A compactness result concerning direct products of models

by

M. Makkai (Budapest)

Introduction. The results of the present paper are motivated by the following theorem of R. Vaught (see [10], Theorem 2b)

THEOREM OF VAUGHT. *If* $\mathbf{K}$ *is a class of similar relational systems and* $\mathbf{K} \in \mathsf{PC}_\Delta$, *then* $\mathsf{SP}(\mathbf{K}) \in \mathsf{UC}_\Delta$ (¹).

The following theorem was proved in [6] and [9].

THEOREM OF ŁOŚ AND TARSKI. *If* $\mathbf{K} \in \mathsf{PC}_\Delta$, *then* $\mathsf{S}(\mathbf{K}) \in \mathsf{UC}_\Delta$.

This theorem can be derived from the Compactness Theorem of the first order predicate calculus by the standard method of diagrams (²). Our main result (Theorem 1 in § 2) is a "compactness" result concerning classes $\mathsf{P}(\mathbf{K})$ from which the theorem of Vaught can be derived in a quite similar way by the method of diagrams. To formulate our theorem, let us call a class $\mathbf{K}$ of similar relational systems *compact* if the following holds: for any set Σ of sentences appropriate for the systems of $\mathbf{K}$, if every finite subset of Σ is satisfiable by a system of $\mathbf{K}$, then Σ is itself satisfiable by a system of $\mathbf{K}$. Then our assertion is that if $\mathbf{K}$ is compact, then the class $\mathsf{P}(\mathbf{K})$ of all direct products of systems in $\mathbf{K}$ is also compact.

Let $\mathsf{Dp}(\mathbf{K})$ denote the class of all direct powers of systems of $\mathbf{K}$ We infer that $\mathbf{K} \in \mathsf{PC}_\Delta$ implies $\mathsf{SDp}(\mathbf{K}) \in \mathsf{UC}_\Delta$ (Corollary 3). This will be derived by the method of diagrams from a more general compactness result (Theorem 2). Another special case of Theorem 2 that if $\mathbf{K}$ is compact, then $\mathsf{Dp}(\mathbf{K})$ is also compact.

We give some remarks concerning direct products and cardinal sums related to these results, and finally we state a compactness result concerning countably weak direct products.

Our main tools in the proofs are a criterion of [4] (stated in § 1 of the present paper) for a sentence holding in a given direct product, and the Compactness Theorem.

§ 1. Preliminaries. We sum up the notions and notations to be used in the paper. For more details, see [9] and [7].

(¹) For definition of the notions, see § 1.
(²) See e.g. [6], § 4.

A *language* is a set of which certain elements are specified as *predicate symbols* and others as *function symbols*. We denote predicate symbols by P, Q, R, S; function symbols by f, languages by $\mathbf{L}$, possibly with several indices. $P \in \mathbf{L}$ and $f \in \mathbf{L}$ will always mean that P is a predicate symbol, f is a function symbol of the language $\mathbf{L}$. With each P or f ($P, f \in \mathbf{L}$) there is associated a natural number $\nu(P)$ and $\nu(f) \geqslant 0$ and P, f are called a $\nu(P)$-ary predicate symbol and a $\nu(f)$-ary function symbol, respectively. (3)

$\mathfrak{A}$ is a relational system, or briefly a system of $\mathbf{L}$ (notation: $\mathfrak{A} \in \mathfrak{S}(L)$) if $\mathfrak{A}$ is an ordered pair of a non-empty set A (denoted by $|\mathfrak{A}|$) and a function with domain $\mathbf{L}$ such that, the value of this function for arguments P and f ($P, f \in \mathbf{L}$) being denoted by $P_{\mathfrak{A}}$ and $f_{\mathfrak{A}}$ respectively, $P_{\mathfrak{A}}$ is a $\nu(P)$-ary relation on A and $f_{\mathfrak{A}}$ is a $\nu(f)$-ary operation on A. (4)

If $\mathbf{L}, \mathbf{L}'$ are languages, $\mathbf{L}' \supset \mathbf{L}$ and $\mathfrak{A} \in \mathfrak{S}(\mathbf{L}')$ then $\mathfrak{A}|\mathbf{L}$ denotes the $\mathbf{L}$-reduct of $\mathfrak{A}$, i.e. the system $\mathfrak{B}$ of $\mathbf{L}$ such that $|\mathfrak{B}| = |\mathfrak{A}|$ and, for any P, $f \in \mathbf{L}$, $P_{\mathfrak{B}} = P_{\mathfrak{A}}$ and $f_{\mathfrak{B}} = f_{\mathfrak{A}}$. If $\mathbf{K} \subset \mathfrak{S}(\mathbf{L}')$, then $\mathbf{K}|\mathbf{L} = \{\mathfrak{A}|\mathbf{L}: \mathfrak{A} \in \mathbf{K}\}$. Let $\mathbf{K} \subset \mathfrak{S}(\mathbf{L})$. Then $\mathbf{K}[\mathbf{L}']$ will denote the class of all systems $\mathfrak{B}$, $\mathfrak{B} \in \mathfrak{S}(\mathbf{L}')$ such that $\mathfrak{B}|\mathbf{L} \in \mathbf{K}$.

We suppose that the notions of the first order predicate calculus are known. We use the following notations for the logical operations: $\neg$ (negation), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies), $\longleftrightarrow$ (equivalent), (x) (for all x...), $(\exists x)$ (there exists an x such that ...). $\bigwedge_{i...} F_i$ and $\bigvee_{i...} F_i$ stand for the conjunction and the disjunction, respectively, of the formulas F_i such that i satisfies the condition "i...". The (first order) formulas of $\mathbf{L}$ are built up in the well-known way from the symbols of $\mathbf{L}$, the identity symbol $=$, of the fixed infinite sequence $v_0, v_1, \ldots$ of (individual) variables and the logical operations. The set of all formulas of $\mathbf{L}$ and the set of all sentences (formulas having no free variable) of $\mathbf{L}$ are denoted by $\mathfrak{F}(\mathbf{L})$ and $\mathfrak{F}_0(\mathbf{L})$, respectively. If the sentence F has the form $(v_0)\ldots(v_n)\Phi$, Φ containing no quantifiers, then F is called a *universal sentence of* $\mathbf{L}$. If $F \in \mathfrak{F}(\mathbf{L})$ and F contains no free variable except the distinct variables $x_1, \ldots, x_n$; $\mathfrak{A} \in \mathfrak{S}(\mathbf{L})$ and $a_1, \ldots, a_n \in |\mathfrak{A}|$, then

$$\mathfrak{A} \left|\frac{x_1, \ldots, x_n}{a_1, \ldots, a_n}\right. F$$

will mean that $a_1, \ldots, a_n$ satisfy F in $\mathfrak{A}$ under the correspondence $x_1 \to a_1, \ldots$ $\ldots, x_n \to a_n$. In particular, if $F \in \mathfrak{F}_0(\mathbf{L})$, $\mathfrak{A} \vdash F$ means that F holds (is true) in $\mathfrak{A}$. If $F \in \mathfrak{F}_0(\mathbf{L})$ then $\mathsf{M}_{\mathbf{L}}(F)$ denotes the class of all systems $\mathfrak{A}$ such that $\mathfrak{A} \in \mathfrak{S}(\mathbf{L})$ and $\mathfrak{A} \vdash F$. If $\Sigma \subset \mathfrak{F}_0(\mathbf{L})$, then $\mathsf{M}_{\mathbf{L}}(\Sigma) = \bigcup_{F \in \Sigma} \mathsf{M}_{\mathbf{L}}(F)$. The systems $\mathfrak{A}$ and $\mathfrak{B}$ of the same language $\mathbf{L}$ are said to be *elementarily equivalent* (notation: $\mathfrak{A} \equiv \mathfrak{B}$) if for any F, $F \in \mathfrak{F}_0(\mathbf{L})$, $\mathfrak{A} \vdash F$ if and only if $\mathfrak{B} \vdash F$. An (*elementary*) *equivalence type* is a class of the form $\{\mathfrak{B}: \mathfrak{B} \equiv \mathfrak{A}\}$ for some fixed system $\mathfrak{A}$.

