of uniform convergence. Further we put \( F = \{ f : f \in G \text{ and } \{ f \} = Z \} \) and we define the map \( \varphi \) of \( G \) onto \( F \) by the formula: \( \varphi(g) = gZ \). It is easy to check that the triple \( G, F, \varphi \) satisfies all the conditions of Theorem 1, whence \( F = \varphi(G) \) is Borel.

Finally let us observe that, in fact, the Lemma of [3] yields the following general proposition.

**Proposition.** If a continuous map \( \varphi \) from a metric, separable and complete space \( X \) into a metric space satisfies the condition

\[
(C') \quad \varphi^{-1}(U) \text{ is open for every open } U \subseteq X,
\]

then \( \varphi(X) \) is an absolutely Borel set.

This proposition yields Theorem 1 of this paper since \( (C') \) implies \( (C) \).

References


Some theorems of set theory and their topological consequences

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It is generally known that numerous theorems of topology have a set-theoretical content, and some even prove to be theorems of set theory formulated in a different language. In such a situation it is often possible to formulate the corresponding theorem of set theory and prove it, and finally — by applying some substitution — pass to a topological theorem. This usually permits a simplification of the proof and a better understanding of the content of the theorem.

The object of the present paper is to give such proofs of several theorems of general topology concerning Cartesian products, among them the well-known theorems of Bockstein and Mareczewski (and some of their generalizations), and the Hewitt-Marczewski-Pondiczery theorem on dense subsets of Cartesian products \( \dagger \). All topological theorems of the present paper are grouped in the second part and are deduced from three theorems of set theory which are presented in the first part and which, as we think, may be interesting in themselves.

1. **Three theorems of set theory.** In the proof of Theorem 1 we shall use Theorem I(ii) of [4] (a simple proof of this theorem is given in [10]), which permits one to estimate the power of a family of sets with the help of its elements and the powers of its quasi-disjoint subfamilies. Recall that a family of sets \( \mathcal{F} \) is called quasi-disjoint if \( \mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset \) for distinct \( \mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{F} \). The theorem of Erdős-Rado mentioned above tells us that if a family \( \mathcal{G} \) is composed of sets of power at most \( n \) and the power of every quasi-disjoint subfamily \( \mathcal{G} \subseteq \mathcal{F} \) does not exceed \( m' \geq n \), then \( \mathcal{G} \subseteq m' \) \( \dagger \). \( \dagger \) By the symbol \( I(\mathcal{G}) \) we mean the intersection of all sets belonging to \( \mathcal{G} \). The union of all sets belonging to \( \mathcal{W} \) is denoted by \( S(\mathcal{W}) \).

\( \dagger \) Small Gothic letters denote finite and infinite cardinal numbers. The symbol \( \mathcal{X} \) denotes the power of the set \( X \).
**Theorem 1.** If families $\mathcal{A}$ and $\mathcal{B}$ are composed of sets of power not exceeding $n$ and $m$ respectively, where $n \geq m \geq n_0$ and

\[
A \cap B \neq \emptyset \quad \text{for all} \quad A \in \mathcal{A}, \ B \in \mathcal{B},
\]

then there exists a set $N$ of power not greater than $m^*$ such that

\[
A \cap N \neq \emptyset \quad \text{for all} \quad A \in \mathcal{A}, \ B \in \mathcal{B}.
\]

**Proof.** Let us suppose that the family $\mathcal{A}$ is well ordered by the relation $\subseteq$. Let

\[
\mathcal{B}^* = \{ T \subset S(\mathcal{A}) : T \cap B \neq \emptyset \text{ for all } B \in \mathcal{B} \}.
\]

For all $A_4 \in \mathcal{A}$ the family

\[
\mathcal{U}(A_4) = \{ A \in \mathcal{A} : A \hat{\cap} A_4 \in \mathcal{B}^* \}
\]

is non-empty, since it contains $A_4$. Denote by $A_4^*$ the first set (with respect to the ordering $\hat{\subseteq}$) belonging to it and let

\[
T_d = A_4 \hat{\cap} A_4^*;
\]

clearly $T_d \in \mathcal{B}^*$ for all $A_4 \in \mathcal{A}$.

