A compactness result concerning direct products of models

by

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Introduction. The results of the present paper are motivated by the following theorem of R. Vaught (see [10], Theorem 2b)

**Theorem of Vaught.** If $K$ is a class of similar relational systems and $K \in PC_{\lambda}$, then $SP(K) \in UC_{\lambda}$ (1).

The following theorem was proved in [6] and [9].

**Theorem of Łoś and Tarski.** If $K \in PC_{\lambda}$, then $S(K) \in UC_{\lambda}$.

This theorem can be derived from the Compactness Theorem of the first order predicate calculus by the standard method of diagrams (*). Our main result (Theorem 1 in § 2) is a "compactness" result concerning classes $P(K)$ from which the theorem of Vaught can be derived in a quite similar way by the method of diagrams. To formulate our theorem, let us call a class $K$ of similar relational systems **compact** if the following holds: for any set $\Sigma$ of sentences appropriate for the systems of $K$, if every finite subset of $\Sigma$ is satisfiable by a system of $K$, then $\Sigma$ is itself satisfiable by a system of $K$. Then our assertion is that if $K$ is compact, then the class $P(K)$ of all direct products of systems in $K$ is also compact.

Let $D_{\alpha}(K)$ denote the class of all direct powers of systems of $K$. We infer that $K \in PC_{\lambda}$ implies $SD_{\alpha}(K) \in UC_{\lambda}$ (Corollary 3). This will be derived by the method of diagrams from a more general compactness result (Theorem 2). Another special case of Theorem 3 that if $K$ is compact, then $D_{\alpha}(K)$ is also compact.

We give some remarks concerning direct products and cardinal sums related to these results, and finally we state a compactness result concerning countably weak direct products.

Our main tools in the proofs are a criterion of [4] (stated in § 1 of the present paper) for a sentence holding in a given direct product, and the Compactness Theorem.

**§ 1. Preliminaries.** We sum up the notions and notations to be used in the paper. For more details, see [9] and [7].

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(1) For definition of the notions, see § 1.

(*) See e.g. [10], § 4.
A language \( \mathcal{L} \) is a set of which certain elements are specified as predicate symbols and others as function symbols. We denote predicate symbols by \( P, Q, R; \) function symbols by \( f, g, h \); a language \( \mathcal{L} \) possibly with several indices. \( P \in \mathcal{L} \) and \( f \in \mathcal{L} \) will always mean that \( P \) is a predicate symbol, \( f \) is a function symbol of the language \( \mathcal{L} \). With each \( P \) or \( f \) \((P, f \in \mathcal{L})\) there is associated a natural number \( \nu(P) \) and \( \nu(f) \geq 0 \) and \( P, f \) are called a \( \nu(P) \)-ary predicate symbol and a \( \nu(f) \)-ary function symbol, respectively. \(^{(1)} \)

\( \mathcal{M} \) is a relational system, or briefly a system of \( \mathcal{L} \) (notation: \( \mathcal{M} \models \phi(\mathcal{L}) \)) if \( \mathcal{M} \) is an ordered pair of a non-empty set \( A \) (denoted by \( \mathfrak{A}(\mathcal{L}) \)) and a function \( \mathfrak{B}(\mathcal{L}) \) with domain \( \mathcal{L} \) such that, the value of this function for arguments \( P \) and \( f \) \((P, f \in \mathcal{L})\) being denoted by \( \mathfrak{B}(P) \) and \( \mathfrak{B}(f) \) respectively, \( \mathfrak{B}(P) \) is a \( \nu(P) \)-ary relation on \( A \) and \( \mathfrak{B}(f) \) is a \( \nu(f) \)-ary operation on \( A \). \(^{(4)} \)

