the class of all operations f defined as

$$f(x_1, x_2, \ldots, x_n) = g(x_j) \quad (1 \leqslant j \leqslant n)\,, \tag{3.11}$$

$$f(x_1, x_2, \ldots, x_n) = a\,, \tag{3.12}$$

where $g \in \mathcal{S}$ and $a \in A_0$, then $(A; \boldsymbol{F})$ is a v^*-algebra.

Indeed, it is very easy to verify that each algebraic operation in $(A; \boldsymbol{F})$ is of the form (3.11) or (3.12). Moreover, if $g_1, g_2 \in \mathcal{S}$ and the equation

$$g_1(x_1) = g_2(x_2) \tag{3.13}$$

depends on a certain variable, then $g_1(A) \cap g_2(A) \neq \emptyset$ and, consequently, $g_1 = g_2 g$ or $g_2 = g_1 g$, where g is an element of $\mathcal{S}$. Since the transformations from $\mathcal{S}$ are one-to-one, equation (3.13) is equivalent to the equation $x_2 = g(x_1)$ in the case $g_1 = g_2 g$ and to the equation $x_1 = g(x_2)$ in the opposite case.

If $g_1, g_2 \in \mathcal{S}$ and the equation

$$g_1(x) = g_2(x) \tag{3.14}$$

depends on the variable x, then $g_1(A) \cap g_2(A) \neq \emptyset$. Consequently, one of the transformations g_1, g_2 is left-divisible by the other one. Without loss of generality we may assume that $g_1 = g_2 g$, where $g \in \mathcal{S}$. Of course, g is not the identical transformation. Moreover, equation (3.14) is equivalent to the equation $x = g(x)$, i.e. an element x from A satisfies (3.14) if and only if it is a fixed point of the transformation g. Thus there exists an element $c \in A_0$ such that (3.14) holds if and only if $x = c$.

Finally, suppose that $g \in \mathcal{S}$, $a \in A_0$ and the equation

$$g(x) = a \tag{3.15}$$

depends on the variable x. Since the transformation g is one-to-one, there exists an element $c \in A$ such that equation (3.15) holds if and only if $x = c$. Furthermore, $c \in A_0$ because of the inclusion $g^{-1}(A_0) \subset A_0$. Thus $(A; \boldsymbol{F})$ is a v_*-algebra.

4. A representation theorem. In the preceding section we presented a construction of three types of v_*-algebras. Now we shall prove that each v_*-algebra can be obtained in this way. Namely, we shall prove the following representation theorem.

THEOREM 4.1. *Let $(A; \boldsymbol{F})$ be a v_*-algebra.*

(i) *If $A^{(0)} \neq \emptyset$ and $A^{(3)} \neq A^{(3,1)}$, then A is a unital left-module satisfying the cancellation law over an associative ring $\mathfrak{R}$ with the unit element, without divisors of zero such that for any pair of elements of $\mathfrak{R}$ at least one element is left-divisible by the other one. Moreover, there exists a divisible submodule A_0 of A such that the class of algebraic operations is the class of all operations defined as*

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \lambda_j x_j + a\,,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathfrak{R}$ and $a \in A_0$.

(ii) *If $A^{(0)} = \emptyset$ and $A^{(3)} \neq A^{(3,1)}$, then A is a unital left-module satisfying the cancellation law over an associative ring $\mathfrak{R}$ with the unit element, without divisors of zero and such that for any pair of elements of $\mathfrak{R}$ at least one element is left-divisible by the other one. Moreover, there exists an admis-*

sible subset B of $\mathfrak{R}\times A$ such that the class of algebraic operations is the class of all operations defined as

$$f(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} \lambda_j x_j + a,$$

where $\langle \sum_{j=1}^{n} \lambda_j, a\rangle \in B$.

(iii) *If $A^{(3)} = A^{(3,1)}$, then there is a semigroup $\mathcal{S}$ of one-to-one transformations of the set A into itself containing the identical transformation and satisfying the following conditions:*

(*) *each transformation that is not the identical transformation has at most one fixed point.*

(**) *if $g_1, g_2 \in \mathcal{S}$ and $g_1(A) \cap g_2(A) \neq \emptyset$, then at least one element of the pair g_1, g_2 is left-divisible by the other one. Moreover, there exists a subset A_0 of the set A containing all fixed points of transformations from $\mathcal{S}$ that are not the identical transformation and satisfying the conditions $g(A_0) \subset A_0$, $g^{-1}(A_0) \subset A_0$ for all $g \in \mathcal{S}$ such that the class of algebraic operations is the class of all operations defined as*

$$f(x_1, x_2, \dots, x_n) = g(x_j) \qquad (1 \leqslant j \leqslant n),$$
$$f(x_1, x_2, \dots, x_n) = a,$$

where $g \in \mathcal{S}$ and $a \in A_0$.

Before proving the Theorem we shall prove some lemmas. We assume that all algebras considered in this section are v_*-algebras.

LEMMA 4.1. *If $f, g \in A^{(n)}$ $(n \geqslant 3)$ and $f(x_1, x_2, \dots, x_n) = g(x_1, x_2, \dots, x_n)$ whenever $x_1 = x_2$ or $x_1 = x_3$, then $f = g$.*

Proof. Suppose the contrary. Then the equation

$$f(x_1, x_2, \dots, x_n) = g(x_1, x_2, \dots, x_n) \tag{4.1}$$

depends on a certain variable. Consequently, there exist an index k $(1 \leqslant k \leqslant n)$ and an operation $h \in A^{n-1}$ such that (4.1) is equivalent to the equation

$$x_k = h(x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_n).$$

First consider the case $k = 1$. Then equation (4.1) is equivalent to the equation

$$x_1 = h(x_2, x_3, \dots, x_n).$$

Since equation (4.1) holds for $x_1 = x_2$, we have the formula

$$x_2 = h(x_2, x_3, \dots, x_n) \tag{4.2}$$

for all elements $x_2, x_3, \dots, x_n \in A$. On the other hand, equation (4.1) holds for $x_1 = x_3$ and, consequently, $x_3 = h(x_2, x_3, \dots, x_n)$ for all elements $x_2, x_3, \dots, x_n \in A$. But this contradicts (4.2), which completes the proof in the case $k = 1$.