Let us now consider sets $A_4, A_5 \in \mathcal{A}$ such that $T_d \cap T_{d_1} \in \mathcal{B}^*$. Then we have

\[
T_d \cap T_{d_1} = A_4 \hat{\cap} A_4^* \hat{\cap} A_5 \hat{\cap} A_5^* = (A_4 \hat{\cap} A_5) \hat{\cap} (A_4^* \hat{\cap} A_5^*),
\]

from which it follows, in view of the fact that $\mathcal{B}^*$ contains all sets $S(\mathcal{A})$ which include it, that $A_4^* \hat{\subseteq} A_5^*$ and $A_5^* \hat{\subseteq} A_4^*$. Hence

\[
\text{if} \quad T_d \cap T_{d_1} \in \mathcal{B}^*, \text{ then } A_4^* = A_5^*;
\]

Let $\mathcal{I} = \{ T_d \}_{d \in T}$ and let $\mathcal{I}_d$ be an arbitrary quasi-disjoint subfamily of $\mathcal{I}$. We consider first the case

\[
\text{(i) } I(\mathcal{I}_d) \in \mathcal{B}^*.
\]

Since $T_d \cap T_{d_1} = I(\mathcal{I}_d)$ for distinct $T_d, T_{d_1} \in \mathcal{I}_d$, on the basis of (3) and the definition of the sets $T_d$ there exists an $A_4 \in \mathcal{A}$ such that

\[
T_d \cap A_4^* = A_4 \cap A_4^* = A_4 \cap A_d = A_d.
\]

From this it follows that $S(\mathcal{I}_d) \subset A_4$ and — in view of the quasi-disjointness of the family $\mathcal{I}_d$ and the condition $A_d \leq n \leq m$ — that $\mathcal{I}_d \leq m^*$.

If (i) does not hold, then

\[
\text{(ii) } I(\mathcal{I}_d) \notin \mathcal{B}^*, \text{ i.e. there exists a } B_4 \in \mathcal{B} \text{ such that } I(\mathcal{I}_d) \cap B_4 = \emptyset.
\]

Since $T_d \cap B_4 \neq \emptyset$ for all $T_d \in \mathcal{I}_d$, we have

\[
(T_d \setminus I(\mathcal{I}_d)) \cap B_4 \neq \emptyset \quad \text{for } T_d \in \mathcal{I}_d;
\]

and hence in view of the quasi-disjointness of the family $\mathcal{I}_d$ and the condition $\mathcal{I}_d \leq m$ we conclude that $\mathcal{I}_d \leq m$.

Hence the power of a quasi-disjoint subfamily $\mathcal{I} \subset \mathcal{I}$ does not exceed $m$ and from the theorem of Erdős-Rado we have $I \leq m^*$, from which it follows that the set $N = S(\mathcal{I})$ satisfies the conclusion of the theorem.

**Corollary.** If families $\mathcal{A}$ and $\mathcal{B}$ are composed respectively of finite sets and sets of power at most $m \geq n_0$ and satisfy condition (1), then there exists a set $N$ of power at most $m$ which satisfies condition (2).

**Proof.** Observe that $\mathcal{A} = \bigcup_{i=1}^n \mathcal{A}_i$, where

\[
\mathcal{A}_i = \{ A \in \mathcal{A} : A \hat{\subseteq} \mathcal{A}_i \text{ has exactly } i \text{ elements} \}.
\]

Using Theorem 1 we obtain a set $N_i$ of power $\leq m^* = m$ such that

\[
A \cap B \cap N_i \neq \emptyset \quad \text{for } A \in \mathcal{A}_i \text{ and } B \in \mathcal{B}.
\]

It is easy to check that the set $N = \bigcup_{i=1}^n N_i$ satisfies the required condition.

In [2] a theorem is proved establishing a special case of the above corollary. By modifying slightly the proof of that theorem one gives a direct proof of the above corollary. A variant of Theorem 1 with $n = m = m_0$ and $m_0 + 1$ in place of $m^*$ is obtained, by using the continuum hypothesis, in [9].