If \( \mathcal{L}, \mathcal{L}' \) are languages, \( \mathcal{L}' \supset \mathcal{L} \) and \( \mathfrak{G} \in \mathfrak{S}(\mathcal{L}) \) then \( \mathfrak{S}(\mathcal{L}) \) denotes the \( \mathcal{L}' \)-reduct of \( \mathfrak{G} \), i.e. the system \( \mathfrak{S} \) of \( \mathcal{L} \) such that \( \mathfrak{S} = \mathfrak{G} \) and, for any \( P, f \) \((P, f \in \mathcal{L})\), \( P \in \mathcal{L}' \) and \( f \in \mathcal{L}' \). If \( \mathcal{L}' \supset \mathcal{L} \), then \( \mathfrak{K}(\mathcal{L}') \subseteq \mathfrak{S}(\mathcal{L}) \). \( \mathfrak{K}(\mathcal{L}') \subseteq \mathfrak{K}(\mathcal{L}) \) if \( \mathfrak{K}(\mathcal{L}') = \mathfrak{K}(\mathcal{L}) \).

Let \( \mathcal{L} \subseteq \mathfrak{S}(\mathcal{L}) \). Then \( \mathfrak{K}(\mathcal{L}) \) will denote the class of all systems \( \mathfrak{S}, \mathfrak{S} \in \mathfrak{S}(\mathcal{L}) \) such that \( \mathfrak{S}(\mathcal{L}) \subseteq \mathfrak{K} \).

We shall use the following notations for the logical operations: \( \neg \) (negation), \( \land \) (and), \( \lor \) (or), \( \rightarrow \) (implies), \( \leftrightarrow \) (equivalent), \( (\forall \pi) \) (for all \( \pi \)), \( \exists \) (there exist an \( \pi \) such that \( \ldots \)). \( \neg \land \exists \neg \rightarrow \) (the first order) formulas of \( \mathcal{L} \) are built up in the well-known way from the symbols of \( \mathcal{L} \), the identity symbol \( = \), of the fixed infinite sequence \( \pi_1, \pi_2, \ldots \) of (individual) variables and the logical operations. The set of all formulas of \( \mathcal{L} \) and the set of all sentences (formulas having no free variable) of \( \mathcal{L} \) are denoted by \( \mathfrak{S}(\mathcal{L}) \) and \( \mathfrak{K}(\mathcal{L}) \), respectively. If the sentence \( \mathcal{F} \) has the form \((\forall_1 \ldots \forall_n) \varphi \), \( \varphi \) containing no quantifiers, then \( \mathcal{F} \) is called a universal sentence of \( \mathcal{L} \). If \( \mathcal{F} \in \mathfrak{S}(\mathcal{L}) \), \( \varphi \) contains no free variable except the distinct variables \( \pi_1, \ldots, \pi_n \); \( \mathfrak{S} \in \mathfrak{S}(\mathcal{L}) \) and \( \pi_1, \ldots, \pi_n \in \mathfrak{S} \), then

\[
\exists \mathfrak{S} = \exists \mathfrak{S} \mathcal{F}
\]

will mean that \( \pi_1, \ldots, \pi_n \) satisfy \( \mathcal{F} \) in \( \mathfrak{S} \) under the correspondence \( \pi_i \rightarrow \pi_i \ldots \pi_n \rightarrow \pi_n \). In particular, if \( \mathfrak{S} \in \mathfrak{S}(\mathcal{L}) \), \( \mathcal{F} \rightarrow \mathcal{F} \) means that \( \mathcal{F} \) holds (is true) \( \mathbf{(1)} \).

For \( \pi \) a predicate symbol, \( \mathfrak{S} \in \mathfrak{S}(\mathcal{L}) \), \( \mathcal{F} \in \mathfrak{S}(\mathcal{L}) \) we call \( \mathcal{F} \) a \( \pi \)-ary relation of \( \mathcal{L} \). A singular relation \( \mathfrak{R} \) is identified in the usual way with a set, i.e. we shall write \( \mathfrak{R} \) \( \equiv \) \( \mathfrak{S} \) equivalently with the statement that \( \mathfrak{R}(\mathfrak{S}) \) holds.
with this interpretation, we can "express" (for \( F \in \mathcal{B}(L) \)) by a sentence \( \phi^F \) a part of the fact that the condition of (FV) holds.