Suppose that $k > 1$. By the assumption equation (4.1) holds whenever $x_j = x$ for all $j \neq k$. Consequently, $x_k = h(x, x, \dots, x)$ for arbitrary $x, x_k \in A$, which gives a contradiction. The lemma is thus proved.

LEMMA 4.2. *If $A^{(3)} \neq A^{(3,1)}$, then there exists exactly one operation $s \in A^{(3)}$ satisfying the condition $s(x, y, y) = s(y, x, y) = x$.*

Proof. Let $f \in A^{(3)} \setminus A^{(3,1)}$. We may assume that the operation $f(x_1, x_2, x_3)$ depends on the variables x_2 and x_3. Consequently, the equation

$$f(x_1, x_2, x_3) = f(x_1, x_4, x_5) \tag{4.3}$$

depends on the variables x_2, x_3, x_4 and x_5. Thus there exist an index k $(1 \leqslant k \leqslant 5)$ and an operation $g \in A^{(4)}$ such that equation (4.3) is equivalent to the equation

$$x_k = g(x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_5).$$

By the symmetry properties of (4.3) we may assume that $1 \leqslant k \leqslant 3$. Indeed, the cases $k = 4$ and $k = 5$ can be reduced, by the substitutions $x_4 = x_2$, $x_2 = x_4$ and $x_5 = x_3$, $x_3 = x_5$, to the cases $k = 2$ and $k = 3$, respectively. Moreover, the case $k = 3$ can be reduced, by the substitution $f_0(x_1, x_2, x_3) = f(x_1, x_3, x_2)$, to the case $k = 2$. Consequently, we may assume that $1 \leqslant k \leqslant 2$.

In the case $k = 1$ equation (4.3) is equivalent to the equation $x_1 = g(x_2, x_3, x_4, x_5)$. Since equation (4.3) holds for $x_2 = x_3 = x_4 = x_5$ and for all x_1, we infer that $x_1 = g(x_2, x_2, x_2, x_2)$ for all $x_1, x_2 \in A$, which is impossible. Thus $k = 2$ and equation (4.3) is equivalent to the equation $x_2 = g(x_1, x_3, x_4, x_5)$. Since equation (4.3) holds for $x_2 = x_4$, $x_3 = x_5$, we have the formula

$$x_2 = g(x_1, x_3, x_2, x_3) \tag{4.4}$$

for all $x_1, x_2, x_3 \in A$. Hence it follows that the operation $g(x_1, x_2, x_3, x_4)$ depends on the variables x_2 and x_4. Indeed, in the opposite case for all $x_1, x_2, x_3, x_4 \in A$ formula (4.4) would imply the equation $x_3 = g(x_1, x_2, x_3, x_4)$, which is equivalent to the equation $f(x_1, x_3, x_2) = f(x_1, x_3, x_4)$. Thus $f(x_1, x_2, x_3)$ would not depend on the variable x_3, which contradicts the assumption. Consequently, $g(x_1, x_2, x_3, x_4)$ depends on both x_2 and x_4. Hence it follows that the equation

$$g(x_1, x_2, x_3, x_4) = g(x_1, x_5, x_3, x_6) \tag{4.5}$$

depends on the variables x_2, x_4, x_5 and x_6. Thus there exist an index j $(1 \leqslant j \leqslant 6)$ and an operation $h \in A^{(5)}$ such that equation (4.5) is equivalent to the equation

$$x_j = h(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_6).$$

By the symmetry properties of (4.5) we may assume that $1 \leqslant j \leqslant 4$. Indeed, the cases $k = 5$ and $k = 6$ can be reduced, by the substitutions $x_5 = x_2$, $x_2 = x_5$ and $x_6 = x_4$, $x_4 = x_6$, to the cases $k = 2$ and $k = 4$ respectively. Moreover, by the substitution $g_0(x_1, x_2, x_3, x_4) = g(x_1, x_4, x_3, x_2)$, which does not change condition (4.4), we can reduce the case $j = 4$ to the case $j = 2$. Consequently, we may assume that $1 \leqslant j \leqslant 3$ and the operation g in (4.5) fulfils (4.4).

First consider the case $j = 1$. Then (4.5) is equivalent to the equation $x_1 = h(x_2, x_3, x_4, x_5, x_6)$. From (4.4) it follows that equation (4.5) holds whenever $x_2 = x_4$ and $x_5 = x_6$. Thus for all $x_1, x_2, x_3, x_5 \in A$ we have the equation $x_1 = h(x_2, x_3, x_2, x_5, x_5)$, which gives a contradiction. Consequently, $2 \leqslant j \leqslant 3$.

Now consider the case $j = 3$. Then (4.5) is equivalent to the equation $x_3 = h(x_1, x_2, x_4, x_5, x_6)$. Since, by (4.4), the equation (4.5) holds whenever $x_2 = x_4$ and $x_5 = x_6$, we have the equation $x_3 = h(x_1, x_2, x_2, x_5, x_5)$ for all $x_1, x_2, x_3, x_5 \in A$. But this is impossible. Consequently, $j = 2$ and equation (4.5) is equivalent to the equation $x_2 = h(x_1, x_3, x_4, x_5, x_6)$.

Since, by (4.4), equation (4.5) holds for $x_2 = x_4$ and $x_5 = x_6$, we have the formula

$$x_2 = h(x_1, x_3, x_2, x_5, x_6) \tag{4.6}$$

for all $x_1, x_2, x_3, x_5 \in A$. Moreover, equation (4.5) holds whenever $x_2 = x_5$ and $x_4 = x_6$. Thus for all $x_1, x_2, x_3, x_4 \in A$ the equation

$$x_2 = h(x_1, x_3, x_4, x_2, x_4) \tag{4.7}$$

holds. Put $s(x, y, z) = h(x, x, x, y, z)$. Of course, $s \in A^{(3)}$ and, by (4.6) and (4.7), the equations $s(x, y, y) = s(y, x, y) = x$ hold. From Lemma 4.1 it follows that these conditions determine the operation s uniquely. The lemma is thus proved.

By $\tilde{A}^{(n)}$ $(n \geqslant 1)$ we shall denote the subclass of $A^{(n)}$ consisting of all operations f satisfying the condition $f(x, x, \ldots, x) = x$.