**Theorem 2.** If the families of sets $\{ A_i \}_{i \in T}$ and $\{ B_i \}_{i \in T}$ are composed of sets of power not exceeding respectively $n$ and $m$, where $n \leq m \leq n_0$ and

\[
A_i \cap B_i \neq \emptyset \quad \text{and} \quad A_i \cap B_{i'} = \emptyset \quad \text{for all } i, i' \in T, i \neq i',
\]

then $T \leq m$.

**Proof.** The families $\mathcal{A} = \{ A_i \}_{i \in T}$ and $\mathcal{B} = \{ B_i \}_{i \in T}$ satisfy the hypotheses of Theorem 1, where in place of $n$ we must take $n+1$. For the set $N$ satisfying the conclusion of that theorem and an arbitrary $t \in T$ we have

\[
(A_t \setminus \{ t \}) \cap (B_t \setminus \{ t \}) = \{ t \} \subset N
\]

and hence $T \leq m$ and $T \leq m^* = m^*$.

The corollary below follows from Theorem 2 in the same way as the previous Corollary follows from Theorem 1; it can also be deduced from the Corollary to Theorem 1.

**Corollary.** If the families of sets $\{ A_i \}_{i \in T}$ and $\{ B_i \}_{i \in T}$ are composed respectively of finite sets and sets of power at most $m \geq n_0$ and satisfy condition (4), then the power of the set $T$ does not exceed $m$. 
Remark. Theorem 2 can be derived directly from the theorem of Erdős-Rado. Let $T_0 \subset T$ be a non-empty set of indices such that the family $(A_t)_{t \in T}$ is quasi-disjoint. Choose $t_0 \in T_0$ and notice that $B_{t_0} \cap \bigcap_{t \in T_0} A_t = \emptyset$. For every $t \in T_0 \setminus \{t_0\}$ we have $(A_t \setminus \bigcap_{t' \in T_0 \setminus \{t_0\}} A_{t'}) \cap B_{t_0} \neq \emptyset$, and hence, in view of the quasi-disjointness of the family $(A_t)_{t \in T}$, the condition $B_{t_0} \subset m$, we conclude that $\bar{T} \leq m - 1 = m$, since $A_t \neq A_{t'}$ for $t \neq t'$. It follows that $\bar{T} \leq m$.

In the proof of Theorem 3 we shall use theorem II of [5], which states that an arbitrary set $A$ of power $m > \aleph_0$ includes $2^m$ independent subsets, i.e., that there exists a family $\mathcal{S}$ of subsets of the set $A$ such that $\mathcal{S} = 2^m$ and
\[ S_1^2 \cap S_2^3 \cap \ldots \cap S_k^k \neq \emptyset, \quad \text{where} \quad S_1 = S, \quad S_2 = A \setminus S, \]
for an arbitrary finite sequence $S_1, S_2, \ldots, S_k$ of distinct sets from $\mathcal{S}$ and an arbitrary finite sequence $i_1, i_2, \ldots, i_k$ of zeros and ones.

**Theorem 3.** For every set $A$ of power $m > \aleph_0$, there exists a family $\mathcal{S}$ of functions mapping the set $A$ into itself such that $\mathcal{S} = 2^m$ and for an arbitrary finite sequence $f_1, f_2, \ldots, f_k$ of distinct functions from $\mathcal{S}$ and an arbitrary finite sequence $n_1, n_2, \ldots, n_k$ of elements of the set $\mathcal{S}$ there exists an $a \in A$ such that
\[ f_i(a) = n_i \quad \text{for} \quad i = 1, 2, \ldots, k. \]

Proof. Let $\mathfrak{M}$ be the family of all sequences $(n_1, n_2, \ldots, n_k; F_1, F_2, \ldots, F_k)$ where $k$ is an arbitrary natural number, $n_1, n_2, \ldots, n_k$ are elements of $A$, and $F_1, F_2, \ldots, F_k$ are finite subsets of $A$. Since $\mathfrak{M} = m$, there exists a one-to-one map $k$ of the set $\mathfrak{M}$ onto the family $\mathcal{S}$.