More precisely, we do the following. Let \( I_0 \) be the language consisting of all \( \mathcal{B}^F_\alpha \) for \( \Theta \in \mathcal{B}(L) \). Let \( R \in \mathcal{L}_n \), \( n \) a natural number. We denote by \( (\mathcal{B}^F_\alpha)(R(x), (\mathcal{B}^F_\alpha)(R(x), (n \geq 1) \), and \( (\mathcal{B}^F_\alpha)(R(x), (\mathcal{B}^F_\alpha)(R(x), \) the following formulas, respectively:

\[
\bigwedge_{i < \alpha} R(v_i) \land \bigvee_{i < \alpha} R(v_i) \land \bigvee_{i < \alpha} R(v_i)
\]

Then, if \( R \in \mathcal{L}_n \) and \( R \in \mathcal{L}_n \) for some \( L \), then \( \mathcal{B}^F_\alpha(R(x)) \) and \( \mathcal{B}^F_\alpha(R(x)) \) (for \( n \geq 0 \)) are equivalent, respectively, to the statement that there exist at least or exactly \( n \) different elements \( x \) in (3) such that \( \mathcal{B}^F_\alpha(R(x)) \).

In all that follows we write \( R^F_\alpha \) instead of \( R^F \). Now we define \( \phi^F \) for any \( F \in \mathcal{B}(L) \) as the following sentence of \( I_0 \):

\[
\bigwedge_{i < \alpha} R(v_i) \land \bigvee_{i < \alpha} R(v_i) \land \bigvee_{i < \alpha} R(v_i)
\]

(for the notations, see (FV) in § 1).

Now we can formulate (FV) equivalently as follows. (3) holds if and only if there exists a system \( \mathcal{B}_n \in \mathcal{L}_n \), such that

\[
\text{(*)} \quad \exists \mathcal{B}_n \in \mathcal{L}_n \quad \forall i \in I \quad \phi^F_i \wedge \phi^F_i \land \phi^F_i \land \phi^F_i
\]

Let us assume that \( \mathbf{K} \) is compact, \( \mathbf{K} \in \mathcal{L}(\mathcal{L}) \) and \( \Sigma \) is a set of sentences of \( \mathcal{L} \) such that \( \mathbf{M}_\mathcal{L}(\Sigma) \land \phi^F \neq 0 \) for any \( \Sigma \in \Sigma \). Then for any \( \mathbf{F} \in \mathcal{F}(\Sigma) \) there exist an indexed family \( \mathcal{B}_n \in \mathcal{B}(\Sigma) \) (3)

\[
\text{(**)} \quad \ast \left( F \in \Sigma \right) \land \mathcal{B}_n \land \mathcal{B}_n \land \mathcal{B}_n \land \mathcal{B}_n
\]

We define \( X_\mathcal{L} \) as the set of sentences \( \mathcal{F} \) satisfying the following two conditions:

\[
\text{(1)} \quad \mathcal{F}_\mathcal{L} \land \mathcal{F}_\mathcal{L} \land \mathcal{F}_\mathcal{L} \land \mathcal{F}_\mathcal{L}
\]
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(i) $\mathcal{P}$ has the form

$$\neg(\exists \mathcal{M}_0 \vee \bigwedge_{\mathcal{F} \in \mathcal{A}} e(\mathcal{F}, \mathcal{J}) \mathcal{E}_\mathcal{F}(\mathcal{V}) \quad \text{for} \quad \mathcal{V} \in \mathcal{A})$$

where $e(\mathcal{F}, \mathcal{J})$ is 1 or $-1$. (*)

(ii) For every $\Sigma'$ such that $\Sigma' \subseteq \Sigma$ and $\Sigma' \supseteq \Sigma$ we have $\mathcal{V}_{\Sigma'} \models \mathcal{P}$.