Let EC_Δ be the family of all classes $\mathbf{K}$ of systems such that $\mathbf{K} = \mathsf{M}_{\mathbf{L}}(\Sigma)$ for some $\mathbf{L}$ and Σ, $\Sigma \subset \mathfrak{F}_0(\mathbf{L})$. If in addition Σ contains only universal sentences, then $\mathbf{K} \in \mathsf{UC}_\Delta$. PC_Δ denotes the family of the classes $\mathbf{K}$ such that $\mathbf{K} = \mathbf{K}'|\mathbf{L}$ for some $\mathbf{K}', \mathbf{L}, \mathbf{L}'$ with $\mathbf{L}' \supset \mathbf{L}$, $\mathbf{K}' \subset \mathfrak{S}(\mathbf{L}')$ and $\mathbf{K}' \in \mathsf{EC}_\Delta$.

We shall repeatedly use the following basic theorem of the theory of models (see e.g. [8]).

COMPACTNESS THEOREM (CTh). *If $\Sigma \subset \mathfrak{F}_0(\mathbf{L})$ for a language $\mathbf{L}$ and for any finite set Σ' of Σ we have $\mathsf{M}_{\mathbf{L}}(\Sigma') \neq 0$, then $\mathsf{M}_{\mathbf{L}}(\Sigma) \neq 0$.*

We shall apply the following natural terminology. A class $\mathbf{K}$ of systems of $\mathbf{L}$ is called *compact* if for any set Σ, $\Sigma \subset \mathfrak{F}_0(\mathbf{L})$, the following hold: if for every finite subset Σ' of Σ we have $\mathsf{M}_{\mathbf{L}}(\Sigma') \cap \mathbf{K} \neq 0$, then $\mathsf{M}_{\mathbf{L}}(\Sigma) \cap \mathbf{K} \neq 0$. CTh implies evidently that a class of EC_Δ or PC_Δ is compact. Let us denote by $\overline{\mathbf{K}}$ the class of all systems elementarily equivalent to some systems of $\mathbf{K}$. Then it is easily seen that $\mathbf{K}$ is compact if and only if $\overline{\mathbf{K}} \in \mathsf{EC}_\Delta$.

Let $\mathsf{S}_{\mathbf{L}}$ denote the set of all equivalence types of $\mathbf{L}$. As is well known (cf. [1], pp. 523 ff), $\mathsf{S}_{\mathbf{L}}$ constitutes a totally disconnected compact topological space if the basis of $\mathsf{S}_{\mathbf{L}}$ is defined as the set of all sets

$$\langle G \rangle = \{\pi\colon \pi \in \mathsf{S}_{\mathbf{L}},\ \pi \subset \mathsf{M}_{\mathbf{L}}(G)\}$$

for any $G \in \mathfrak{F}_0(\mathbf{L})$. CTh is equivalent with the compactness of $\mathsf{S}_{\mathbf{L}}$.

If

$$(1) \qquad \langle \mathfrak{A}^{(i)}: i \in I \rangle$$

is a non-empty indexed family of systems $\mathfrak{A}^{(i)}$ of $\mathbf{L}$ with the index set I, then the direct product of the systems $\mathfrak{A}^{(i)}$ is the system $\mathfrak{B}$ of $\mathbf{L}$ defined by the following stipulations: $|\mathfrak{B}|$ is the cartesian product (5) $\prod_{i \in I} |\mathfrak{A}^{(i)}|$ of the sets $|\mathfrak{A}^{(i)}|$; if $P \in \mathbf{L}$, $f \in \mathbf{L}$ $\varphi_1, \ldots, \varphi_n \in |\mathfrak{B}|$ and $\nu(P) = n$, $\nu(f) = n$ then $P_{\mathfrak{B}}(\varphi_1, \ldots, \varphi_n)$ is equivalent to the statement that for every i, $i \in I$, $P_{\mathfrak{A}^{(i)}}(\varphi_1(i), \ldots, \varphi_n(i))$ holds and $f_{\mathfrak{B}}(\varphi_1, \ldots, \varphi_n)$ is the function $\varphi \in |\mathfrak{B}|$ such that for any i, $i \in I$, $\varphi(i) = f_{\mathfrak{A}^{(i)}}(\varphi_n(i), \ldots, \varphi_n(i))$ (cf. [4]). If, for each i, $\mathfrak{A}^{(i)}$ is the same system $\mathfrak{A}$, then $\prod_{i \in I} \mathfrak{A}^{(i)}$ is called a *direct power* of $\mathfrak{A}$ and denoted by $\mathfrak{A}^I$.

Let us denote (1) briefly by $\mathfrak{A}$, and let $\Theta \in \mathfrak{F}_0(\mathbf{L})$. We write $\mathbf{K}_\Theta^{\mathfrak{A}}$ for the set $\{i\colon i \in I, \mathfrak{A}^{(i)} \vdash \Theta\}$. The following criterion of [4] (pp. 83-84, espe-

(3) More precisely, a language should be considered an ordered pair of two functions, one of them being the function ν restricted to the set of predicate symbols, etc. But our simplified conventions will not cause any confusion.

(4) An n-ary relation on A is a function, defined on the set A^n of the ordered n-tuples of elements of A, with possible values *true* (or 1) and *false* (or 0). An n-ary operation on A is a function defined on A^n with possible values in A. A singular relation R is identified in the usual way with a set, i.e. we shall write $x \in R$ equivalently with the statement that $R(x)$ holds.

(5) I.e. the set of all functions φ such that the domain of φ is I and $\varphi(i) \in |\mathfrak{A}^{(i)}|$ for every $i \in I$.

cially (4) on p. 84) is the basis of our results. It gives a necessary and sufficient condition for a sentence F to hold in a given direct product; roughly speaking, in terms of stipulations on the number of those factors of the product which satisfy certain sentences, the latter depending only of F (see Θ_j^F below). The use of the superscripts F in the formulation seems perhaps unnecessarily complicating; however, it will be useful later when we apply (FV) simultaneously for several F's.