Let $\mathcal{S}$ be a family of power $2^m$ of independent subsets of $A$ and $a_n$ an arbitrary element of $A$. For every $S \in \mathcal{S}$ we define a function $f_S$ mapping the set $A$ into itself. Accordingly we consider for every $S \in \mathcal{S}$ the sequence $h(a) = (a_1, a_2, \ldots, a_k; F_1, F_2, \ldots, F_k)$ and set
\[ f_S(a) = \begin{cases} a_i & \text{if there exists } i \leq k \text{ such that } F_i \subseteq S \\ a_0 & \text{otherwise} \end{cases} \quad \text{for } j \neq i, \]
(5)\[ f_S(a) = a_i \quad \text{if there exists } i \leq k \text{ such that } F_i \subseteq S \quad \text{and} \quad F_j \not\subseteq S \quad \text{for } j \neq i, \]
\[ f_S(a) = a_0 \quad \text{otherwise}. \]

We shall show that the family $\mathcal{S} = \{f_S\}_{S \in \mathcal{S}}$ satisfies the conclusion of the theorem. Let $f_{S_1}, f_{S_2}, \ldots, f_{S_k}$ be an arbitrary finite sequence of functions, where $S_1, S_2, \ldots, S_k$ are distinct sets from $\mathcal{S}$ and $a_1, a_2, \ldots, a_k$ are elements of $A$. From the independence of the sets of the family $\mathcal{S}$ it follows that there exist elements $a_{ij} \in S_i \setminus S_j$ for $i, j = 1, 2, \ldots, k$ and $i \neq j$.

Let $F_i = \{a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k\}$; then
\[ F_i \subseteq S_i \quad \text{and} \quad F_j \not\subseteq S_i \quad \text{for } j \neq i, \]
and hence, in agreement with (5), we conclude that
\[ a = h^{-1}(a_1, a_2, \ldots, a_k; F_1, F_2, \ldots, F_k) \]
we have
\[ f_S(a) = a_i \quad \text{for } i = 1, 2, \ldots, k. \]

Since the functions $f_S$ and $f_{S'}$ are distinct for $S \neq S'$, it follows that
\[ \mathcal{S} = \{f_S\}_{S \in \mathcal{S}} = 2^m. \]

**Remark 1.** The property of a family $\mathcal{S}$ satisfying the conclusion of Theorem 3 may be expressed differently by saying that, for an arbitrary sequence of distinct functions $f_1, f_2, \ldots, f_k$ chosen from the family $\mathcal{S}$, the function attaching to a point $a \in A$ the point $f(a) = (f(a), f(a), \ldots, f(a))$ belonging to the $k$th Cartesian power $A^k$ of $A$, is a map of $A$ onto $A^k$.

**Theorem 2.** Theorem 3 may also be formulated as follows:

For every set $A$ of power $m > \aleph_0$, there exists a family $\mathcal{S}$ of subsets of $A$, $(A_{x1}, x_1 \in T)$, where $S = m$ and $T = 2^m$, such that

\[ A_{x1} \cap A_{x2} = \emptyset \quad \text{for } x \neq x', \quad t \in T, \]
(6)
\[ \bigcup_{x \in T} A_{x1} = A \quad \text{for every } t \in T, \]
(7)
\[ \bigcup_{x \in T} A_{x1} = \emptyset \quad \text{for every } x \in T, \]
(8)
\[ \text{for every function } f: T \rightarrow S \text{ the family } (A_{x1}, x_1 \in T) \text{ is composed of independent (and distinct) subsets of } A. \]

In fact, assuming that Theorem 3 holds, we set $S = A$, $T = \mathbb{N}$, and $A_{x1} = f^{-1}(x)$. It is easy to check that conditions (6), (7), and (8) will then be satisfied. If, on the other hand, there exists a family $(A_{x1}, x_1 \in T)$ of subsets of $A$ satisfying these conditions, then setting $S = A$ and putting for $t \in T$ and $a \in A$
\[ f(a) = x \quad \text{where} \quad a \in A_{x1} \]
(on the basis of (6) and (7) there exists exactly one $x$ satisfying this condition) we obtain a family $\mathcal{S} = \{f(a)\}_{x \in T}$ satisfying the conclusion of Theorem 3.