We define the set $\mathcal{I'}$ of sentences of $\mathcal{L}$ as follows:

$$\mathcal{I'} = \left\{ \mathcal{F} \subseteq \mathcal{F} : \mathcal{F} \models R \mathcal{P} \right\} \cup \bigcup_{\mathcal{V} \in \mathcal{A}} \mathcal{V}_{\Sigma'}.$$

We can easily see by the definition of $\mathcal{V}_{\Sigma'}$ and (**), that $\mathcal{V}_{\Sigma'} \models \mathcal{M}(I')$, which proves our assertion.

By OTM we have a system $\mathcal{B}$ such that

$$\mathcal{B} \models \mathcal{M}(I').$$

Let $|\mathcal{B}| = I$, $i \in I$. We define the set $\mathcal{A}_i$ of sentences of $\mathcal{L}$ by

$$\mathcal{A}_i = \left\{ \mathcal{F} : \mathcal{F} \models R \mathcal{P}, j = \mathcal{F} \right\}, \quad \mathcal{F} \models \mathcal{P}, j < \mathcal{P} / \mathcal{R}_0.$$
for an arbitrary finite subset $A'$ of $\Delta$ such that $\exists \varepsilon \in M_{\Delta}(A') \land P(K)[L_A]$.

Let $\Phi$ be an open formula (containing no quantifiers) of $L$ such that the conjunction of the formulas of $A'$ arises from $\Phi$ by substituting some constants $c_\nu$ for the variables $v_1, \ldots, v_\nu$ in $\Phi$. If no such system $\exists \varepsilon$ existed, then for an arbitrary $\exists \varepsilon \in P(L)$, we should have $\exists \varepsilon (\varepsilon_1 \ldots \varepsilon_\nu)(\neg \Phi)$, whence, by our assumption, $\exists \varepsilon (\varepsilon_1 \ldots \varepsilon_\nu)(\neg \Phi)$, which is obviously false. So we have completed the proof of Corollary 1.\(^{(3)}\)

Theorem 2 below was suggested by Corollary 2, which is a straightforward analogue of Vaught's theorem (Corollary 1).

To formulate the theorem we introduce a new notion. If $K \subseteq L$ and $L'$ is a subset of $L$, then $P_L(K)$ will denote the class of (non-empty) direct products $\prod_{i \in I} \mathcal{M}_i$ for any $i \in I$, and $\mathcal{M}[L']$ is the same system $\mathcal{M}[L']$ for every $i$, $i \in I$.

**Theorem 2.** If $L' \subseteq L$, $K \subseteq L$ and $K \in P_{L'}(K)$, then $P_L(K)$ is compact.

**Lemma 1.** If $F \subseteq P_L(K)$, then we can give a finite set $\mathcal{G}$ of finite sets of sentences of $L$ such that for any $K, K \subseteq L$.

$$P(K) \land M(L) \neq 0$$

is equivalent to the existence of a set $\Gamma, \Gamma \in \mathcal{G}$, so that for every $H, H \in \Gamma$, we have a system $K$ in $\mathcal{M}[H]$.\(^{(7)}\)

**Proof.** We use the notations introduced in the proof of Theorem 1 and at the end of § 1. Put $m = \mathfrak{m}$. Let $E$ be the set of all functions defined on the set $m$ with possible values 0 and 1. Let $\Gamma$ be the set of all functions $\eta$ defined on $E$ with possible values 0 and 1 such that there exists a system $\mathcal{B}$ of $\mathcal{M}(\mathcal{F})$, such that for every $e \in E$.

$$\bigcap e \in E \mathcal{M}(\mathcal{F}) = 0 \ (8)$$

If and only if $\eta(e) = 0$.

Let $\Gamma(\eta)$ be the set of sentences defined by

$$\Gamma(\eta) = \{ \land e \in E \mathcal{M}(\mathcal{F}) : \eta(e) = 1, e \in E \}$$

and put $\mathcal{G} = \{ \Gamma(\eta) : \eta \in E \}$.

