11. Lemma. If every set can be linearly ordered, then $\mathfrak{S}_\\omega$ holds.

The truth of 11 was first observed by Kuratowski. The proof is not difficult and may be found in Sierpiński [6], p. 412.

12. Lemma. The conjunction of $\mathfrak{S}_\omega$ and $\mathfrak{S}_\omega$ is equivalent to $\mathfrak{S}_\omega$.

Proof. Obviously $\mathfrak{S}_\omega$ implies both $\mathfrak{S}_\omega$ and $\mathfrak{S}_\omega$. On the other hand, $\mathfrak{S}_\omega$ says that every set $X$ of non-empty sets there is a function $f$ such that for $X \in X$, $f(X) \in \mathcal{P}_\omega(X)$, while $\mathfrak{S}_\omega$ says there is a choice function $f$ on $\mathcal{P}_\sigma(\bigcup X)$. Since $\mathcal{P}_\sigma(X) \subseteq \mathcal{P}_\sigma(\bigcup X)$ for each $X \in X$, it follows that $g(f)$ is the desired choice function.

We are now in position to prove

13. Theorem. If $\mathfrak{S}$ is consistent, then the axioms of $\mathfrak{S}$ do not imply $\mathfrak{S}_\alpha$ for any $\alpha \in \beta$.

Proof. Since for each $\alpha \in \beta$, $\mathfrak{S}_\alpha$ implies $\mathfrak{S}_\sigma$, it is sufficient to prove that $\mathfrak{S}_\sigma$ is independent of the axioms of $\mathfrak{S}$. In view of Lemmas 12 and 13 we see that $\mathfrak{S}_\sigma$, together with the supposition that every set can be linearly ordered, implies the axiom of choice. Since Mostowski [2] has shown that the axioms of $\mathfrak{S}$, together with the principle of linear ordering do not imply the axiom of choice, it is clear that they cannot imply $\mathfrak{S}_\sigma$. It follows that the axioms of $\mathfrak{S}$ alone cannot imply $\mathfrak{S}_\sigma$.

It is clear from the above independence result that $\mathfrak{S}_\sigma$ is independent of the conjunction of the axioms of $\mathfrak{S}$ and $\mathfrak{S}_\sigma$, and conversely that $\mathfrak{S}_\sigma$ is independent of the conjunction of the axioms of $\mathfrak{S}$ and $\mathfrak{S}_\sigma$. In view of this independence 13 gives a nice decomposition of the axiom of choice into independent, heuristically complementary statements. The feeling that the statements are complementary is strengthened by the fact that if $\mathfrak{S}_\sigma$ is replaced by $\mathfrak{S}_\sigma$ the new conjunction does not imply the axiom of choice.

In view of 9 it would be interesting to know if there is a prime $\mathfrak{p} \neq \mathfrak{p}$ such that $\mathfrak{S}_\sigma$ implies $\mathfrak{S}_{\mathfrak{p}}$, where $\mathfrak{p} = \mathcal{P}(\mathfrak{p})$.

References


An interpolation theorem for denumerably long formulas

by

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O. Introduction. $\mathfrak{L}_\omega$ is the first-order language obtained by modifying the usual formation rules for the finite first-order formulas so that conjunctions and disjunctions of less than $\beta$ individual variables are allowed (thus in particular, $\mathfrak{L}_\omega$ is the usual (finite) first-order language) (1). We say that the interpolation theorem is true for $\mathfrak{L}_\omega$ just in case that for all formulas $\varphi, \phi$ of $\mathfrak{L}_\omega$ if the implication $\varphi \rightarrow \phi$ is valid, then there exists a formula $\pi$ of $\mathfrak{L}_\omega$ such that (1) $\varphi \rightarrow \pi$ and $\pi \rightarrow \phi$ are valid, (2) if a variable occurs free in $\pi$, then it occurs free both in $\varphi$ and in $\phi$, and (3) if a relational symbol occurs (occurs positively, occurs negatively) in $\pi$, then it occurs (occurs positively, occurs negatively) both in $\varphi$ and in $\phi$.

The interpolation theorem is known to be true for $\mathfrak{L}_\omega$ see Craig [3] and Lyndon [10] and, whenever $\theta$ is an inaccessible cardinal, for $\mathfrak{L}_\omega$ see Macintyre-Takeuti [11]". In this paper we show that the interpolation theorem is true for $\mathfrak{L}_\omega^\infty$.

The interpolation theorem is obtained as a consequence of the completeness of a Gentzen type formalization for $\mathfrak{L}_\omega$. The essential property of our rules of inference, in addition to the usual subformula property associated with Gentzen type systems, is that the number of variables that occur free in the premise(s) and do not occur free in the conclusion (i.e. are quantified out) is always finite (2).