**Remark 3.** It is not difficult to check that the above proof of Theorem 3 allows us to obtain the following stronger formulation:

For infinite cardinal numbers $m$ such that $r < m$ and an arbitrary set $A$ of power $\sum_{n=1}^{m} r^n$ there exists a family $\mathcal{S}$ of functions mapping $A$ into itself such that $\mathcal{S} = 2^m$ and for any index set $S$, where $\mathcal{S} \subseteq S$, any family $(f_{x1})_{x1 \in T}$ composed of distinct functions from $\mathcal{S}$ and any family $(a_{x1})_{x1 \in T}$ of elements of $A$, there exists $a \in A$ such that $f(a) = a_i$ for all $i \in S$.

Analogously we can also strengthen the formulation of Theorem 3 given in remarks 1 and 2.

Theorem 3 establishes a generalization of theorem I of Hausdorff [5] and, as follows from remark 2, also of the theorem about the existence
of independent sets. Strengthening Theorem 3 (in the formulation given in remark 2) by the method indicated in remark 3 we obtain a generalization of lemma 3.16 of Tarski [19].

2. Topological consequences. We shall denote the Cartesian product (with the Tychonoff topology) of the family \( \{ X_s \}_{s \in S} \) of topological spaces by the symbol \( P X_s \). For every \( K \subseteq S \) the projection \( p_K : \prod_{s \in S} P X_s \rightarrow P X_K \) is defined and continuous. In particular for each \( s_0 \in S \) the continuous transformation \( p_{s_0} : P X_s \rightarrow X_{s_0} \) is defined, the projection on the \( s_0 \)-th axis of the product \( P X_s \). Subsets of the product \( P X_s \) of the form

\[
K = \prod_{s \in S} K_s, \quad \text{where} \quad K_s \subseteq X_s,
\]

we shall call cubes; the set \( X_s \) will be called the \( s \)-th face of the cube \( K \) and the set

\[
D(K) = \{ s : K_s \neq X_s \}
\]

will be called the set of its distinguished indexes. A criterion for the disjointness of cubes is given by the following easily provable

**Lemma 1.** For non-empty cubes \( K = \prod_{s \in S} K_s \) and \( L = \prod_{s \in S} L_s \) of the Cartesian product \( P X_s \) the following equivalence holds:

\[
( K \wedge L = 0 ) = ( \text{there exists } s \in D(K) \cap D(L) \text{ such that } K_s \cap L_s = 0 ).
\]

Let us consider now the class \( K = (B_s)_{s \in S} \), where \( B_s \) is a family of sets \( s \)-th axis of the space \( X_s \), and set

\[
A(K) = \{ \bar{K} : s \in D(K) \},
\]

\[
B(K, R) = \bigcup_{s \in \bar{K}} \{ R \in B_s : R \cap X_s = 0 \},
\]

where, as above, \( K \) is a cube \( P X_s \). Thus

\[
A(K) = S_{\bar{K}} \quad \text{and} \quad B(K) = S_{\bar{K}}.
\]

**Lemma 1** immediately implies

**Lemma 2.** Suppose we are given a family \( \{ X_s \}_{s \in S} \) of disjoint topological spaces and the class \( K = (B_s)_{s \in S} \), where \( B_s \) is a family of subsets of the space \( X_s \). For non-empty cubes \( K = \prod_{s \in S} K_s \) and \( L = \prod_{s \in S} L_s \) of the Cartesian product \( P X_s \), where \( K_s \in B_s \) for \( s \in D(K) \), the following equivalence holds:

\[
( K \wedge L = 0 ) = ( A(K) \cap B(L) \cap R \neq 0 )
\]

A subset of a topological space \( X \) which is the intersection of \( m \geq n \) open sets is called a \( G^*_m \) set, and the union of an arbitrary number of \( G^*_m \) sets is called a \( G^*_m \) set. Instead of \( G^*_m \) and \( G^*_m \) we shall write \( G_t \) and \( G_{t_X} \),

For each \( s \in S \) let \( \mathcal{B}_s \) be a base for \( X_s \). A cube \( K = P X_s \), where \( D(K) \leq n \) and \( K_s \in \mathcal{B}_s \) for \( s \in D(K) \), will be called a basic cube with respect to the class \( B = (\mathcal{B}_s)_{s \in S} \). Clearly every open set in \( P X_s \) is the union of basic cubes. A cube \( K = P X_s \), where \( D(K) \leq m \) and \( K_s \) for every \( s \in S \) the intersection of at most \( m \) elements of the base \( \mathcal{B}_s \), will be called an \( m \)-cube \((t)\) with respect to the class \( B = (\mathcal{B}_s)_{s \in S} \). Every \( m \)-cube is a \( G^*_m \) set in \( P X_s \).