We may easily see that this $\mathcal{G}$ satisfies the requirements of the lemma. Suppose first (7). Then, as we proved in the proof of Theorem 1, we have a system $\mathcal{B}, \mathcal{B} \subseteq M(\mathcal{F})$, and an indexed family $K = (\mathcal{M}: e \in E)$, such that (8) holds. Let $\eta$ be defined by (8) for this given $\mathcal{B}$ and suppose

$$\eta(e) = 1.$$
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\[ P(\pi[L] \land K) \land M_L(L) \not= \emptyset, \text{ i.e. we have an indexed family } \langle \pi^i : i \in I \rangle \text{ such that } \pi^i \in K, \pi^i[L] = \pi \text{ for every } i \text{ and} \]

\[ \bigwedge_{i \in I} \pi^i \land M_L(L). \]

We show that there exists a \( \langle \pi^i : i \in I \rangle \) with \( \pi^i \in K \) (\( i \in I \)), \( \pi^i[L] \) being always the same system \( B \) of \( L' \) for each \( i \) and

\[ \pi^i = \pi. \]

To do this we apply a theorem of Bûcher and Craig [2] (see also [3]) a special case of which is that, for a family \( \langle K^i : i \in I \rangle \) of \( P_C^* \)-classes \( K^i \) of systems of a fixed language \( L \), if \( \bigcap_{i \in I} C(K^i) \neq \emptyset \) is non-empty then \( \bigcap_{i \in I} K^i \) is also non-empty. Hence \( C(K) \) meets the intersection of all \( E_C^* \)-classes containing \( K \). To use the theorem, let \( K = K \) be the class \( \langle B[L] : B \in K \rangle \). Since \( K \in P_C^* \), it is easily seen that \( K \in E_C^* \). Now \( C(K) = \pi \) if \( i \in I \) and thus \( \bigcap_{i \in I} C(K^i) = \pi \neq \emptyset. \]

Hence we have a system \( B \) such that \( B \in \bigcap_{i \in I} K^i \).

The existence of the required systems \( \pi^i \) now follows from the definition of \( K^i \), and from (11), (12) and the fact that direct product preserves elementary equivalence (which follows from (FY)) also \( \bigwedge_{i \in I} \pi^i \land M_L(L) \not= \emptyset. \)

We note that if \( L \) contains only individual constants then the theorem of Bûcher and Craig can be replaced by a simple argument involving GTh. This case takes place in Corollary 2.

**Corollary 2.** If \( K \in P_C \) then \( \mathcal{D}P(K) \Delta \mathfrak{U}C \).

The proof is similar to that of Corollary 1, the identity \( \mathcal{D}P(K)[L] = \mathcal{D}P(K)[L] \) being now used.

We remark that Corollary 2 can be proved in the same manner as Corollary 1 was proved in [10]. Indeed, if \( Sb(K) \) denotes the class of all direct powers of systems of \( K \), then \( Sb(K) = \mathcal{S}sb(K) \) is \( \mathcal{S}sb(P) \) (\( \mathcal{S}sb(P) \)) and if \( K \in P_C \) then \( Sb(K) \neq P_C \). The last statement can be proved in the same way as Theorem 2 (a) in [10]. If we suppose that \( L \) contains no predicate symbol, then the new proof can be obtained from the one in [10] by adding to \( \langle \mathfrak{I}, \mathfrak{F} \rangle \) a five-place operation \( \varphi(x, y, \varphi', \varphi', \sigma) \) and (3) the condition that for any fixed \( x, y, z, z', \sigma \in \mathfrak{I} \) \( \varphi(x, y, z, z', \sigma) \) produces an isomorphism of \( \mathfrak{I}[\varphi] \) onto \( \mathfrak{I}[\varphi] \). For general \( L \), this proof may be amended in the same way, as indicated in the last paragraph of [10], p. 231.

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(4) To be precise, \( \mathcal{D}P(K) \) ought to be modified as \( \mathcal{D}P(K) \) in (7).

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**Corollary 3.** If \( K \) is compact then \( \mathcal{D}P(K) \) is also compact.