LEMMA 4.3. *Let s be the ternary algebraic operation satisfying the condition*

$$s(x, y, y) = s(y, x, y) = x\,. \tag{4.8}$$

Then the following equations are true:

$$s(x_1, x_2, x_3) = s(x_2, x_1, x_3)\,, \tag{4.9}$$

$$s\big(s(x_1, x_2, x_3), x_4, x_3\big) = s\big(x_1, s(x_2, x_4, x_3), x_3\big)\,, \tag{4.10}$$

$$f\big(s(x_1, x_2, x_3), x_3\big) = s\big(f(x_1, x_3), f(x_2, x_3), x_3\big) \text{ for any } f \in \tilde{A}^{(2)}\,, \tag{4.11}$$

$$f(x_1, x_2, \ldots, x_n) = s\big(f(x_1, x_1, x_3, x_4, \ldots, x_n), f(x_1, x_2, x_1, x_4, \ldots, x_n), f(x_1, x_1, x_1, x_4, \ldots, x_n)\big) \text{ for any } f \in \tilde{A}^{(n)} \quad (n \geqslant 3)\,. \tag{4.12}$$

Proof. From (4.8) it follows that equation (4.9) holds whenever $x_3 = x_1$ or $x_3 = x_2$. Thus, in view of Lemma 4.1, it holds for all $x_1, x_2, x_3 \in A$.

Further, from (4.8) we get the equations

$$s\big(s(x_1, x_2, x_2), x_4, x_2\big) = s(x_1, x_4, x_2)\,,$$

$$s\big(x_1, s(x_2, x_4, x_2), x_2\big) = s(x_1, x_4, x_2)\,,$$

$$s\big(s(x_1, x_2, x_4), x_4, x_4\big) = s(x_1, x_2, x_4)\,,$$

$$s\big(x_1, s(x_2, x_4, x_4), x_4\big) = s(x_1, x_2, x_4)\,,$$

which imply that (4.10) holds whenever $x_3 = x_2$ or $x_3 = x_4$. Hence, in virtue of Lemma 4.1, we get formula (4.10) for all $x_1, x_2, x_3, x_4 \in A$.

From the equations

$$f\big(s(x_1, x_2, x_1), x_1\big) = f(x_2, x_1)\,,$$

$$s\big(f(x_1, x_1), f(x_2, x_1), x_1\big) = s\big(x_1, f(x_2, x_1), x_1\big) = f(x_2, x_1)\,,$$

$$f\big(s(x_1, x_2, x_2), x_2\big) = f(x_1, x_2)\,,$$

$$s\big(f(x_1, x_2), f(x_2, x_2), x_2\big) = s\big(f(x_1, x_2), x_2, x_2\big) = f(x_1, x_2)\,,$$

where $f \in \tilde{A}^{(2)}$, it follows that equation (4.11) holds whenever $x_3 = x_1$ or $x_3 = x_2$. Thus it holds for all $x_1, x_2, x_3 \in A$.

Finally, taking into account formula (4.8), we have for every operation $f \in \tilde{A}^{(n)}$ the equations

$$s\big(f(x_2, x_2, x_3, x_4, \ldots, x_n), f(x_2, x_2, x_2, x_4, \ldots, x_n), f(x_2, x_2, x_2, x_4, \ldots, x_n)\big) = f(x_2, x_2, x_3, x_4, \ldots, x_n)\,,$$

$$s\big(f(x_3, x_3, x_3, x_4, \ldots, x_n), f(x_3, x_2, x_3, x_4, \ldots, x_n), f(x_3, x_3, x_3, x_4, \ldots, x_n)\big) = f(x_3, x_2, x_3, x_4, \ldots, x_n)\,.$$

Hence it follows that equation (4.12) holds whenever $x_1 = x_2$ or $x_1 = x_3$, which implies, in virtue of Lemma 4.1, that equation (4.12) holds for all $x_1, x_2, \ldots, x_n \in A$. The lemma is thus proved.

In the sequel we shall denote by $\mathfrak{K}$ the class $\tilde{A}^{(2)}$. Elements of $\mathfrak{K}$ will be denoted by small Greek letters.

LEMMA 4.4. *If $A^{(3)} \neq A^{(3,1)}$, then $\mathfrak{K}$ is an associative ring with respect to the operations*

$$(\alpha+\beta)(x, y) = s\big(\alpha(x, y), \beta(x, y), y\big)\,, \tag{4.13}$$

$$(\alpha\beta)(x, y) = \alpha\big(\beta(x, y), y\big)\,, \tag{4.14}$$

where s is a ternary algebraic operation satisfying the condition

$$s(x, y, y) = s(y, x, y) = x\,. \tag{4.15}$$

Moreover, the ring $\mathfrak{R}$ has the unit element and has no divisors of zero. The element $-\alpha$ is given by the formula

$$(4.16)\qquad (-\alpha)(x, y) = s\big(y, y, \alpha(x, y)\big).$$

Proof. First of all we note that the existence of the operation s satisfying condition (4.15) follows from Lemma 4.2.

We define the zero-element and the unit element by the formulas $0(x, y) = y$, $1(x, y) = x$. Obviously, $0 \neq 1$. From (4.13) and (4.15) it follows that

$$(\alpha+0)(x, y) = s\big(\alpha(x, y), y, y\big) = \alpha(x, y),$$

$$(\alpha 1)(x, y) = \alpha(x, y), \quad (1\alpha)(x, y) = \alpha(x, y).$$

Consequently, $\alpha+0 = \alpha$, $\alpha 1 = 1\alpha = \alpha$ for every $\alpha \in \mathfrak{R}$.