(FV) *With every sentence* $F \in \mathfrak{F}_0(\mathrm{L})$ *we can associate the natural numbers* m^F, M^F; *the sequence*

$$\langle q_j^{(0),F}\colon j < m^F\rangle, \langle q_j^{(1),F}\colon j < m^F\rangle, \ldots, \langle q_j^{(M^F-1),F}\colon j < m^F\rangle$$

of M^F *sequences of natural numbers; subsets* $s_0^F, \ldots, s_{M^F-1}^F$ *of* m^F (6); *the sequence* $\Theta_0^F, \ldots, \Theta_{m^F-1}^F$ *of* m^F *sentences of* $\mathbf{L}$ *such that for any non-empty indexed family* (1)

(2) $$\prod_{i\in I} \mathfrak{A}^{(i)} \vdash F$$

if and only if there exists a k, $k < M^F$, *such that for any* j, $j < m^F$, $\mathbf{K}_{\Theta_j^F}^{\mathfrak{A}}$ *has exactly* $q_j^{(k),F}$ *elements if* $j \in s_k^F$, *and at least* $q_j^{(k),F}$ *elements if* $j \notin s_k^F$.

The formulas Θ_j^F may be chosen so that for any different j_1, j_2 ($j_1, j_2 < m^F$) we have $\mathsf{M}_{\mathbf{L}}(\Theta_{j_1}^F) \cap \mathsf{M}_{\mathbf{L}}(\Theta_{j_2}^F) = 0$; but we shall not use this because it does not simplify the proof. (FV) was stated in [4] only for the case of $\mathbf{L}$ containing only predicate symbols, but we can strengthen the original form to the general case by a trivial argument, using the "representing predicates" of the functions.

For a class $\mathbf{K}$, $\mathbf{K} \subset \mathfrak{S}(\mathbf{L})$, we denote by $\mathsf{P}(\mathbf{K})$ the class of all direct products of non-empty indexed families of systems in $\mathbf{K}$. (7) $\mathsf{Dp}(\mathbf{K})$ denotes the class of all (non-empty) direct powers of systems of $\mathbf{K}$.

§ 2. THEOREM 1. *For any language* $\mathbf{L}$, *and class* $\mathbf{K}$, $\mathbf{K} \in \mathfrak{S}(\mathbf{L})$, *if* $\mathbf{K}$ *is compact then* $\mathsf{P}(\mathbf{K})$ *is also compact.*

Proof. We associate a unary predicate symbol R^{Θ} with every sentence Θ of $\mathbf{L}$, for different Θ's the R^{Θ}'s being different. To explain in broad terms our purpose in doing so, let I be an "index" set and let $\mathfrak{A}_i$ be a system of $\mathbf{L}$ for any $i \in I$. The intended interpretation of R^{Θ} will be the set of indices i in I such that $\mathfrak{A}_i$ satisfies Θ. Using the symbols R^{Θ}

(6) An ordinal number (in particular a natural number) is considered as the set of all smaller ordinal numbers.

(7) See [9]. In [10] Vaught uses this notation in a slightly different sense: he allows the empty direct product (thus including in every class $\mathsf{P}(\mathbf{K})$ the one-element system $\mathfrak{A}_0$ in which each $P_{\mathfrak{A}_0}$ is identically true for each $P \in \mathbf{L}$) and he includes in his $\mathsf{P}(\mathbf{K})$ all systems isomorphic to some system of $\mathsf{P}(\mathbf{K})$ in our sense. Our results hold for either definition of $\mathsf{P}(\mathbf{K})$.

with this interpretation, we can "express" (for $F \in \mathfrak{F}_0(\mathbf{L})$) by a sentence Φ^F a part of the fact that the condition of (FV) holds.

More precisely, we do the following. Let $\mathbf{L}_0$ be the language consisting of all R^{Θ} for $\Theta \in \mathfrak{F}_0(\mathbf{L})$. Let $R \in \mathbf{L}_0$, n be a natural number. We denote by $(\exists^n x)R(x)$, $(\exists!^n x)R(x)$, $(n \geqslant 1)$, and $(\exists^0 x)R(x)$, $(\exists!^0 x)R(x)$ the following formulas, respectively:

$$(\exists v_0)\ldots(\exists v_{n-1})\Big(\bigwedge_{\substack{i\neq k\\ i,k<n}} v_i \neq v_k \wedge \bigwedge_{i<n} R(v_i)\Big),$$

$$(\exists v_0)\ldots(\exists v_{n-1})(v_n)\Big(\bigwedge_{\substack{i\neq k\\ i,k<n}} v_i \neq v_k \vee \bigwedge_{i<n} R(v_i) \vee \big(R(v_n) \to \bigvee_{i<n} v_n = v_i\big)\Big),$$

$$(v_0)(v_0 = v_0),$$

$$(v_0)\big(\neg R(v_0)\big).$$

Then, if $R \in \mathbf{L}'$ and $\mathfrak{B} \in \mathfrak{S}(\mathbf{L}')$ for some $\mathbf{L}'$, then $\mathfrak{B} \vdash (\exists^n x)R(x)$ and $\mathfrak{B} \vdash (\exists!^n x)R(x)$ $(n \geqslant 0)$ are equivalent, respectively, to the statement that there exist at least or exactly n different elements x in $|\mathfrak{B}|$ such that $\mathfrak{B} \models_{x}^{v_0} R(v_0)$.

In all that follows we write R_j^F instead of $R^{\Theta_j^F}$. Now we define Φ^F for any $F \in \mathfrak{F}_0(\mathbf{L})$ as the following sentence of $\mathbf{L}_0$:

$$\bigvee_{k<M^F} \Big(\bigwedge_{i\in s_k^F} (\exists!^{q_j^{(k),F}} x) R_j^F(x) \wedge \bigwedge_{j\in M^F/s_k^F} (\exists^{q_j^{(k),F}} x) R_j^F(x)\Big)$$

(for the notations, see (FV) in § 1).

Now we can formulate (FV) equivalently as follows. (2) holds if and only if there exists a system $\mathfrak{B}$, $\mathfrak{B} \in \mathfrak{S}(\mathbf{L}_0)$, such that

(*) $$|\mathfrak{B}| = I, \quad (R_j^F)_{\mathfrak{B}} = K_{\Theta_j^F}^{\mathfrak{A}} \text{ for } j < m^F, \quad \text{and} \quad \mathfrak{B} \vdash \Phi^F.$$

Let us assume that $\mathbf{K}$ is compact, $\mathbf{K} \subset \mathfrak{S}(\mathbf{L})$ and Σ is a set of sentences of $\mathbf{L}$ such that $\mathsf{M}_{\mathbf{L}}(\Sigma') \cap \mathsf{P}(\mathbf{K}) \neq 0$ for any $\Sigma' \in S_\omega(\Sigma)$. (8) Then for any $\Sigma' \in S_\omega(\Sigma)$ there exist an indexed family $\mathfrak{A}_{\Sigma'} = \langle \mathfrak{A}_{\Sigma'}^{(i)}\colon i \in I_{\Sigma'}\rangle$ of systems in $\mathbf{K}$ and a system $\mathfrak{B}_{\Sigma'}$ in $\mathfrak{S}(\mathbf{L}_0)$ such that

(**) (*) *holds for each* $F \in \Sigma'$, *with* $\mathfrak{B}_{\Sigma'}$ *and* $\mathfrak{A}_{\Sigma'}$ *instead of* $\mathfrak{B}$ *and* $\mathfrak{A}$, *respectively.*

We define $X_{\Sigma'}$ as the set of sentences Ψ satisfying the following two conditions:

(8) $S_\omega(X)$ denotes the set of all (non-empty) finite subsets of the set X.