If \( K \in P_C \) then the conclusion follows at once from Theorem 2 by taking \( L = L' \). If, more generally, \( K \) is compact, then we use the relation \( \mathcal{D}P(K) = \mathcal{D}P(K) \), which follows trivially from the fact that the direct powers preserve elementary equivalence. Now \( K \in P_C \), and so \( \mathcal{D}P(K) = \mathcal{D}P(K) \), indeed.

We remark that Corollary 3 might be proved directly in a simple way. Now we give a few remarks, supplementing our results, in connection with direct products and cardinal sums. If \( a \) is a cardinal number, we denote by \( \mathcal{D}P(K) \) the class of all direct products of exactly \( a \) systems of \( K \). Then, for a finite cardinal \( n \), if \( K \in P_C \), then \( \mathcal{D}P(K) \in P_C \).

If, more generally, \( K \) is compact then \( \mathcal{D}P(K) \) is also compact; this follows from the last statement in the same (trivial) way as the general case in Corollary 3. Let us suppose that \( \pi(L) \) (the cardinality of \( L \)) is \( a \). If \( K \in P_C \) then \( \mathcal{D}P(K) \) is compact. This can be proved in the same way as Theorem 1, for in this case we clearly can require the set \( L = [\mathfrak{I}] \) of the proof of Theorem 1 to be of power \( a \). In this case \( \bigcup_{x \in \mathfrak{I}} \mathcal{D}P(K) = \mathcal{D}P(K) \), as easily follows from Theorem 6.8 of [4] after extending it to languages of arbitrary power.

Let \( \mathcal{D}P(K) \) denote, analogously, the class of all direct powers \( \mathfrak{I} \) of systems \( K \) in \( K \) such that \( \pi(L) = a \). If \( a \) is finite and \( K \in P_C \), then, as easily seen, \( \mathcal{D}P(K) \in P_C \). From (FY) we can easily infer that for every \( \mathfrak{I} \), \( \mathfrak{I} \in \mathcal{D}P(K) \), a sentence \( \varphi \in \mathfrak{I} \) can be given such that if \( \varphi \in \mathfrak{I} \) and \( x(L) \geq \mathfrak{I} \) then \( \mathfrak{I} \not= \mathfrak{I} \) and only if \( \mathfrak{I} \not= \mathfrak{I} \). In particular, \( \mathfrak{I} \) is \( \mathfrak{I} \) if \( x(L) \geq \mathfrak{I} \) (independently of \( x(L) \)). It follows that the mapping \( \varphi \) of \( \mathfrak{I} \) into itself defined by \( \varphi(x) = \mathfrak{I} \), \( \mathfrak{I} \in \mathfrak{I} \), \( x(L) \geq \mathfrak{I} \) is a continuous one. The image of a closed subset of a compact space under a continuous mapping is closed; applying this to \( \varphi \), we infer that \( \mathcal{D}P(K) \) is compact (for arbitrary \( L \)).

If \( \mathfrak{A} \) and \( L \in P \) is an indexed family of systems of a language \( L \) containing only predicate symbols, then any system \( \mathfrak{A} \) obtained as follows, is called a cardinal sum of the systems \( \mathfrak{A} \). Let \( \mathfrak{A} \) (for each \( i \in I \)) is a system isomorphic to \( \mathfrak{A} \) such that the sets \( \mathfrak{A} \) are mutually disjoint. Then let \( \mathfrak{A} = \bigcup_{i \in I} \mathfrak{A} \) and \( \mathfrak{A} = \bigcup_{i \in I} \mathfrak{A} \) for any \( P \in L \). If the systems \( \mathfrak{A} \) are all equal to a fixed \( \mathfrak{A} \), then the cardinal sum goes over to a cardinal of \( \mathfrak{A} \). For a class \( K \) of systems, let \( \mathfrak{C}(K) \) and \( \mathfrak{C}(K) \) be the class of all cardinal sums and multiples, respectively, of systems of \( K \).