**Lemma 3.** Every \( G^*_m \) set in a Cartesian product \( P X_s \) is the union of \( m \)-cubes with respect to an arbitrary class \( B = (\mathcal{B}_s)_{s \in S} \).

**Proof.** It suffices to consider \( G^*_m \) sets. Hence let

\[
M = \bigcap_{t \in T} H_t, \quad \text{where} \quad T \leq m \quad \text{and} \quad H_t \quad \text{is open in} \quad P X_s,
\]

be an arbitrary \( G^*_m \) set. For every \( p \in M \) and \( t \in T \) let \( K(p, t) \) be a basic cube with respect to the class \( B \) satisfying the condition

\[
p \in K(p, t) \subseteq H_t.
\]

We then have

\[
p \in \bigcap_{t \in T} K(p, t) \subseteq M \quad \text{and} \quad M = \bigcup_{p \in M} \bigcap_{t \in T} K(p, t).
\]

The lemma follows from the fact that the intersection of at most \( m \) basic cubes is an \( m \)-cube.

**Theorem 4.** Let \( \{ X_s \}_{s \in S} \) be an arbitrary family of topological spaces such that \( w(X_s) \leq m \) for \( s \in S \) (t) and let \( U, V \subseteq P X_s \) be \( G^*_m \) and \( G^*_m \) sets respectively where \( n \leq m \). If \( U \cap V = 0 \), then there exists a set \( S_s \subseteq S \) such that \( S_s \leq \{ m \} \) \( G^*_m \) and \( p \in S \), \( U \cap p \not\in S \).

**Proof.** Without loss of generality we may assume that \( X_s \cap X_t = 0 \) for \( s \neq t \). Choose for each \( s \in S \) a base \( \mathcal{B}_s \) of the space \( X_s \) such that \( \mathcal{B}_s \leq m \). From Lemma 3 we have

\[
U = \bigcup_{t \in T} K_t \quad \text{and} \quad V = \bigcup_{s \in S} L_s,
\]

where \( K_t \) and \( L_s \) are respectively non-empty \( n \) and \( m \)-cubes with respect to the class \( B = (\mathcal{B}_s)_{s \in S} \). In view of the disjointness of sets \( U \) and \( V \) we have

\[
K_t \wedge L_s = 0 \quad \text{for} \quad t \in T, \quad s \in S
\]

(9) The intersection of the empty subfamily of \( \mathcal{B}_s \) is the whole space \( X_s \).

(10) The symbol \( w(X) \) denotes the weight of the space \( X \), i.e. the power of the least numerable base of the space.
We denote by $\mathfrak{B}$ the family of all subsets of $X_t$, which are intersections of not more than $n$ elements of the base $\mathfrak{B}$, and put $R = (\mathfrak{B})_{s \geq t}$. In particular the $s$th faces of the cubes $(K_t)_{s \geq t}$ belong to $\mathfrak{B}$. From condition (9) and Lemma 2 it follows that the families

$$\mathfrak{A} = (A(K_t))_{s \geq t} \quad \text{and} \quad \mathfrak{B} = (B(L_t, R))_{s \geq t}$$

satisfy condition (1) of Theorem 1. Since $A(K_t) \subseteq N$ for all $t \in T$ and $B(L_t, R) \subseteq m$ for every $t \in T$, it follows from Theorem 1 that there exists a set $N$ of power at most $(m - w)^n = (m - w)^n$ such that

$$A(K_t) \cap B(L_t, R) = \emptyset \quad \text{for} \quad t \in T \quad \text{and} \quad s \in S$$

Using Lemma 2 it is not difficult to check that the set

$$S_0 = \{ s \in S : \text{the $s$th face of some cube of the family $(K_t)_{t \in T}$ belongs to $N$} \}$$

satisfies the conclusion of the theorem.