The associative law for multiplication is a direct consequence of definition (4.14). Taking into account assertions (4.9), (4.10) and (4.11) of Lemma 4.3, we have the equations

$$(\alpha+\beta)(x, y) = s\big(\alpha(x, y), \beta(x, y), y\big) = s\big(\beta(x, y), \alpha(x, y), y\big) = (\beta+\alpha)(x, y),$$

$$\big((\alpha+\beta)+\gamma\big)(x, y) = s\Big(s\big(\alpha(x, y), \beta(x, y), y\big), \gamma(x, y), y\Big) = s\Big(\alpha(x, y), s\big(\beta(x, y), \gamma(x, y), y\big), y\Big) = \big(\alpha+(\beta+\gamma)\big)(x, y),$$

$$\big(\alpha(\beta+\gamma)\big)(x, y) = \alpha\Big(s\big(\beta(x, y), \gamma(x, y), y\big), y\Big) = s\Big(\alpha\big(\beta(x, y), y\big), \alpha\big(\gamma(x, y), y\big), y\Big) = (\alpha\beta+\alpha\gamma)(x, y),$$

which imply that the addition is commutative, associative and the left-distributive law holds. Further, the following equations are a direct consequence of definitions (4.13) and (4.14):

$$\big((\beta+\gamma)\alpha\big)(x, y) = s\Big(\beta\big(\alpha(x, y), y\big), \gamma\big(\alpha(x, y), y\big), y\Big) = (\beta\alpha+\gamma\alpha)(x, y).$$

Thus the right-distriburive law is true.

Setting $f = s$ into (4.12) and taking into account (4.9) and (4.15), we get the formula

$$s(x_1, x_2, x_3) = s\big(s(x_1, x_1, x_3), x_2, x_1\big) = s\big(x_2, s(x_1, x_1, x_3), x_1\big).$$

Hence and from (4.16) the equation

$$\big(\alpha+(-\alpha)\big)(x, y) = s\Big(\alpha(x, y), s\big(y, y, \alpha(x, y)\big), y\Big) = s\big(y, \alpha(x, y), \alpha(x, y)\big) = y = 0(x, y)$$

follows. Thus, $\alpha+(-\alpha) = 0$ for every $\alpha \in \mathfrak{R}$ and, consequently, $\mathfrak{R}$ is an associative ring with the unit element.

Suppose that $\alpha\beta = 0$, i.e.

$$(4.17)\qquad \alpha\big(\beta(x, y), y\big) = y$$

for all $x, y \in A$. We shall prove that at least one of the elements α, β is equal to 0. From (4.17) and from the equation $\alpha\big(\beta(x, x), y\big) = \alpha(x, y)$ it follows that the equation

$$(4.18)\qquad \alpha\big(\beta(x, y), z\big) = \alpha(y, z)$$

holds whenever $y = x$ or $y = z$. Consequently, by Lemma 4.1, it holds for all $x, y, z \in A$. Suppose that $\alpha \neq 0$, i.e. the operation $\alpha(x, y)$ depends on the variable x. Then the equation

$$(4.19)\qquad \alpha(x_1, x_3) = \alpha(x_2, x_3)$$

depends on the variables x_1 and x_2. Consequently, there exist an index k $(1 \leqslant k \leqslant 3)$ and an operation $h \in A^{(2)}$ such that equation (4.19) is equivalent to the equation

$$x_k = h(x_i, x_j),$$

where $1 \leqslant i < j \leqslant 3$, $i \neq k$ and $j \neq k$.

If $k = 3$, then (4.19) is equivalent to the equation $x_3 = h(x_1, x_2)$. Since (4.19) holds whenever $x_1 = x_2$, we infer that $x_3 = h(x_1, x_1)$ for all $x_1, x_3 \in A$, which gives a contradiction. Thus $1 \leqslant k \leqslant 2$. By the substitution $x_2 = x_1$ and $x_1 = x_2$ into (4.19) the case $k = 2$ can be reduced to the case $k = 1$. Therefore we may assume that $k = 1$. Consequently, equation (4.19) is equivalent to the equation $x_1 = h(x_2, x_3)$. Since (4.19) holds whenever $x_1 = x_2$, we have the equation $x_1 = h(x_1, x_3)$ for all $x_1, x_3 \in A$. Thus equation (4.19) holds if and only if $x_1 = x_2$. Hence and from (4.18) we get the equation $\beta(x, y) = y$. Consequently $\beta = 0$, which shows that the ring $\mathfrak{R}$ has no divisors of zero. The lemma is thus proved.

LEMMA 4.5. *If $A^{(3)} \neq A^{(3,1)}$, then A is a unital left-module satisfying the cancellation law over $\mathfrak{R}$ with respect to the operations*

$$(4.20)\qquad x+y = s(x, y, \Theta) \quad (x, y \in A),$$

$$(4.21)\qquad \alpha x = \alpha(x, \Theta) \quad (\alpha \in \mathfrak{R}, x \in A),$$

where Θ is an element of $A^{(0)}$ if $A^{(0)} \neq \emptyset$ and is an arbitrary element of A if $A^{(0)} = \emptyset$. The operation s is defined by Lemma 4.2.

Proof. The element Θ is the zero-element of A. In fact, $x+\Theta = s(x, \Theta, \Theta) = x$. Further, we have, in virtue of Lemma 4.3, the following equations:

$$x+y = s(x, y, \Theta) = s(y, x, \Theta) = y+x,$$

$$(x+y)+z = s\big(s(x, y, \Theta), z, \Theta\big) = s\big(x, s(y, z, \Theta), \Theta\big) = x+(y+z),$$

$$\alpha(x+y) = \alpha\big(s(x, y, \Theta), \Theta\big) = s\big(\alpha(x, \Theta), \alpha(y, \Theta), \Theta\big) = \alpha x+\alpha y$$

for any $x, y, z \in A$ and $\alpha \in \mathfrak{R}$. Moreover, the equations

$$\alpha(\beta x) = \alpha\big(\beta(x, \Theta), \Theta\big) = (\alpha\beta)x\,,$$

$$1x = x\,,$$

$$(\alpha+\beta)x = s\big(\alpha(x, \Theta), \beta(x, \Theta), \Theta\big) = \alpha x + \beta x$$

are true for any $x \in A$ and $\alpha, \beta \in \mathfrak{R}$. Hence, setting $-x = (-1)x$, we get the equation $x + (-x) = 0x = \Theta$.