(1) These kinds of infinitary languages (i.e. infinitary languages $\mathfrak{L}_\omega$ where $\omega$ need not be equal to $\beta$) have been studied by C. R. Karp [4].

(2) Added in proof. J. Malitz has shown that the interpolation theorem for $\mathfrak{L}_\omega$ is false, cf. Notices Amer. Math. Soc. 12 (1965), p. 373.

(3) E. Engeler [3] has obtained a formalization for $\mathfrak{L}_\omega^\infty$ which has the subformula property, however his formalization does not have the property mentioned above and thus it is unsuitable for deriving the interpolation theorem (the author did not learn Fundamenta Mathematicae, 57/88).
After giving some immediate applications of the interpolation theorem for \( L_{\omega1\omega} \), we show that unlike the case of \( L_{\omega1\omega} \), the interpolation theorem is not true for sets of formulas of \( L_{\omega1\omega} \) classes which can not be separated by a class which is closed under \( L_{\omega1\omega} \) elementary equivalence (cf. Theorem 6.3).

1. Preliminaries. We will distinguish between classes and sets, namely, a set is a class which is a member of some other class. We assume the axiom of choice and any of its equivalent formulations. For the most part standard set-theoretical notation will be used. Thus, for example, \( \epsilon \) is the membership relation and \( \cup, \cap, \cup, \cup, \cap \) respectively the relation of inclusion, the empty set, and the operations of forming unions (interscetions) of two and arbitrary many classes. \( X \rightarrow Y \) is the set-theoretical difference of the classes \( X \) and \( Y \) and if \( Z \) is a set then \( P(Z) \) is the set of all subsets of \( Z \). If \( t(z) \) is a term containing \( z \) as a free variable, then a symbolic expression of the form \( t(t(a) \in \Phi) \) where \( \Phi \) is to be replaced by any formula containing \( z \) as free variable denotes the class of all \( t(z) \) for which \( z \) satisfies the given formula. \( (x, y) \) is the ordered pair with \( x \) as its first term and \( y \) as its second term. A function is a set of ordered pairs satisfying the usual conditions and if \( f \) is a function then \( Df \) and \( Df \) respectively are the domain and the range of \( f \). If \( f \) is a function and \( x \in Df \), then \( f(x) \) or sometimes \( f_x \) is the value of \( f \) at \( x \); thus \( f = \{ (x, f(x)); x \in Df \} \). \( A^X \) is the class of all functions whose domain \( A \) and whose range is included in \( X \). If \( X \) is the restriction of \( f \) to \( X \), is the class \( \{ (x, f(x)); x \in X \cap Df \} \). Note that the domain of \( f \setminus X \) is always \( X \).

Ordinals are assumed to have been defined in such a way that an ordinal is the set of all smaller ordinals and we shall use the letters: \( \mu, \lambda, \gamma, \delta \) to represent arbitrary ordinals. The formulas \( \mu < \nu, \mu \geq \nu \) and \( \mu \in \nu \) are equivalent and will be used interchangeably. \( \mu + \nu \) is the ordinal addition of \( \mu \) and \( \nu \). If a function \( f \) has an ordinal \( \mu \) as its domain, then \( f \) is called a \( \mu \)-sequence (or simply a \( \mu \)-sequence) and \( \langle \mu, f \rangle \subseteq \mu \times \nu \). The letters: \( \epsilon, \delta, \gamma, \delta \) will be used to represent natural numbers (i.e. finite ordinals) and \( \langle \epsilon_0, \ldots, \epsilon_n, \rangle \) is the \( n \)-sequence \( \langle \epsilon_0, \ldots, \epsilon_n, \rangle \). If \( \epsilon \in \langle \epsilon_0, \ldots, \epsilon_n, \rangle \) and \( t = \langle \epsilon_0, \ldots, \epsilon_n, \rangle \), then concatenation of \( \epsilon \) and \( t \) in symbols: \( \epsilon \circ t \), is the \( \epsilon + \theta \)-sequence \( \epsilon \circ t = \langle \epsilon + \mu, \lambda \rangle; \mu < \delta \rangle \). If \( \langle \epsilon, f \rangle \subseteq \langle \epsilon, \delta \rangle \), then \( \langle \epsilon, f \rangle \subseteq \langle \epsilon, \delta \rangle \).

By a cardinal it is understood an initial ordinal, i.e. an ordinal whose power exceeds the power of each smaller ordinal. \( \omega \) is the smallest infinite cardinal and \( \omega_1 \) is the least cardinal greater than \( \omega \) (i.e. the smallest uncountable ordinal), \( |A| \) is the cardinality of the set \( A \).

(1) In Engelking’s axiomatization till after obtaining the interpolation theorem. For another axiomatization of \( L_{\omega1\omega} \), see Karp (4).