(i) Ψ has the form

$$\neg(\exists v_0)\bigwedge_{\substack{F\in\Sigma'\\ j<m^F}}\varepsilon(F,j)R_j^F(v_0) \tag{3}$$

where $\varepsilon(F,j)$ is 1 or -1. [9]

(ii) For every Σ'' such that $\Sigma''\in S_\omega(\Sigma)$ and $\Sigma''\supseteq\Sigma'$ we have $\mathfrak{B}_{\Sigma''}\vdash\Psi$.

We define the set Γ of sentences of $\mathbf{L}_0$ as follows:

$$\Gamma=\{\Phi^F\colon F\in\Sigma\}\cup\bigcup_{\Sigma'\in S_\omega(\Sigma)}X_{\Sigma'}. \tag{4}$$

We assert that Γ is consistent, i.e. every finite subset of Γ is satisfiable. Indeed, every finite subset of Γ is contained in a set Γ'' obtained as follows: Let $\Sigma''\in S_\omega(\Sigma)$ and $\Gamma''=\{\Phi^F\colon F\in\Sigma''\}\cup\bigcup_{\Sigma'\subset\Sigma''}X_{\Sigma'}$. We can easily see by the definition of $X_{\Sigma'}$ and (**) that $\mathfrak{B}_{\Sigma''}\in\mathsf{M}_{\mathbf{L}_0}(\Gamma'')$, which proves our assertion.

By CTh we have a system $\mathfrak{B}$ such that

$$\mathfrak{B}\in\mathsf{M}_{\mathbf{L}_0}(\Gamma). \tag{5}$$

Let $|\mathfrak{B}|=I$, $i\in I$. We define the set Λ_i of sentences of $\mathbf{L}$ by

$$\Lambda_i=\{\Theta_j^F\colon F\in\Sigma,\ j<m^F,\ i\in(R_j^F)_{\mathfrak{B}}\}\cup$$
$$\cup\{\neg\Theta_j^F\colon F\in\Sigma,\ j<m^F,\ i\notin(R_j^F)_{\mathfrak{B}}\}.$$

We assert that there exists a system $\mathfrak{A}^{(i)}$ such that

$$\mathfrak{A}^{(i)}\in\mathsf{M}_{\mathbf{L}}(\Lambda_i)\cap\mathbf{K}. \tag{6}$$

To prove this it suffices, by the compactness of $\mathbf{K}$, to show that for any finite subset Λ of Λ_i we have a system $\mathfrak{A}$ with $\mathfrak{A}\in\mathsf{M}_{\mathbf{L}}(\Lambda)\cup\mathbf{K}$.

It is sufficient to consider sets Λ obtained as follows. Let $\Sigma'\in S_\omega(\Sigma)$ and let Λ be defined as Λ_i was, with Σ' instead of Σ. We define the function ε such that $\varepsilon(F,j)=1$ if $i\in(R_j^F)_{\mathfrak{B}}$ and $\varepsilon(F,j)=-1$ if $i\notin(R_j^F)_{\mathfrak{B}}$. Let $G=\bigwedge_{\substack{F\in\Sigma'\\ j<m^F}}\varepsilon(F,j)R_j^F(v_0)$. Thus obviously $\mathfrak{B}\models G\left[\frac{v_0}{i}\right]$. Hence, if we take $\Psi=\neg(\exists v_0)G$, then $\mathfrak{B}\vdash\neg\Psi$; consequently by (4) and (5) $\Psi\notin X_{\Sigma'}$.

By the definition of $X_{\Sigma'}$, this means that there exists a Σ'', $\Sigma''\in S_\omega(\Sigma)$, with $\Sigma'\subset\Sigma''$ and such that $\mathfrak{B}_{\Sigma''}\vdash\neg\Psi$. By the definition of Ψ and (**) this is equivalent to the existence of an i', $i'\in I_{\Sigma''}$, such that

$$i'\in(R_j^F)_{\mathfrak{B}_{\Sigma''}}=K_{\Theta_j^F}^{\mathfrak{A}_{\Sigma''}}\quad\text{if}\quad\varepsilon(F,j)=1$$

and

$$i'\notin(R_j^F)_{\mathfrak{B}_{\Sigma''}}=K_{\Theta_j^F}^{\mathfrak{A}_{\Sigma''}}\quad\text{if}\quad\varepsilon(F,j)=-1.$$

But by the definition of $K_\Theta^{\mathfrak{A}}$ this implies that $\mathfrak{A}_{\Sigma''}^{(i')}\in\mathsf{M}_{\mathbf{L}}(\Lambda)\cap\mathbf{K}$, which was to be shown.

Thus we have an indexed family $\mathfrak{A}=\langle\mathfrak{A}^{(i)}\colon i\in I\rangle$ with (6). By the definition of Λ_i and (6) we have obviously $K_{\Theta_j^F}^{\mathfrak{A}}=(R_j^F)_{\mathfrak{B}}$ for any $F\in\Sigma$, $j<m^F$ and by (4) and (5) $\mathfrak{B}\vdash\Phi^F$ for every $F\in\Sigma$. Consequently by the assertion given before (*) and (6)

$$\prod_{i\in I}\mathfrak{A}^{(i)}\in\mathsf{M}_{\mathbf{L}}(\Sigma)\cap\mathsf{P}(\mathbf{K}),\qquad\text{i.e.}\qquad\mathsf{M}_{\mathbf{L}}(\Sigma)\cap\mathsf{P}(\mathbf{K})\neq 0\qquad\text{q.e.d. [10].}$$

COROLLARY 1. (Theorem of Vaught). *If* $\mathbf{K}\in\mathsf{PC}_\Delta$ *then* $\mathsf{SP}(\mathbf{K})\in\mathsf{UC}_\Delta$.

This can be derived from Theorem 1 by the standard method of diagrams or descriptions of models (see [6], § 4). For the sake of completeness we give this proof here. Let $\mathbf{K}\subset\mathfrak{S}(\mathbf{L})$ and let Σ be the set of all universal sentences of $\mathbf{L}$ holding in every system of $\mathsf{P}(\mathbf{K})$. It suffices to show that for an arbitrary system $\mathfrak{A}$ such that $\mathfrak{A}\in\mathsf{M}_{\mathbf{L}}(\Sigma)$ we have $\mathfrak{A}\in\mathsf{SP}(\mathbf{K})$. Let $|\mathfrak{A}|=A$ and let us associate a new (individual) constant c_a with every element a of A such that $c_a\notin\mathbf{L}$ and c_{a_1}, c_{a_2} are different for different a_1, a_2. By adjoining c_a for every $a\in A$ to $\mathbf{L}$ we obtain the language $\mathbf{L}_A$. Now we define the diagram of $\mathfrak{A}$ as the set $\Delta_{\mathfrak{A}}$ of the following sentences of $\mathbf{L}_A$.