Suppose that $x \neq \Theta$ and $\alpha x = \Theta$. If $\alpha \neq 0$, i.e. the operation $\alpha(x, y)$ depends on the variable x, then the equation $\alpha(x, y) = y$ depends on the variable x. Consequently, there exists an operation $h \in A^{(1)}$ such that this equation is equivalent to one of the equations $x = h(y)$ and $y = h(x)$. Since $\alpha(x, x) = x$ for all $x \in A$, we have the equation $h(x) = x$ for all $x \in A$. Thus the equation $\alpha(x, y) = y$ holds if and only if $x = y$. In particular, $\alpha(x, \Theta) = \Theta$ implies the equation $x = \Theta$, which contradicts the assumption $x \neq \Theta$. Consequently, $\alpha = 0$, which completes the proof.

LEMMA 4.6. *If $A^{(3)} \neq A^{(3,1)}$, then the class $\tilde{A}^{(n)}$ $(n \geqslant 2)$ consists of all operations*

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \lambda_j x_j\,,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathfrak{R}$ and $\sum_{j=1}^{n} \lambda_j = 1$.

Proof. First we shall prove the formulas

$$(4.23) \qquad \lambda(y, x) = (1-\lambda)(x, y)\,,$$

$$(4.24) \qquad \lambda(x, y) = \lambda x + (1-\lambda)y$$

for any $\lambda \in \mathfrak{R}$, i.e. for any operation $\lambda \in \tilde{A}^{(2)}$. Setting $f(x_1, x_2, x_3) = \lambda(x_2, x_3)$ into formula (4.12) of Lemma 4.3, we get the equation

$$(4.25) \qquad \lambda(x_2, x_3) = s\big(\lambda(x_1, x_3), \lambda(x_2, x_1), x_1\big)\,.$$

Replacing in this equation x_2 and x_3 by x and x_1 by y we obtain the formula

$$x = s\big(\lambda(y, x), \lambda(x, y), y\big)\,.$$

Hence, according to the definition of the unit element and the addition in $\mathfrak{R}$, we get equation (4.23). Further, setting $x_1 = \Theta$ into (4.25) and replacing x_2 by x and x_3 by y, we infer that

$$\lambda(x, y) = s\big(\lambda(\Theta, y), \lambda(x, \Theta), \Theta\big) = s\big(\lambda(x, \Theta), (1-\lambda)(y, \Theta), \Theta\big)$$
$$= \lambda x + (1-\lambda)y\,,$$

which completes the proof of (4.24).

From formulas (4.13) and (4.16) of Lemma 4.4 it follows that

$$(4.26) \qquad \big(1+(-1)\big)(x, y) = s\big(x, s(y, y, x), y\big) = 0(x, y) = y\,.$$

Put

$$(4.27) \qquad g(x_1, x_2, x_3, x_4) = s\big(s(x_1, x_2, x_4), s(x_4, x_4, x_3), x_4\big)\,.$$

By formula (4.10) of Lemma 4.3 we have the equation

$$g(x_1, x_2, x_3, x_4) = s\Big(x_1, s\big(x_2, s(x_4, x_4, x_3), x_4\big), x_4\Big)\,.$$

Hence, by (4.15) and (4.26), we obtain the equations

$$g(x_1, x_2, x_2, x_4) = s\Big(x_1, s\big(x_2, s(x_4, x_4, x_2), x_4\big), x_4\Big) = s(x_1, x_4, x_4)$$
$$= x_1 = s(x_1, x_2, x_2)\,,$$

$$g(x_1, x_2, x_4, x_4) = s\Big(x_1, s\big(x_2, s(x_4, x_4, x_4), x_4\big), x_4\Big)$$
$$= s\big(x_1, s(x_2, x_4, x_4), x_4\big) = s(x_1, x_2, x_4)\,.$$

Thus the equation $g(x_1, x_2, x_3, x_4) = s(x_1, x_2, x_3)$ holds whenever $x_3 = x_2$ or $x_3 = x_4$. Consequently, by Lemma 4.1, it holds for all $x_1, x_2, x_3, x_4 \in A$. In particular, we have the equation

$$s(x, y, z) = g(x, y, z, \Theta)$$

for all $x, y, z \in A$. Hence and from (4.27) we get, in virtue of the definitions (4.16), (4.20) and (4.21), the formula

$$(4.28) \qquad s(x, y, z) = x + y - z\,.$$

Now we shall prove by induction with respect to n that each operation of the form (4.22) is algebraic. For $n = 2$ it is a consequence of formula (4.24). Suppose that for an integer $n \geqslant 2$ this statement is true. Let $\lambda_1, \lambda_2, \ldots, \lambda_{n+1} \in \mathfrak{R}$ and $\sum_{j=1}^{n+1} \lambda_j = 1$. By the inductive assumption the operations

$$(4.29) \qquad f_1(x_1, x_3, \ldots, x_{n+1}) = (\lambda_1+\lambda_2)x_1 + \sum_{j=3}^{n+1} \lambda_j x_j\,,$$

$$(4.30) \qquad f_2(x_1, x_2, x_4, \ldots, x_{n+1}) = (\lambda_1+\lambda_3)x_1 + \lambda_2 x_2 + \sum_{j=4}^{n+1} \lambda_j x_j\,,$$

$$(4.31) \qquad f_3(x_1, x_4, \ldots, x_{n+1}) = (\lambda_1+\lambda_2+\lambda_3)x_1 + \sum_{j=4}^{n+1} \lambda_j x_j$$

are algebraic. Thus the composition

$$f(x_1, x_2, \ldots, x_{n+1})$$
$$= s\big(f_1(x_1, x_3, \ldots, x_{n+1}), f_2(x_1, x_2, x_4, \ldots, x_{n+1}), f_3(x_1, x_4, \ldots, x_{n+1})\big)$$

is also algebraic. By (4.28), (4.29), (4.30) and (4.31) we have the equation

$$f(x_1, x_2, \ldots, x_{n+1}) = \sum_{j=1}^{n+1} \lambda_j x_j .$$

Consequently, all operations of the form (4.22) are algebraic.