2. The language \( L_{\omega1\omega}. \) The language \( L_{\omega1\omega} \) can briefly be described as follows. The symbols of \( L_{\omega1\omega} \) include a set \( V \) of \( \omega \) distinct individual variables and we shall use the letters (with or without subscripts): \( v, w, x, y, \) and \( z \) to denote the elements of \( V \) (i.e. individual variables). The equality symbol \( = \) and the relational symbols: \( R_0, \ldots, R_n, \ldots \) are included amongst the symbols of \( L_{\omega1\omega} \). The number and type of relational symbols is determined by the similarity type \( (t) \). Hence the atomic formulas of \( L_{\omega1\omega} \) are of the form \( v = w \) or \( F_0 \ldots F_{n-1} \). \( L_{\omega1\omega} \) is the least class \( W \) such that:

(i) every atomic formula is in \( W \),
(ii) if \( \phi \) is in \( W \), then its negation \( \neg \phi \) is in \( W \).
(iii) if \( X = (\rho; i \in I) \subseteq W \) and \( |I| < \alpha_1 \), then the conjunction of \( X \) in symbols: \( \Pi X \) (also written \( \prod_{i \in I} \phi_i \)) is also in \( W \).

(iv) if \( \rho \) is in \( W \), then its universal generalization \( (\forall \nu) \rho \) is also in \( W \).

When writing formulas of \( L_{\alpha_0} \) we will sometimes use the existential quantifiers \( (\exists \nu) \rho \) (countable) infinite disjunctions \( \bigvee \nu \phi_i \) (or \( \sum X \)) and the usual finite sentential connectives: \( \neg, \rightarrow, \land, \lor, \land \); we shall assume that they have been defined in the usual way in terms of \( \neg, \prod \), \( (\forall \nu) \).

Note that in (iii) it is not assumed that \( X \neq \emptyset \) and thus the empty conjunction \( \prod \emptyset \) is a formula of \( L_{\alpha_0} \). We let \( T = \prod \emptyset \) and call \( T \) the truth symbol.

We assume it to be known what it means for an \( \alpha_0 \)-sequence \( s \) of elements from the set \( A \) to satisfy the formula \( \varphi \in L_{\alpha_0} \) in the relational system \( M \), in symbols: \( (M, s) \models \varphi \) (see (4)). In the case that \( \varphi \) is satisfied by all \( \alpha_0 \)-sequences from \( A \) we say that \( \varphi \) is true in \( M \) (or valid in \( M \)) and we express this condition by: \( M \models \varphi \).

It is clear that \( T \) is a valid formula.

DEFINITION 2.1. If \( \varphi \in L_{\alpha_0} \), then the set of variables occurring free in \( \varphi \), in symbols: \( \text{VF}(\varphi) \), is defined as follows:

(i) if \( \varphi = t = w \), then \( \text{VF}(\varphi) = \{ t \} \),
(ii) if \( \varphi = P_0 \varphi_0 \ldots \varphi_{n-1} \), then \( \text{VF}(\varphi) = \{ \nu_0, \ldots, \nu_{n-1} \} \),
(iii) if \( \varphi = \neg \varphi \), then \( \text{VF}(\varphi) = \text{VF}(\varphi) \),
(iv) if \( \varphi = (\forall \nu) \Phi \), then \( \text{VF}(\varphi) = \text{VF}(\Phi) \cup \{ \nu \} \),
(v) if \( \varphi = (\exists \nu) \Phi \), then \( \text{VF}(\varphi) = \text{VF}(\Phi) \cup \{ \nu \} \).

In a similar way we can define the set of variables occurring bound in \( \varphi \), in symbols: \( \text{BD}(\varphi) \), and then \( \text{VS}(\varphi) = \text{VF}(\varphi) \cup \text{BD}(\varphi) \) is the set of variables occurring in \( \varphi \).

DEFINITION 2.3. If \( \varphi \in L_{\alpha_0} \), then the set of subformulas of \( \varphi \), in symbols: \( \text{SF}(\varphi) \), is the least set \( W \) containing \( \varphi \) and such that whenever it contains

(i) \( \Phi \), it contains \( \Phi \),
(ii) \( (\forall \nu) \Phi \), it contains \( \Phi \),
(iii) \( \Pi X \), it includes \( X \).

DEFINITION 2.4. A relational symbol \( P_0 \) (equality symbol \( = \)) has a positive (negative) occurrence in a formula \( \varphi \in L_{\alpha_0} \) just in case that there exists a finite sequence \( \langle \varphi_i \rangle_{i \leq \alpha_0} \) of formulas such that:

(i) \( \varphi_0 = \varphi \) and \( \varphi_\alpha = \varphi \),
(ii) for all \( i < \alpha \), \( \varphi_i \in \varphi \).

DEFINITION 2.5. A formula \( \varphi \in L_{\alpha_0} \) is a "positive formula" just in case that neither the equality symbol nor any relational symbol has a negative occurrence in \( \varphi \).