$c_{a_1}\neq c_{a_2}$ for any $a_1,a_2\in A$ such that $a_1\neq a_2$;

$P(c_{a_1},\ldots,c_{a_{\nu(P)}})$ for any $a_1,\ldots,a_{\nu(P)}\in A$ such that $P_{\mathfrak{A}}(a_1,\ldots,a_{\nu(P)})$ holds;

$\neg P(c_{a_1},\ldots,c_{a_{\nu(P)}})$ for any $a_1,\ldots,a_{\nu(P)}\in A$ such that $P_{\mathfrak{A}}(a_1,\ldots,a_{a_{\nu(P)}})$ does not hold;

$f(c_{a_1},\ldots,c_{a_{\nu(f)}})=c_{a_{\nu(f)+1}}$ for any $a_1,\ldots,a_{\nu(f)},a_{\nu(f)+1}$ such that $f_{\mathfrak{A}}(a_1,\ldots,a_{\nu(f)})=a_{\nu(f)+1}$.

It is well known (and trivial) that for any system $\mathfrak{B}$, the fact that $\mathfrak{A}$ is isomorphic to a subsystem of $\mathfrak{B}$ is equivalent to the existence of a system $\mathfrak{B}'$ such that $\mathfrak{B}'\in\mathsf{M}_{\mathbf{L}_A}(\Delta_{\mathfrak{A}})$ and $\mathfrak{B}'|\mathbf{L}=\mathfrak{B}$. Consequently, to show $\mathfrak{A}\in\mathsf{SP}(\mathbf{K})$ it suffices to prove that there exists a system $\mathfrak{B}'$ such that $\mathfrak{B}'\in\mathsf{M}_{\mathbf{L}_A}(\Delta_{\mathfrak{A}})\cap\mathsf{P}(\mathbf{K})[\mathbf{L}_A]$. Evidently if $\mathbf{K}\in\mathsf{PC}_\Delta$ then also $\mathbf{K}[\mathbf{L}_A]\in\mathsf{PC}_\Delta$ and, furthermore, for any $\mathbf{K}$, $\mathsf{P}(\mathbf{K}[\mathbf{L}_A])=\mathsf{P}(\mathbf{K})[\mathbf{L}_A]$. Hence, by applying Theorem 1 to $\mathbf{K}[\mathbf{L}_A]$, it is sufficient to show that there exists a system $\mathfrak{B}'$

[9] $1\cdot G$ and $(-1)\cdot G$ mean G and $\neg G$, respectively.

[10] If $\mathsf{P}(\mathbf{K})$ is understood in the modified meaning of footnote (7) then Theorem 1 remains true. That follows by the trivial facts that for any $\mathbf{K}'$, $\mathbf{K}'$ is compact if and only if $\mathbf{K}'\cap\{\mathfrak{A}_0\}$ is compact, and $\mathbf{K}'$ is compact if and only if the closure of $\mathbf{K}'$ under isomorphism is compact.

for an arbitrary finite subset Δ' of $\Delta_{\mathfrak{A}}$ such that $\mathfrak{B}' \in \mathsf{M}_{\mathbf{L}_\Delta}(\Delta') \cap \mathsf{P}(\mathbf{K})[\mathbf{L}_\Delta]$. Let Φ be an open formula (containing no quantifiers) of $\mathbf{L}$ such that the conjunction of the formulas of Δ' arises from Φ by substituting some constants c_α for the variables $v_0, \ldots, v_n$ in Φ. If no such system $\mathfrak{B}'$ existed, then for an arbitrary $\mathfrak{B}$, $\mathfrak{B} \in \mathsf{P}(\mathbf{K})$, we should have $\mathfrak{B} \vdash (v_0)\ldots(v_n)(\neg\Phi)$, whence, by our supposition, $\mathfrak{A} \vdash (v_0)\ldots(v_n)(\neg\Phi)$, which is obviously false. So we have completed the proof of Corollary 1. ([11])

Theorem 2 below was suggested by Corollary 2, which is a straightforward analogue of Vaught's theorem (Corollary 1).

To formulate the theorem we introduce a new notion. If $\mathbf{K} \subset \mathfrak{S}(\mathbf{L})$ for a language $\mathbf{L}$ and $\mathbf{L}'$ is a subset of $\mathbf{L}$, then $\mathsf{P}_{\mathbf{L}'}(\mathbf{K})$ will denote the class of (non-empty) direct products $\prod_{i \in I} \mathfrak{A}^{(i)}$ such that $\mathfrak{A}^{(i)} \in \mathbf{K}$ for any $i \in I$, and $\mathfrak{A}^{(i)}|\mathbf{L}'$ is the same system $\mathfrak{A}$ of $\mathbf{L}'$ for every i, $i \in I$.

THEOREM 2. *If $\mathbf{L}' \subset \mathbf{L}$, $\mathbf{K} \subset \mathfrak{S}(\mathbf{L})$ and $\mathbf{K} \in \mathsf{PC}_\Delta$, then $\mathsf{P}_{\mathbf{L}'}(\mathbf{K})$ is compact.*

First we prove the following elementary consequence of (FV):

LEMMA 1. *If $F \in \mathfrak{F}_0(\mathbf{L})$, then we can give a finite set $\mathfrak{H}$ of finite sets of sentences of $\mathbf{L}$ such that for any $\mathbf{K}$, $\mathbf{K} \subset \mathfrak{S}(\mathbf{L})$,*

$$\mathsf{P}(\mathbf{K}) \cap \mathsf{M}_{\mathbf{L}}(F) \neq 0 \tag{7}$$

is equivalent to the existence of a set Γ, $\Gamma \in \mathfrak{H}$, so that for every H, $H \in \Gamma$, we have a system $\mathfrak{A}$ in $\mathbf{K}$ with $\mathfrak{A} \vdash H$.

Proof. We use the notations introduced in the proof of Theorem 1 and at the end of § 1. Put $m = m^F$. Let E be the set of all functions defined on the set m with possible values 1 and -1. Let X be the set of the functions η defined on E with possible values 0 and 1 such that there exists a system $\mathfrak{B}$, $\mathfrak{B} \in \mathsf{M}_{\mathbf{L}_0}(\Phi^F)$, such that for every $\varepsilon \in E$

$$\bigcap_{j<m} \varepsilon(j)(R_j^F)_{\mathfrak{B}} = 0 \text{ ([12])} \quad \text{if and only if} \quad \eta(\varepsilon) = 0\,. \tag{8}$$

Let $\Gamma(\eta)$ be the set of sentences defined by

$$\Gamma(\eta) = \{\bigwedge_{j<m} \varepsilon(j)\Theta_j^F \colon\ \eta(\varepsilon) = 1,\ \varepsilon \in E\}$$

and put $\mathfrak{H} = \{\Gamma(\eta) \colon \eta \in X\}$.

We may easily see that this $\mathfrak{H}$ satisfies the requirements of the lemma. Suppose first (7). Then, as we saw in the proof of Theorem 1, we have a system $\mathfrak{B}$, $\mathfrak{B} \in \mathsf{M}_{\mathbf{L}_0}(\Phi^F)$, and an indexed family $\mathfrak{A} = \langle \mathfrak{A}^{(i)} \colon i \in I\rangle$ such that $(*)$ holds. Let η be defined by (8) for this given $\mathfrak{B}$ and suppose

([11]) See ([7]). It is easy to see that $\mathsf{S}(\mathbf{K}) \in \mathsf{UC}_\Delta$ if and only if $\mathsf{S}(\mathbf{K} \cup \{\mathfrak{A}_0\}) \in \mathsf{UC}_\Delta$, whence Vaught's theorem is really equivalent to Corollary 1.