Finally, we shall prove by induction with respect to n that each operation from $\tilde{A}^{(n)}$ is of the form (4.22). For $n = 2$ it is a consequence of formula (4.24). Suppose that this assertion is true for $\tilde{A}^{(n-1)}$ $(n \geqslant 3)$. Given $f \in \tilde{A}^{(n)}$, we have, by the inductive assumption, the equations

$$f(x_1, x_1, x_3, x_4, \ldots, x_n) = \sum_{j=1}^{n} a_j x_j ,$$

$$f(x_1, x_2, x_1, x_4, \ldots, x_n) = \sum_{j=1}^{n} \beta_j x_j ,$$

$$f(x_1, x_1, x_1, x_4, \ldots, x_n) = \sum_{j=1}^{n} \gamma_j x_j ,$$

where

$$\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} \beta_j = \sum_{j=1}^{n} \gamma_j = 1 .$$

Hence and from (4.12) and (4.28) it follows that

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \lambda_j x_j ,$$

where $\lambda_j = a_j + \beta_j - \gamma_j$ $(j = 1, 2, \ldots, n)$ and, consequently, $\sum_{j=1}^{n} \lambda_j = 1$. The lemma is thus proved.

LEMMA 4.7. *Let $A^{(3)} \neq A^{(3,1)}$. If a and β belong to the ring $\mathfrak{R}$ of binary operations from $\tilde{A}^{(2)}$, then there exists an element $\gamma \in \mathfrak{R}$ such that $a = \beta\gamma$ or $\beta = a\gamma$.*

Proof. If $\beta = 0$, then $\beta = a \cdot 0$. Suppose that $\beta \neq 0$. By Lemma 4.6 the operations

$$f(x_1, x_2, x_3, x_4, x_5) = ax_1 - ax_2 + \beta x_3 + (1-\beta)x_5$$

and

$$g(x_1, x_2, x_3, x_4, x_5) = \beta x_4 + (1-\beta)x_5$$

belong to $\tilde{A}^{(5)}$. Since $\beta \neq 0$, the equation

(4.32) $$f(x_1, x_2, x_3, x_4, x_5) = g(x_1, x_2, x_3, x_4, x_5)$$

depends on the variables x_3 and x_4. Consequently, there exist an index k $(1 \leqslant k \leqslant 5)$ and an operation $h \in A^{(4)}$ such that equation (4.32) is equivalent to the equation

(4.33) $$x_k = h(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_5)$$

It is obvious that equation (4.32) does not depend on the variable x_5. Consequently, $1 \leqslant k \leqslant 4$. Moreover, the equation $f(x, x, x, x, x) = g(x, x, x, x, x) = x$ implies the equation $h(x, x, x, x) = x$. Thus $h \in \tilde{A}^{(4)}$ and, consequently, by Lemma 4.6

$$h(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_5) = \sum_{\substack{j=1 \\ j \neq k}}^{5} \mu_j x_j .$$

Setting $x_{5-k} = x$, $x_k = \mu_{5-k} x$ and $x_j = \Theta$ for indices j different from k and $5-k$ we get a system of elements satisfying equation (4.33) and, consequently, equation (4.32). Hence it follows that $a\mu_{5-k}x = \beta x$ if $k = 1$ or 2 and $\beta\mu_{5-k}x = ax$ if $k = 3$ or 4. Since x is an arbitrary element of A, we infer, by Lemma 4.5, that one of the elements a, β is left-divisible by the other one, which completes the proof.

LEMMA 4.8. *If $A^{(3)} = A^{(3,1)}$, then $A^{(n)} = A^{(n,1)}$ for all $n \geqslant 3$.*

Proof. We shall prove the lemma by induction with respect to n. By the assumption the equation $A^{(n)} = A^{(n,1)}$ holds if $n = 3$. Suppose that $n \geqslant 4$ and

(4.34) $$A^{(n-1)} = A^{(n-1,1)} .$$

Let $f \in A^{(n)}$. Put $g(x) = f(x, x, \ldots, x)$. For each pair i, j $(i \neq j; i, j = 1, 2, \ldots, n)$ replacing x_i by x_j in $f(x_1, x_2, \ldots, x_n)$ we obtain an $(n-1)$-ary operation. From the inductive assumption (4.34) it follows that there exists an index $r(i, j)$ $(1 \leqslant r(i, j) \leqslant n)$ different from i such that

$$f(x_1, x_2, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n) = g(x_{r(i,j)}) .$$

If g is a constant operation, i.e. $g(x) = c$, where $c \in A^{(0)}$ then, by Lemma 4.1, the equation $f(x_1, x_2, \ldots, x_n) = c$ holds for all $x_1, x_2, \ldots, x_n \in A$ and, consequently, $f \in A^{(n,1)}$. Therefore we may assume that the operation g is not constant.

First consider the case $r(1, 2) = 2$ and $r(1, 3) = 3$. Then we have the equations

$$f(x_2, x_2, x_3, x_4, \ldots, x_n) = g(x_2) ,$$

$$f(x_3, x_2, x_3, x_4, \ldots, x_n) = g(x_3) ,$$

which show that the equation $f(x_1, x_2, \ldots, x_n) = g(x_1)$ holds whenever $x_1 = x_2$ or $x_1 = x_3$. Thus, by Lemma 4.1, it holds everywhere, which implies $f \in A^{(n,1)}$.

Now consider the case $r(1,2) = p \neq 2$. Setting $r(1,3) = q$ we have equations

$$f(x_2, x_2, x_3, x_4, \ldots, x_n) = g(x_p)\,, \tag{4.35}$$

$$f(x_3, x_2, x_3, x_4, \ldots, x_n) = g(x_q)\,. \tag{4.36}$$

Setting $x_2 = x_3$ into equation (4.35) and taking into account the inequality $p \neq 2$, we get the equation

$$f(x_3, x_3, x_3, x_4, \ldots, x_n) = g(x_p)\,. \tag{4.37}$$

Further, setting $x_2 = x_3$ into equation (4.36), we get the equation

$$f(x_3, x_3, x_3, x_4, \ldots, x_n) = \begin{cases} g(x_q) & \text{if} \quad q \neq 2\,, \\ g(x_3) & \text{if} \quad q = 2\,. \end{cases}$$