As can be read from [4], for the cardinal sum there exists an exactly analogous criterion as (FY). By the help of this criterion we might prove similar results for cardinal sums to those we proved for direct products. But all these results can be obtained by the following stronger statement.
If $K \in \text{PC}_3$, then (a) $\text{CS}(K) \in \text{PC}_3$ and (b) $\text{CM}(K) \in \text{PC}_3$.

The proofs of these assertions are very simple: we only sketch them in a few words. Let us assume $K \in \text{PC}_3$. In order to ensure that $K \in \text{CS}(K)$ we require the existence of a binary relation $S(x, y)$ on $\mathbb{N}$ such that:

- $S(x, y)$ holds for every $x, y \in \mathbb{N}$; any two of the sets $S_x = \{ y : S(x, y) \text{ holds} \}$ are equal or disjoint; the subsystem of $\mathbb{N}$ with domain $S_x$ belongs to $K$ for any $x \in \mathbb{N}$; and finally, $P_S(x_1, \ldots, x_n) \in \mathbb{N}$ holds only if $a_1, \ldots, a_n$ are elements of the same set $S_x$.

It is easily seen that this requirement is equivalent, first, to $S \in K$ for a certain $K \in \text{CS}_3$, and secondly, to $S \in \text{CS}(K)$. In the case of (b) we add to our requirement that there should exist a ternary operation $\varphi(x, y, z)$ such that for any fixed $x, y \in \mathbb{N}$, $\varphi(x, y, z)$ is an isomorphism of $S_x$ onto $S_y$.

In order to be able to extend all the results proved for direct products to cardinal sums, we only have to add the remark that the cardinal sum preserves elementary equivalence (see Theorem 5.1 in [1]).

Finally we give a compactness result concerning countably weak direct products. We define this notion as it was given in [4] (p. 71, 4.3, using 4.2). Let $L$ be an arbitrary language, $E(a)$ a formula of $L$ containing no free variable except $x$, $\Phi^L(x) : i L$ a (non-empty) indexed family of systems of $L$. The countably weak direct product of the systems $\mathfrak{M}_i$ (relative to $E(a)$) is the subsystem of $\mathfrak{M}$ of $\prod_{i \in \mathbb{N}} \mathfrak{M}_i$ such that $\mathfrak{M}$ is the set of indices $i$ with $\mathfrak{M}_i \models E(a)$, for which the set of indices $i$ with $\mathfrak{M}_i \models E(a)$ is at most countable.

The countably weak direct power of a system $K$ is defined in the natural way. Let us denote by $\text{CW}(K)$ and $\text{CWP}(K)$ the classes of all systems isomorphic to some countably weak direct product and power, respectively, of systems in $K$.

**Theorem 3.** If the language $L$ is at most countable, $K \subseteq L$, and $K$ is compact, then (a) $\text{CW}(K)$ and (b) $\text{CWP}(K)$ are compact.

**Proof.** Consider (a). This is how we can modify the proof of Theorem 1 to yield this result: First we can apply an analogue of the criterion (FV).

The new criterion can be given with the help of [4], namely the final remarks of § 3 at the bottom of p. 89, Theorems 7.1 and 3.2 and the discussion 4.3 of § 4. Using this criterion we replace $\Phi'$ in the proof of Theorem 1 by a formula $\Phi''_L$, $\Phi'_L$ playing a similar role in the new proof to that of $\Phi'$ in the old one, such that $\Phi''_L$ is a formula of the predicate calculus $\Lambda$ of Frehken [3]. $L$ is obtained by adding the quantifier "there exists at most countably many ..." to the first order logic based on the language $L$. We then apply a result of Frehken [5] which says that if $\Gamma$ is a set of sentences of $L$ and every finite subset is satisfiable, then $\Gamma'$ is itself satisfiable. We use this instead of CT to prove the analogue $\Gamma'$ of the set $\Gamma$ (see (3)) in the present situation is satisfiable. We can easily see that Lemma 1 holds also in this case and hence that the proof of Theorem 2 can be extended to prove (b) or a more general result like Theorem 2.

**References**


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