([12]) If A is a set, $1\cdot A$ and $(-1)A$ mean A and the complement of A, respectively.

$\eta(\varepsilon) = 1$. Then by definition there exists an i with $i \in \bigcap_{j<m} \varepsilon(j)(R_j^F)_{\mathfrak{B}}$ and by $(*)$ this obviously implies $\mathfrak{A}^{(i)} \vdash \bigwedge_{j<m} \varepsilon(j)\Theta_j^F$. Consequently $\Gamma = \Gamma(\eta) \in \mathfrak{H}$ satisfies the requirements, proving the first part of the lemma.

Secondly let us assume that for $\Gamma = \Gamma(\eta) \in \mathfrak{H}$ there exists a system $\mathfrak{A}^H$, $\mathfrak{A}^H \in \mathbf{K}$, for every H, $H \in \Gamma$, such that

$$\mathfrak{A}^H \vdash H\,. \tag{9}$$

Since $\Gamma(\eta) \in \mathfrak{H}$ we have a $\mathfrak{B}$, $\mathfrak{B} \in \mathsf{M}_{\mathbf{L}_0}(\Phi^F)$ satisfying (8). Let $I = |\mathfrak{B}|$ and $\mathfrak{A} = \langle \mathfrak{A}^{(i)} \colon i \in I\rangle$ such that $\mathfrak{A}^{(i)} = \mathfrak{A}^H$ if $i \in \bigcap_{j<m} \varepsilon(j)(R_j^F)_{\mathfrak{B}}$ and $H = \bigwedge_{j<m} \varepsilon(j)\Theta_j^F$ (H is uniquely determined).

It is easily seen by (9) that now $(*)$ holds, i.e. $\prod \mathfrak{A}^{(i)} \in \mathsf{P}(\mathbf{K}) \cap \mathsf{M}_{\mathbf{L}}(F)$, whence (7) holds, q.e.d.

LEMMA 2. *Let $\mathbf{K} \subset \mathfrak{S}(\mathbf{L})$, $\mathbf{L}' \subset \mathbf{L}$, and let $\mathbf{K}$ be compact. Let $F \in \mathfrak{F}_0(\mathbf{L})$ and denote the set of all points π of the space $\mathsf{S}_{\mathbf{L}'}$ such that $\mathsf{P}(\pi[\mathbf{L}] \cap \mathbf{K}) \cap \mathsf{M}_{\mathbf{L}}(F) \neq 0$ by X_F. Then X_F is a closed subset of $\mathsf{S}_{\mathbf{L}'}$.*

Proof. Suppose that π is a limit point of X_F, i.e. for every G, $G \in \mathfrak{F}_0(\mathbf{L}')$, such that $\pi \in \langle G\rangle_{\mathbf{L}'}$ we have $\langle G\rangle_{\mathbf{L}'} \cap X_F \neq 0$. We have to show $\pi \in X_F$.

Assume on the contrary that $\mathsf{P}(\pi[\mathbf{L}] \cap \mathbf{K}) \cap \mathsf{M}_{\mathbf{L}}(F) = 0$. Then by Lemma 1 for each $\Gamma \in \mathfrak{H}$ we have a sentence H_Γ, $H_\Gamma \in \Gamma$, such that $\pi[\mathbf{L}] \cap \mathbf{K} \cap M_{\mathbf{L}}(H_\Gamma) = 0$. If we use the compactness of $\mathbf{K}$, this implies the existence of a sentence G_Γ, $G_\Gamma \in \mathfrak{F}(\mathbf{L}')$, such that $\pi \in \langle G_\Gamma\rangle_{\mathbf{L}'}$ and $\mathsf{M}_{\mathbf{L}}(G_\Gamma) \cap \mathbf{K} \cap \mathsf{M}_{\mathbf{L}}(H_\Gamma) = 0$. Take $G = \bigwedge_{\Gamma \in \mathfrak{H}} G_\Gamma$. Now we have $\pi \in \langle G\rangle_{\mathbf{L}'}$, and $\mathsf{M}_{\mathbf{L}}(G) \cap \mathbf{K} \cap \mathsf{M}_{\mathbf{L}}(H_\Gamma) = 0$ for every $\Gamma \in \mathfrak{H}$. But this implies by Lemma 1 that $\mathsf{P}(\mathsf{M}_{\mathbf{L}}(G) \cap \mathbf{K}) \cap \mathsf{M}_{\mathbf{L}}(F) = 0$, which contradicts the fact that π is a limit point of X_F. Q.e.d.

Proof of Theorem 2. Let us assume that the hypotheses of Theorem 2 hold and let Σ be a set of sentences of $\mathbf{L}$ such that for any Σ', $\Sigma' \in S_\omega(\Sigma)$ we have

$$\mathsf{P}_{\mathbf{L}'}(\mathbf{K}) \cap \mathsf{M}_{\mathbf{L}}(\Sigma') \neq 0\,. \tag{10}$$

Let us denote by $X_{\Sigma'}$ the set X_F of Lemma 2 if F is the conjunction of the sentences of Σ'. Then (10) and the definition of $\mathsf{P}_{\mathbf{L}'}(\mathbf{K})$ obviously imply that $X_{\Sigma'}$ is non-empty for every Σ', $\Sigma' \in S_\omega(\Sigma)$. Using Lemma 2 and the compactness of $\mathsf{S}_{\mathbf{L}'}$ we get $\bigcap_{\Sigma' \in S_\omega(\Sigma)} X_{\Sigma'} \neq 0$. This means that there exists π, $\pi \in \mathsf{S}_{\mathbf{L}'}$, such that $\mathsf{P}(\pi[\mathbf{L}] \cap \mathbf{K}) \cap \mathsf{M}_{\mathbf{L}}(\Sigma') \neq 0$ for every $\Sigma' \in S_\omega(\Sigma)$. Since $\mathbf{K} \in \mathsf{PC}_\Delta$ and so $\mathbf{K}$ is compact, we have $\pi[\mathbf{L}] \cap \mathbf{K}$ is obviously also compact. Applying Theorem 1 for $\pi[\mathbf{L}] \cap \mathbf{K}$ instead of $\mathbf{K}$ we obtain

$\mathsf{P}(\pi[\mathbf{L}]\cap\mathbf{K})\cap\mathsf{M}_{\mathbf{L}}(\Sigma)\neq 0$, i.e. we have an indexed family $\langle\mathfrak{A}^{(i)}: i\in I\rangle$ such that $\mathfrak{A}^{(i)}\in\mathbf{K}$, $\mathfrak{A}^{(i)}|\mathbf{L}'\in\pi$ for every i and

(11) $$\prod_{i\in I}\mathfrak{A}^{(i)}\in\mathsf{M}_{\mathbf{L}}(\Sigma).$$

We show that there exists a $\langle\mathfrak{B}^{(i)}: i\in I\rangle$ with $\mathfrak{B}^{(i)}\in\mathbf{K}$ $(i\in I)$, $\mathfrak{B}^{(i)}|\mathbf{L}'$ being always the same system $\mathfrak{B}$ of $\mathbf{L}'$ for each i and