**Corollary.** For every pair $U, V$ of disjoint $G_\delta$ sets in the Cartesian product $P X_t$ of a family of topological spaces $(X_t)_{t \in T}$, such that $w(X_t) \leq 2^n$ for $s \leq t$, there exists a set $S_0 \subseteq S$ such that $S_0 \subseteq 2^n$ and $p_{S_0}(U) \cap p_{S_0}(V) = \emptyset$.

**Remark.** The following example shows that the power of the set $S_0$ of the last corollary cannot be reduced.

Let $I$ denote the closed interval $[0,1]$ and for each $t \in I$ let $D_t$ be a copy of the two-point discrete space $(0,1)$. Let $p_t$ denote the projection of the product $I \times P D_t$ on the $I$-axis and $p_t$ the projection of this product on the $D_t$-axis. For every $t \in I$ consider the $\mathfrak{B}_t$-cubes

$$K_t = p_t^{-1}(0) \quad \text{and} \quad L_t = p_t^{-1}(1)$$

It is not difficult to check that the $G_\delta$ sets

$$U = \bigcup_{t \in I} K_t \quad \text{and} \quad V = \bigcup_{t \in I} L_t$$

are disjoint but their projections on $P D_t$ and $I \times P D_t$, where $I \subseteq T$, have a non-empty intersection.

The spaces considered in the above example have countable bases, whence our corollary cannot be strengthened even in this case. We have not succeeded, however, in finding an example of two disjoint $G_\delta$ sets in the Cartesian product of the family $(X_t)_{t \in T}$ of spaces with a countable base whose projections on every product $P X_t$ of spaces $X_t$ are $G_{\delta}$ sets, where $S_0 \subseteq S$ and $S_0 \subseteq 2^n$ have a non-empty intersection (compare Corollary 1 to Theorem 5 below). That such a situation is possible in a Cartesian product of spaces of weight $2^n$ is shown by the example considered above in which $I$ must be regarded as a discrete space of power $2^n$; the sets $U$ and $V$ in this case will be open.

**Theorem 6.** Let $(X_t)_{t \in S}$ be a family of topological spaces such that $w(X_t) \leq m$ for $s \leq t$ and let $U, V \subseteq P X_t$ be a open set and $G_\delta$ set respectively. If $U \cap V = \emptyset$, then there exists a set $S_0 \subseteq S$ such that $S_0 \subseteq 2^n$ and $p_{S_0}(U) \cap p_{S_0}(V) = \emptyset$.

**Proof.** We proceed as in the proof of Theorem 4 except that the corollary to Theorem 1 is used in place of the theorem itself.

**Corollary.** Any two disjoint open sets $U, V$ in the Cartesian product $P X_t$ of spaces with countable bases there exists a set $S_0 \subseteq S$ such that $S_0 \subseteq 2^n$ and $p_{S_0}(U) \cap p_{S_0}(V) = \emptyset$.

**Corollary.** Any closed $G_\delta$ set in the Cartesian product $P X_t$ of spaces with countable bases is of the form $p_{S_0}(U)$, where $S_0 \subseteq S$, $S_0 \subseteq 2^n$, and $U$ is an open $G_\delta$ set in the product $P X_t$.

Notice that the proof of Theorem 5 requires only the corollary to Theorem 1 and hence may be carried out without the use of the Erdős-Rado theorem, modifying the proof of the theorem in [2]. Corollary 2 to Theorem 5 is proved (in a somewhat different formulation) in [1].

Let us turn now to the application of Theorem 2.

**Theorem 6.** Any family $(K_t)_{t \in T}$ of non-empty and pairwise disjoint $G_\delta$ sets in the Cartesian product $P X_t$ of topological spaces such that $w(X_t) \leq m \geq n$ has power at most $w^n$.