Hence and from (4.37) it follows that $p = q$ if $q \neq 2$ and $p = 3$ if $q = 2$. Consequently, if $q \neq 2$, then, according to (4.35) and (4.36), the equation $f(x_1, x_2, \ldots, x_n) = g(x_p)$ holds whenever $x_1 = x_2$ or $x_1 = x_3$. Thus, by Lemma 4.1, it holds everywhere, which implies the relation $f \in A^{(n,1)}$. Now we shall prove that the equation $q = 2$ is impossible. Indeed, if $q = 2$ and, consequently, $p = 3$, then, by (4.35) and (4.36), the equations

$$f(x_2, x_2, x_3, x_4, \ldots, x_n) = g(x_3)\,, \tag{4.38}$$

$$f(x_3, x_2, x_3, x_4, \ldots, x_n) = g(x_2) \tag{4.39}$$

hold. Setting $r(4,3) = m$, we have the equation

$$f(x_1, x_2, x_3, x_3, x_5, \ldots, x_n) = g(x_m)\,. \tag{4.40}$$

Setting $x_1 = x_2$ into (4.40) we get the formula

$$f(x_2, x_2, x_3, x_3, x_5, \ldots, x_n) = \begin{cases} g(x_m) & \text{if} \quad m \neq 1\,, \\ g(x_2) & \text{if} \quad m = 1\,. \end{cases}$$

Hence and from (4.38) it follows that $m = 3$. Further, setting $x_1 = x_3$ into (4.40) we obtain the equation

$$f(x_3, x_2, x_3, x_3, x_5, \ldots, x_n) = g(x_3)\,,$$

which contradicts equation (4.39). Consequently, the case $q = 2$ never holds.

Finally the case $r(1,2) = 2$ and $r(1,3) \neq 3$ can be reduced to the previous one by the transposition of the variables x_2 and x_3 in the operation $f(x_1, x_2, \ldots, x_n)$, which completes the proof.

Proof of Theorem 4.1. Suppose that $A^{(3)} \neq A^{(3,1)}$. By virtue of Lemmas 4.4 and 4.7 there exists an associative ring $\mathfrak{R}$ with the unit element, without divisors of zero and such that for each pair $\alpha, \beta \in \mathfrak{R}$ at least one of the elements α, β is left-divisible by the other one. Moreover, by Lemma 4.5, the set A is unital left-module over $\mathfrak{R}$ satisfying the cancellation law. Further, by Lemma 4.6, the operation $h(x_1, x_2, x_3) = x_1 + x_2 - x_3$ is algebraic. Consequently, for any operation $f \in A^{(n)}$ $(n \geqslant 1)$ the operation

$$g(x_1, x_2, \ldots, x_{2n}) = f(x_1, x_2, \ldots, x_n) + x_{2n} - f(x_{n+1}, x_{n+1}, \ldots, x_{2n}) \tag{4.41}$$

is algebraic. Moreover, $g(x, x, \ldots, x) = x$ and, consequently, $g \in \tilde{A}^{(2n)}$. Thus, by Lemma 4.6, there exist elements $\lambda_1, \lambda_2, \ldots, \lambda_{2n} \in \mathfrak{R}$ with $\sum_{j=1}^{2n} \lambda_j = 1$ for which the formula

$$g(x_1, x_2, \ldots, x_{2n}) = \sum_{j=1}^{2n} \lambda_j x_j \tag{4.42}$$

holds. Setting $x_{n+1} = x_{n+2} = \ldots = x_{2n} = \Theta$ into (4.41) and (4.42) we get the equation

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \lambda_j x_j + a\,, \tag{4.43}$$

where $a = f(\Theta, \Theta, \ldots, \Theta)$.

Now consider the case $A^{(0)} \neq \emptyset$ and $A^{(3)} \neq A^{(3,1)}$. Then, by Lemma 4.5, $\Theta \in A^{(0)}$. Put $A_0 = A^{(0)}$. We have proved that each algebraic n-ary operation is of the form (4.43), where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathfrak{R}$ and $a \in A_0$. Since the addition and the scalar-multiplication in A are, by the definition, algebraic operations, we infer that each operation of the form (4.43) with arbitrary coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ from $\mathfrak{R}$ and a from A_0 is algebraic. Moreover, it is obvious that A_0 is a submodule of A. If $\alpha \in \mathfrak{R}$, $\alpha \neq 0$, $a \in A$ and $\alpha a \in A_0$, i.e. αa is an algebraic constant, then the equation $\alpha x = \alpha a$ depends on the variable x and, consequently, has a unique solution $x = a$ belonging to $A^{(0)}$. Thus $a \in A_0$ and, consequently, the submodule A_0 is divisible, which completes the proof of assertion (i).

Now suppose that $A^{(0)} = \emptyset$ and $A^{(3)} \neq A^{(3,1)}$. We have previously proved that each algebraic operation is of the form (4.43). Let B be the subset of the product $\mathfrak{R} \times A$ consisting of all pairs $\langle \sum_{j=1}^{n} \lambda_j, a \rangle$, where the elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathfrak{R}$ and the element a of A appear in the representation formula (4.43) for algebraic operations. We note that for each $\langle \lambda, a \rangle \in B$ the unary operation $\lambda x + a$ is algebraic.

We shall prove that the set B is admissible. Since the trivial operation $f(x) = x$ is always algebraic, we infer that $\langle 1, 0 \rangle \in B$. Further, let $\langle \lambda, a \rangle \in B$. If λ is left-divisible by $1 - \lambda$, i.e. $\lambda = (1-\lambda)\alpha$, where $\alpha \in \mathfrak{R}$, then $(1-\lambda)(\alpha+1) = 1$ and, consequently, the element $1 - \lambda$ has the right inverse in $\mathcal{R}$. Since the ring $\mathcal{R}$ has no divisors of zero, the element $1-\lambda$ is invertible in $\mathcal{R}$. The operations $f(x) = x$ and $g(x) = \lambda x + a$ are algebraic. Moreover, $f\big((1-\lambda)^{-1}a\big) = g\big((1-\lambda)^{-1}a\big)$ and $f \neq g$. Consequently, the equation $f(x) = g(x)$ has a unique solution belonging to $A^{(0)}$, which contradicts the assumption $A^{(0)} = \emptyset$. Thus λ is not left-divisible by $1-\lambda$ and, consequently, by the divisibility properties of $\mathcal{R}$, $1-\lambda$ is left-divisible by λ, i.e. $1-\lambda = \lambda\beta$, where $\beta \in \mathcal{R}$. Hence we get the equation $\lambda(1+\beta) = 1$, which shows that λ has a right inverse in $\mathcal{R}$ and, consequently, is invertible in $\mathcal{R}$ because the ring $\mathcal{R}$ has no divisors of zero. Condition (ii) for admissible sets is thus proved.