(12) $$\mathfrak{B}^{(i)}\equiv\mathfrak{A}^{(i)}.$$

To do this we apply a theorem of Büchi and Craig [2] (see also [3]) a special case of which is that, for a family $\langle\mathbf{K}^{(i)}: i\in I\rangle$ of PC_Δ-classes $\mathbf{K}^{(i)}$ of systems of a fixed language $\mathbf{L}$, if $\bigcap_{i\in I}\mathsf{C}(\mathbf{K}^{(i)})$ is non-empty then $\bigcap_{i\in I}\mathbf{K}^{(i)}$ is also non-empty. Here $\mathsf{C}(\mathbf{K})$ means the intersection of all EC_Δ-classes containing $\mathbf{K}$. To use the theorem, let $\mathbf{K}^{(i)}$ be the class $\{\mathfrak{B}|\mathbf{L}': \mathfrak{B}\in\mathbf{K}, \mathfrak{B}\equiv\mathfrak{A}^{(i)}\}$. Since $\mathbf{K}\in\mathsf{PC}_\Delta$, it is easily seen that $\mathbf{K}^{(i)}\in\mathsf{PC}_\Delta$. Now $\mathsf{C}(\mathbf{K}^{(i)})=\pi$ if $i\in I$ and thus $\bigcap_{i\in I}\mathsf{C}(\mathbf{K}^{(i)})=\pi\neq 0$.

Hence we have a system $\mathfrak{B}$ such that $\mathfrak{B}\in\bigcap_{i\in I}\mathbf{K}^{(i)}$.

The existence of the required systems $\mathfrak{B}^{(i)}$ now follows from the definition of $\mathbf{K}^{(i)}$, and from (11), (12) and the fact that direct product preserves elementary equivalence (which follows from (FV)) also $\prod_{i\in I}\mathfrak{B}^{(i)}\in\mathsf{M}_{\mathbf{L}}(\Sigma)$. Hence $\mathsf{P}_{\mathbf{L}'}(K)\cap\mathsf{M}_{\mathbf{L}}(\Sigma)\neq 0$. Q.e.d.

We note that if $\mathbf{L}\setminus\mathbf{L}'$ contains only individual constants then the theorem of Büchi and Craig can be replaced by a simple argument involving CTh. This case takes place in Corollary 2.

Corrolary 2. *If* $\mathbf{K}\in\mathsf{PC}_\Delta$ *then* $\mathsf{SDp}(\mathbf{K})\in\mathsf{UC}_\Delta$.

The proof is similar to that of Corollary 1, the identity $\mathsf{Dp}(\mathbf{K})[\mathbf{L}_\Delta]=\mathsf{P}_{\mathbf{L}}(\mathbf{K}[\mathbf{L}_\Delta])$ being now used.

We remark that Corollary 2 can be proved in the same manner as Corollary 1 was proved in [10]. Indeed, if $\mathsf{Sbdp}(\mathbf{K})$ denotes the class of all subdirect powers of systems of $\mathbf{K}$, then $\mathsf{SDp}(\mathbf{K})=\mathsf{SSbdp}(\mathbf{K})=\mathsf{SSbdpS}(\mathbf{K})$, [13] and if $\mathbf{K}\in\mathsf{PC}_\Delta$ then $\mathsf{Sbdp}(\mathbf{K})\in\mathsf{PC}_\Delta$. The last statement can be proved in the same way as Theorem 2 (a) in [10]. If we suppose that $\mathbf{L}$ $(\mathbf{K}\subset\mathfrak{S}(\mathbf{L}))$ contains no predicate symbol, then the new proof can be obtained from the one in [10] by adding to $(\mathfrak{A},F)$ a five-place operation $\varphi(x,y,x',y',z)$ and to (3) the condition that for any fixed $x,y,y',x'\in\mathfrak{A}$ $\varphi(x,y,x',y',z)$ produces an isomorphism of $\mathfrak{A}/F_{xy}$ onto $\mathfrak{A}/F_{x'y'}$. For general $\mathbf{L}$, this proof may be amended in the same way, as indicated in the last paragraph of [10], p. 231.

[13] To be precise, $\mathsf{Dp}(\mathbf{K})$ ought to be modified as $\mathsf{P}(\mathbf{K})$ in (7).

Corollary 3. *If* $\mathbf{K}$ *is compact then* $\mathsf{Dp}(\mathbf{K})$ *is also compact.*

If $\mathbf{K}\in\mathsf{PC}_\Delta$ then the conclusion follows at once from Theorem 2 by taking $\mathbf{L}=\mathbf{L}'$. If, more generally, $\mathbf{K}$ is compact, then we use the relation $\overline{\mathsf{Dp}(\mathbf{K})}=\overline{\mathsf{Dp}(\overline{\mathbf{K}})}$, which follows trivially from the fact that the direct powers preserve elementary equivalence. Now $\overline{\mathbf{K}}\in\mathsf{EC}_\Delta$, and so $\overline{\mathsf{Dp}(\mathbf{K})}=\mathsf{Dp}(\overline{\mathbf{K}})\in\mathsf{EC}_\Delta$ indeed.

We remark that Corollary 3 might be proved directly in a simple way.

Now we give a few remarks, supplementing our results, in connection with direct products and cardinal sums. If α is a cardinal number, we denote by $\mathsf{P}^\alpha(\mathbf{K})$ the class of all direct products of exactly α systems of $\mathbf{K}$. Then, for a finite cardinal n, if $\mathbf{K}\in\mathsf{PC}_\Delta$ then $\mathsf{P}^n(\mathbf{K})\in\mathsf{PC}_\Delta$.

If, more generally, $\mathbf{K}$ is compact then $\mathsf{P}^n(\mathbf{K})$ is also compact; this follows from the last statement in the same (trivial) way as the general case in Corollary 3. Let us suppose that $\varkappa(\mathbf{L})$ (the cardinality of $\mathbf{L}$) $\leqslant\alpha$. If $\mathbf{K}\subset\mathfrak{S}(\mathbf{L})$ and $\mathbf{K}$ is compact, then $\mathsf{P}^\alpha(\mathbf{K})$ is compact. This can be proved in the same way as Theorem 1, for in this case we clearly can require the set $I=|\mathfrak{B}|$ of the proof of Theorem 1 to be of power α. In this case $\overline{\bigcup_{\beta\geqslant\alpha}\mathsf{P}^\beta(\mathbf{K})}=\overline{\mathsf{P}^\alpha(\mathbf{K})}$, as easily follows from Theorem 6.8 of [4] after extending it to languages of arbitrary power.

