References

[1] E. W. Beth, *On Padoa's method in the theory of definition*, Indag. Math. 15 (1953), pp. 330-339.

[2] W. Craig, *Linear reasoning. A new form of the Herbrand-Gentzen theorem*, Jour. Symb. Logic 22 (1957), pp. 250-268.

[3] E. Engeler, *A reduction-principle for infinite formulas*, Math. Ann. 151 (1963), pp. 296-303.

[4] C. R. Karp, *Completeness proofs in the predicate logic with infinitely long expressions*, To appear in The Journal of Symbolic Logic.

[5] — *Finite-quantifier equivalence*, To appear in the Proceedings of the Symposium on Model Theory held at Berkeley in 1963.

[6] H. J. Keisler, *Ultraproducts and elementary classes*, Indag. Math. 23 (1961), pp. 477-495.

[7] S. C. Kleene, *Introduction to Metamathematics*, Van Nostrand, New York, 1952.

[8] C. Kuratowski, *Topologie I*, 4-ème éd., Warszawa 1958.

[9] E. G. K. Lopez-Escobar, *Infinitely long formulas with countable quantifier degrees*, Doctoral Dissertation at the Univ. of California, Berkeley, 1964.

[10] R. Lyndon, *Properties preserved under homomorphisms*, Pacific Journ. Math. 9 (1955), pp. 143-154.

[11] S. Maehara and G. Takeuti, *A formal system of first-order predicate calculus with infinitely long expressions*, Journ. Math. Soc. Japan 13 (1961), pp. 357-370.

[12] — *The first order predicate logic with infinitely long expressions with equality*, To appear in the Journal of the Math. Soc. of Japan.

[13] D. S. Scott, *Logic with denumerably long formulas and finitary quantifiers*, To appear on the Proceedings of the Symposium on Model Theory held at Berkeley in 1963.

Re u par la Rédaction le 22. 10. 1964

On a Freedman's problem

by

C. Ryll-Nardzewski (Wrocław)

Given a compact and metrisable space X, let us consider the space 2^X of all closed subsets of X (cf. [1], p. 106) and an arbitrary fixed $Z \in 2^X$. Let $Z^0 \subset 2^X$ denote the set of all homeomorphs of Z contained in X. D. Freedman has conjectured that Z^0 is always a Borel set in 2^X and, in fact, for the case where X is the Cantor dyadic set this was proved by D. Scott ([2], pp. 126-128). Our aim now is to prove the general statement for an arbitrary X. This is based on a refinement of the method of [3] (1).

THEOREM 1. *If F is an arbitrary group of autohomeomorphisms of a separable topological space G admitting a complete metrization* (2) *and a continuous map φ from G into a metric space satisfies the condition*

(C) *the class of all level-sets of φ and the class of all F-orbits are identical, i.e.*

$$\big\{\{g\colon \varphi(g)=\varphi(g_0)\}\colon g_0\in G\big\} = \big\{\{gf\colon f\in F\}\colon g\in G\big\},$$

then $\varphi(G)$ is an absolutely Borel set (i.e. every homeomorph of $\varphi(G)$ in any metric space is Borel).

Proof. The decomposition of G given by $F_g=\{gf\colon f\in F\}$ is open in the sense of [3] since

$$\{g\colon F_g\cap U\neq 0\}=\bigcup_{f\in F}\{g\colon gf\in U\}.$$

Let S be a Borel selector given by the Lemma (see [3], p. 129). The continuous mapping φ is one-to-one on S and $\varphi(S)=\varphi(G)$ (S is a selector), whence $\varphi(G)$ is an absolutely Borel set (cf. [1], p. 396).

THEOREM 2. *The set Z^0* (introduced at the begining) *is Borel.*

Proof. The set G of all homeomorphisms of Z into X is a G_δ set in the space X^Z of all continuous maps of Z into X with the topology

(1) [2] and [3] give information on other topics similar to those presented in this note.

(2) Let us recall that every G_δ set in a complete metric space always admits a complete metrization topologically equivalent to the original one ([1], p. 316), e.g. the set $N!$ of [2] is such a set in the space N^N of [2].

of uniform convergence. Further we put $F = \{f: f \in G \text{ and } fZ = Z\}$ and we define the map φ of G onto Z^0 by the formula: $\varphi(g) = gZ$. It is easy to check that the triple G, F, φ satisfies all the conditions of Theorem 1, whence $Z^0 = \varphi(G)$ is Borel.

Finally let us observe that, in fact, the Lemma of [3] yields the following general proposition.

PROPOSITION. *If a continuous map φ from a metric, separable and complete space X into a metric space satisfies the condition*

(C′) $\varphi^{-1}\varphi(U)$ *is open for every open* $U \subseteq X$,

then $\varphi(X)$ *is an absolutely Borel set.*

This proposition yields Theorem 1 of this paper since (C) implies (C′).

References

[1] C. Kuratowski, *Topologie I*, Warszawa-Wrocław 1958.

[2] Dana Scott, *Invariant Borel Sets*, Fund. Math. 56 (1964), pp. 117-128.

[3] C. Ryll-Nardzewski, *On Borel measurability of orbits*, ibidem 56 (1964), pp. 129-130.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 6. 11. 1964

Some theorems of set theory and their topological consequences

by

R. Engelking and **M. Karłowicz** (Warszawa)

It is generally known that numerous theorems of topology have a set-theoretical content, and some even prove to be theorems of set theory formulated in a different language. In such a situation it is often possible to formulate the corresponding theorem of set theory and prove it, and finally — by applying some substitution — pass to a topological theorem. This usually permits a simplification of the proof and a better understanding of the content of the theorem.

The object of the present paper is to give such proofs of several theorems of general topology concerning Cartesian products, among them the well-known theorems of Bockstein and Marczewski (and some of their generalizations), and the Hewitt-Marczewski-Pondiczery theorem on dense subsets of Cartesian products (1). All topological theorems of the present paper are grouped in the second part and are deduced from three theorems of set theory which are presented in the first part and which, as we think, may be interesting in themselves.

1. Three theorems of set theory. In the proof of Theorem 1 we shall use theorem I(ii) of [4] (a simple proof of this theorem is given in [10]), which permits one to estimate the power of a family of sets with the help of the powers of its elements and the powers of its quasi-disjoint subfamilies. Recall that a family of sets $\mathfrak{A}$ is called *quasi-disjoint* if $A_1 \cap A_2 = \mathbf{I}(\mathfrak{A})$ for distinct $A_1, A_2 \in \mathfrak{A}$ (2). The theorem of Erdös-Rado mentioned above tells us that if a family $\mathfrak{A}$ is composed of sets of power at most $\mathfrak{n}$ and the power of every quasi-disjoint subfamily $\mathfrak{A}_0 \subset \mathfrak{A}$ does not exceed $\mathfrak{m} \geqslant \aleph_0$, then $\overline{\overline{\mathfrak{A}}} \leqslant \mathfrak{m}^{\mathfrak{n}}$. (3)

(1) Recently K. A. Ross and A. H. Stone gave in [12] a very simple and elegant proof of the theorem of Bockstein and certain of its generalizations. It appears (see [3]) that by a similar method one can also obtain some results of the present paper.

(2) By the symbol $\mathbf{I}(\mathfrak{A})$ we mean the intersection of all sets belonging to $\mathfrak{A}$. The union of all sets belonging to $\mathfrak{A}$ is denoted by $\mathbf{S}(\mathfrak{A})$.

(3) Small Gothic letters denote finite and infinite cardinal numbers. The symbol $\overline{\overline{A}}$ denotes the power of the set A.