**Proof.** Without loss of generality we may assume that $X_t \cap X_{t'} = \emptyset$ for $s \neq s'$. Choose for every $s \leq t$ a base $\mathfrak{B}_s$ of $X_t$ such that $\mathfrak{B}_s \subseteq \mathfrak{B}_t$. On the basis of Lemma 3 to Theorem 4, we can suppose that the family $(K_t)_{t \in T}$ is composed of $n$-cubes with respect to the class $B = (\mathfrak{B}_t)_{t \in T}$. Denote by $\mathfrak{B}_t$ the family of all subsets of $X_t$ which are intersections of not more than $n$ elements of the base $\mathfrak{B}_t$ and set $R = (\mathfrak{B}_t)_{t \in T}$. From Lemma 2 to Theorem 4 it follows that the families

$$A(K_t)_{t \in T} \quad \text{and} \quad B(K_t, R)_{t \in T}$$

(For further consequences of Theorem 5, see Theorems 4 and 6 in [3]).
satisfy the conditions of Theorem 2 if for \( m \) we take \( n \cdot \omega = \omega^n \) and hence \( \varphi_i \approx (\omega^n)^{m} = \omega^n \).

**Corollary.** Any family of non-empty and pairwise disjoint \( G \) sets in the Cartesian product \( P X_i \) of topological spaces, such that \( w(X_i) < 2^n \) for \( s \in S \), has power at most \( 2^m \).

**Theorem 7.** Any family of non-empty and pairwise disjoint open sets in the Cartesian product \( P X_i \) of a family of topological spaces \( (X_i)_{i \in I} \) such that \( w(X_i) \leq \omega \geq \omega_i \) for \( s \in S \), has power at most \( \omega \).

**Proof.** We proceed as in the proof of Theorem 6 except that the corollary to Theorem 2 is used in place of the Theorem itself.

**Corollary.** Any family of non-empty and pairwise disjoint open sets in the Cartesian product \( P X_i \) of spaces with countable bases is countable.

Notice that the proof of Theorem 7 requires only the corollary to Theorem 3 (which follows from the corollary to Theorem 1) and hence may be carried out without the use of the Erdős-Rado theorem. Theorem 1.2 of [10] is similar in character to our Theorem 6, but it appears to us that Theorem 6 does not follow from it. The corollary to Theorem 7 was first proved in [7].

From the final theorem of the first part we derive:

**Theorem 8.** The Cartesian product of not more than \( 2^n \) topological spaces which contain dense subsets of power \( \omega \) contains a dense subset of power \( \omega \).

**Proof.** Such a product contains a dense subset which is a continuous image of the product of \( 2^n \) copies of the discrete space of power \( m \) in a cube, the faces of which have power \( \omega \) and are dense in the respective spaces. We consider a set \( A \) of power \( \omega \) and a family \( \varphi \) of functions mapping \( A \) into itself which satisfies the conclusion of Theorem 3. It suffices to show that the product \( P X_f \), where for all \( f \in \varphi, X_f = A \) is the discrete space of power \( m \), contains a dense subset of power \( m \).

Let the function \( \varphi : A \to P X_f \) attach to a point \( a \in A \) the point \( \{f(a)\}_f \) of the product \( P X_f \). The power of the set \( \varphi(A) \subseteq P X_f \) does not exceed \( m \). To prove that the set \( \varphi(A) \) is dense in \( P X_f \), it suffices to check that for any finite sequence \( f_1, f_2, \ldots, f_k \) of distinct functions from \( \varphi \) and any finite sequence \( a_1, a_2, \ldots, a_k \) of elements of \( A \) there exists an \( f \in A \) such that \( \varphi(a_i) = a_i \), i.e., that \( f(a_i) = a_i \), for \( i = 1, 2, \ldots, k \).

The existence of such a point \( a \) is ensured by Theorem 3.

Theorem 9 was proved in [11] and [6] and, for the case \( m = \omega \), in [8]. It is not difficult to check that the Cartesian product of more than \( 2^n \) spaces of more than one point does not have a dense subset of power \( m \) (see [11] and [8]).

Finally, we note that from the theorems of the first part we can also derive theorems analogous to those above for \( p \)-box topologies in the product \( P X_f \), i.e., for topologies defined by a base composed of sets of the form \( \bigcup_{\omega} \bigcup_{U \in \mathcal{U}} U \), where \( U \) is an open set in \( X_f \) and \( \mathcal{U} < p \). We shall not formulate these theorems here since they are less interesting, but the reader, if he wishes, will be able to do so without the least difficulty.

**References**


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