Let $\mathsf{Dp}^\alpha(\mathbf{K})$ denote, analogously, the class of all direct powers $\mathfrak{A}^I$ of systems $\mathfrak{A}$ in $\mathbf{K}$ such that $\varkappa(I)=\alpha$. If n is finite and $\mathbf{K}\in\mathsf{PC}_\Delta$ then, as is easily seen, $\mathsf{Dp}^n(\mathbf{K})\in\mathsf{PC}_\Delta$. From (FV) we can easily infer that for every F, $F\in\mathfrak{F}_0(\mathbf{L})$, a sentence G_F of $\mathfrak{F}_0(\mathbf{L})$ can be given such that if $\mathfrak{A}\in\mathfrak{S}(\mathbf{L})$ and $\varkappa(I)\geqslant\omega$ then $\mathfrak{A}^I\vdash F$ if and only if $\mathfrak{A}\vdash G_F$. In particular, $\mathfrak{A}^I\equiv\mathfrak{A}^{I'}$ if $\varkappa(I)$, $\varkappa(I')\geqslant\omega$ (independently of $\varkappa(\mathbf{L})$). It follows that the mapping φ of $\mathsf{S}_{\mathbf{L}}$ into itself defined by: $\varphi(\pi)=\overline{\{\mathfrak{A}^I: \mathfrak{A}\in\pi, \varkappa(I)=\omega\}}$ is a continuous one. The image of a closed subset of a compact spaces under a continuous mapping is closed; applying this to φ, we infer that $\mathsf{Dp}^\omega(\mathbf{K})$ is compact if $\mathbf{K}$ is compact (for arbitrary $\mathbf{L}$).

If $\langle\mathfrak{A}^{(i)}: i\in I\rangle$ is an indexed family of systems of a language $\mathbf{L}$ containing only predicate symbols, then any system $\mathfrak{B}$, obtained as follows, is called a *cardinal sum* of the systems $\mathfrak{A}^{(i)}$. Let $\mathfrak{B}^{(i)}$ (for each $i\in I$) be a system isomorphic to $\mathfrak{A}^{(i)}$ such that the sets $|\mathfrak{B}^{(i)}|$ are mutually disjoint. Then let $|\mathfrak{B}|=\bigcup_{i\in I}|\mathfrak{B}^{(i)}|$ and $P_{\mathfrak{B}}=\bigcup_{i\in I}P_{\mathfrak{B}(i)}$ for any $P\in\mathbf{L}$. If the systems $\mathfrak{A}^{(i)}$ are all equal to a fixed $\mathfrak{A}$, then the cardinal sum goes over to a cardinal multiple of $\mathfrak{A}$. For a class $\mathbf{K}$ of systems, let $\mathsf{Cs}(\mathbf{K})$ and $\mathsf{Cm}(\mathbf{K})$ be the class of all cardinal sums and multiples, respectively, of systems of $\mathbf{K}$.

As can be read from [4], for the cardinal sum there exists an exactly analogous criterion as (FV). By the help of this criterion we might prove similar results for cardinal sums to those we proved for direct products. But all these results can be obtained by the following stronger statement.

If $\mathbf{K} \in \mathsf{PC}_\Delta$ *then* (a) $\mathsf{Cs}(\mathbf{K}) \in \mathsf{PC}_\Delta$ *and* (b) $\mathsf{Cm}(\mathbf{K}) \in \mathsf{PC}_\Delta$.

The proofs of these assertions are very simple: we only sketch them in a few words. Let us assume $\mathbf{K} \in \mathsf{PC}_\Delta$. In order to ensure that $\mathfrak{A} \in \mathsf{Cs}(\mathbf{K})$ we require the existence of a binary relation $S(x, y)$ on $\mathfrak{A}$ such that: $S(x, x)$ holds for every x, $x \in |\mathfrak{A}|$; any two of the sets $S_x = \{y\colon S(x, y)$ holds$\}$ are equal or disjoint; the subsystem of $\mathfrak{A}$ with domain S_x belongs to $\mathbf{K}$, for any $x \in |\mathfrak{A}|$; and finally, $P_{\mathfrak{A}}(a_1, \ldots, a_n)$ holds only if $a_1, \ldots, a_n$ are elements of the same set S_x. It is easily seen that this requirement is equivalent, first, to $\mathfrak{A} \in \mathbf{K}'$ for a certain $\mathbf{K}' \in \mathsf{PC}_\Delta$, and secondly, to $\mathfrak{A} \in \mathsf{Cs}(\mathbf{K})$. In the case of (b) we add to our requirement that there should exist a ternary operation $\varphi(x_1, x_2, y)$ such that for any fixed $x_1, x_2 \in |\mathfrak{A}|$ $\varphi(x_1, x_2, y)$ is an isomorphism of S_{x_1} onto S_{x_2}.

In order to be able to extend all the results proved for direct products to cardinal sums, we only have to add the remark that the cardinal sum preserves elementary equivalence (see Theorem 5.1 in [4]).

Finally we give a compactness result concerning countably weak direct products. We define this notion as it was given in [4] (p. 71, 4.3, using 4.2). Let $\mathbf{L}$ be an arbitrary language, $E(x)$ a formula of $\mathbf{L}$ containing no free variable except x, $\langle \mathfrak{A}^{(i)}\colon i \in I\rangle$ a (non-empty) indexed family of systems of $\mathbf{L}$. The countably weak direct product of the systems $\mathfrak{A}^{(i)}$ (relative to $E(x)$) is the subsystem $\mathfrak{B}$ of $\prod_{i \in I} \mathfrak{A}^{(i)}$ such that $|\mathfrak{B}|$ is the set of functions φ, $\varphi \in \prod_{i \in I} |\mathfrak{A}^{(i)}|$, for which the set of indices i with $\mathfrak{A}^{(i)} \models_{\varphi(i)}^{x} \neg E(x)$ is at most countable.

The countably weak direct power of a system $\mathfrak{A}$ is defined in the natural way. Let us denote by $\mathsf{Cw}(\mathbf{K})$ and $\mathsf{Cwp}(\mathbf{K})$ the classes of all systems isomorphic to some countably weak direct product and power, respectively, of systems in $\mathbf{K}$.

THEOREM 3. *If the language* $\mathbf{L}$ *is at most countable,* $\mathbf{K} \subset \mathfrak{S}(\mathbf{L})$ *and* $\mathbf{K}$ *is compact, then* (a) $\mathsf{Cw}(\mathbf{K})$ *and* (b) $\mathsf{Cwp}(\mathbf{K})$ *are compact.*

Proof. Consider (a). This is how we can modify the proof of Theorem 1 to yield this result. First we can apply an analogue of the criterion (FV). The new criterion can be given with the help of [4], namely the final remarks of § 7 at the bottom of p. 89, Theorems 7.1 and 3.2 and the discussion 4.3 of § 4. Using this criterion we replace Φ^F in the proof of Theorem 1 by a formula Φ'^F, Φ'^F playing a similar role in the new proof to that of Φ^F in the old one, such that Φ'^F is a formula of the predicate calculus $\mathbf{L}_1$ of Fuhrken [5]. $\mathbf{L}_1$ is obtained by adding the quantifier "there exists at most countably many..." to the first order logic based on the language $\mathbf{L}_0$. We then apply a result of Fuhrken [5] which says that if Γ' is a set of sentences of $\mathbf{L}_1$ and every finite subset is satisfiable, then Γ' is itself satisfiable. We use this instead of CTh to prove that the analogue Γ' of the set Γ (see (3)) in the present situation is satisfiable. We can easily see that Lemma 1 holds also in this case and hence that the proof of Theorem 2 can be extended to prove (b) or a more general result like Theorem 2.

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