Let $\mu_1, \mu_2, \ldots, \mu_n$ be a system of elements of $\mathcal{R}$ satisfying the condition $\sum_{j=1}^{n} \mu_j = 1$. Let $\langle \lambda_j, a_j \rangle \in B$ $(j = 1, 2, \ldots, n)$. By Lemma 4.6 the operation $g(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \mu_j x_j$ is algebraic. Moreover, the unary operations $f_j(x) = \lambda_j x + a_j$ $(j = 1, 2, \ldots, n)$ are algebraic. Thus the composition

$$g\big(f_1(x), f_2(x), \ldots, f_n(x)\big) = \sum_{j=1}^{n} \mu_j \lambda_j x + \sum_{j=1}^{n} \mu_j a_j$$

is algebraic and, consequently, $\langle \sum_{j=1}^{n} \mu_j \lambda_j, \sum_{j=1}^{n} \mu_j a_j \rangle \in B$, which completes the proof of condition (iii) for admissible sets.

Finally suppose that $a, \lambda \in \mathcal{R}$, $a \neq 0$, $a \in A$ and $\langle 1 + a\lambda - a, aa \rangle \in B$. Then the operation $f(x) = (1 + a\lambda - a)x + aa$ is algebraic. Further, by Lemma 4.6, the operation $g(x, y) = (1-a)x + ay$ is also algebraic. Since the equation $f(x) = g(x, y)$ depends on the variable y, we infer that it is equivalent to one of the equations $y = \lambda x + a$, $x = \lambda^{-1}y - \lambda^{-1}a$ with algebraic right-hand side. In the first case we have the relation $\langle \lambda, a \rangle \in B$ and in the second case $\langle \lambda^{-1}, \lambda^{-1}a \rangle \in B$. Setting in the last case $\mu_1 = -\lambda$, $\mu_2 = 1 + \lambda$, $\langle \lambda_1, a_1 \rangle = \langle \lambda^{-1}, -\lambda^{-1}a \rangle$ and $\langle \lambda_2, a_2 \rangle = \langle 1, 0 \rangle$ we have, by the previously proved condition (iii) for admissible sets, the relation $\langle \mu_1\lambda_1 + \mu_2\lambda_2, \mu_1 a_1 + \mu_2 a_2 \rangle \in B$ and, consequently, $\langle \lambda, a \rangle \in B$, which implies condition (iv) for admissible sets. Thus the set B is admissible.

To prove assertion (ii) of the theorem it suffices to prove that for each pair $\langle \lambda, a \rangle \in B$ and each system $\lambda_1, \lambda_2, \ldots, \lambda_n$ of elements of $\mathcal{R}$ with $\lambda_1 + \lambda_2 + \ldots + \lambda_n = \lambda$ the operation $\sum_{j=1}^{n} \lambda_j x_j + a$ is algebraic. Since the element λ is invertible in $\mathcal{R}$, the operation

$$h(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \lambda^{-1} \lambda_j x_j$$

is, by Lemma 4.6, algebraic. Moreover, the operation $g(x) = \lambda x + a$ is algebraic. Thus the composition

$$g\big(h(x_1, x_2, \ldots, x_n)\big) = \sum_{j=1}^{n} \lambda_j x_j + a$$

is algebraic, which completes the proof of assertion (ii).

Suppose now that $A^{(3)} = A^{(3,1)}$. Then, by Lemma 4.8, the class of algebraic operations is the class of all operations f defined as

$$f(x_1, x_2, \ldots, x_n) = h(x_j) \quad (1 \leqslant j \leqslant n),$$

where $h \in A^{(1)}$.

First let us assume that all operations from $A^{(1)}$ are constant. Since $A^{(1)}$ contains the trivial operation, A is a one-point set $A = \{a_0\}$ and, consequently, $f(x_1, x_2, \ldots, x_n) = a_0$ for every operation f. Let $\mathcal{S}$ be the group containing the identical transformation only and $A_0 = \{a_0\}$. Then assertion (iii) of the theorem is obvious.

Now suppose that the set $\mathcal{S}$ of non-constant unary algebraic operations is non-void. For any operation g from $\mathcal{S}$ the equation

$$g(x) = g(y) \tag{4.44}$$

depends on both x and y. Consequently, there exists an operation $h \in A^{(1)}$ such that (4.44) is equivalent to one of the equations $x = h(y)$ and $y = h(x)$. Since (4.44) holds whenever $x = y$, we infer that $h(x) = x$ for all $x \in A$, i.e. (4.44) holds if and only if $x = y$. Thus each operation from $\mathcal{S}$ is one-to-one. Hence it follows that the set $\mathcal{S}$ is a semigroup under the composition $(g_1 g_2)(x) = g_1\big(g_2(x)\big)$. Of course, the identical operation is the unit element of $\mathcal{S}$. Let $g_1, g_2 \in \mathcal{S}$ and $g_1(A) \cap g_2(A) \neq \emptyset$. Then the equation

$$g_1(x) = g_2(y) \tag{4.45}$$

depends on both x and y. Thus, there exists an operation $g \in A^{(1)}$ such that (4.45) is equivalent to one of the equations $x = g(y)$ and $y = g(x)$. Consequently, one of the equations $g_2(y) = g_1\big(g(y)\big)$, $g_1(x) = g_2\big(g(x)\big)$ holds. Hence it follows that $g \in \mathcal{S}$ and at least one element of the pair g_1, g_2 is left-divisible by the other one.

Let A_0 be the set of all algebraic constants. Obviously, $g(A_0) \subset A_0$ for all $g \in \mathcal{S}$. Moreover, if the equation $g(x) = x$ depends on the variable x, then it has a unique solution belonging to A_0. Hence it follows that each transformation that is not the identical transformation has at most one fixed point in A and the set A_0 contains all fixed points. Further, if $c \in A_0$, $g \in \mathcal{S}$ and the equation $g(x) = c$ depends on the variable x, then it has a unique solution belonging to A_0. Consequently, $g^{-1}(A_0) \subset A_0$ for all $g \in \mathcal{S}$. Assertion (iii) of the theorem is thus proved.

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Reçu par la Rédaction le 7. 12. 1964