It is clear that a relational (or the equality) symbol has a negative (positive) occurrence in \( \varphi \) if and only if it has a positive (negative) occurrence in \( \neg \varphi \).

A replacement function is a function \( f \) such that \( D(f) = V \) and \( f \subset V \).

DEFINITION 2.6. If \( f \) is a replacement function, then \( B(f) \) and \( S(f) \) are defined as follows:

(i) \( B(f)(v = w) = S(f)(v = w) = f(v) = f(w) \),
(ii) \( B(f)(P_0 \varphi_0 \ldots \varphi_{n-1}) = S(f)(P_0 \varphi_0 \ldots \varphi_{n-1}) = P_0 f(v_0) \ldots f(v_{n-1}) \),
(iii) \( B(f)(\forall \nu) \Phi = B(f)(\forall \nu) \Phi = \prod_{i < \alpha} B(f)(\nu) \Phi \),
(iv) \( B(f)(\exists \nu) \Phi = \exists \nu (B(f)(\nu) \Phi) = S(f)(\exists \nu) \Phi \).

In a similar way we can define \( S(f) \) and \( V(f) \).

DEFINITION 2.7. If \( \varphi \in L_{\alpha_0} \), then "\( \psi \) is free for \( \nu \) in \( \varphi \)" just in case that either (i) \( \nu \in \text{VF}(\varphi) \) or (ii) for all \( n < \omega \) and all \( \langle \varphi_i \rangle_{i < \alpha} \) of subformulas of \( \varphi \) such that (i) \( \nu_0 = \varphi \), (ii) for all \( i < n \), \( \nu_0 = \varphi \), and (iii) for some \( \nu_0 = (\forall \nu) \Phi \) and \( \varphi \in \text{VF}(\Phi) \), then for some \( i < n \), \( \varphi_i = (\forall \nu) \varphi \).
Definition 2.8. If \( \varphi \in L_{\text{mat}} \) and \( A \) is a set of individual variables and \( \Phi = \varphi[u_0/u_0, \ldots, u_{n-1}/u_{n-1}] \), then \( \Phi \) is a "proper \( A \)-substitution instance of \( \varphi \)" (or simply: a proper substitution of \( \varphi \)) if and only if \( (u_i; 0 < i < n) \subseteq A \) and for all \( 0 < i < n \), \( u_i \) is free for \( u_i \) in \( \varphi \).

Definition 2.9. \( \Phi \) is a normal form of \( \varphi \) just in case that no variable occurs both free and bound in \( \Phi \) and \( \varphi = R_0(\Phi) \), where \( j \) is a replacement function such that (i) \( f(v) = v \) for all \( v \in E(\Phi) \) and (ii) for every subformula of \( \Phi \) of the form \( \langle \varphi \rangle \pi, f(v) \in E(\Phi(\pi)) \).

Intuitively, if \( \Phi \) is a normal form of \( \varphi \), then \( \Phi \) can be obtained from \( \varphi \) by "renaming" the variables that occur bound in \( \varphi \) in such a way that no variable occurs both free and bound in \( \Phi \) (the purpose of clause (ii) is to make sure that if \( \Phi \) is a normal form of \( \varphi \), then \( \Phi \rightarrow \varphi \) is valid). It is clear that to every formula \( \varphi \) there corresponds at least one formula \( \Phi \) such that \( \Phi \) is a normal form of \( \varphi \).

As mentioned in the introduction we will consider a Gentzen type formalization of \( L_{\text{mat}} \). Hence the following definition.

Definition 2.10. If \( X \) and \( Y \) are countable sets of formulas of \( L_{\text{mat}} \) then we let \( X \rightarrow Y = (X, Y) \) and it is called a "\( L_{\text{mat}} \)-sequent" (or: a sequent); \( X \) is the "antecedent" and \( Y \) is the "consequent" of the sequent \( X \rightarrow Y \).

If \( S = X \rightarrow Y \) is a sequent, then the set of variables occurring free in \( S \), in symbols: \( \text{FV}(S) \), is the set \( \bigcup_{\varphi \in S} \text{FV}(\varphi) \). The set of variables occurring bound (occurring) in \( S \), in symbols: \( \text{BD}(S) \), are similarly defined. If \( r \) is either the equality symbol or a relational symbol and \( X \) is a set of formulas, then \( r \) has a positive (negative) occurrence in \( X \) just in case that \( r \) has a positive (negative) occurrence in some formula of \( X \); if \( X \rightarrow Y = S \) is a sequent, then \( r \) has a positive (negative) occurrence in \( S \) if and only if \( r \) has a positive (negative) occurrence in \( (\rightarrow \varphi: \varphi \in X) \cup Y \).

Definition 2.11. \( X \rightarrow Y \) is a "normal form of \( X \rightarrow Y \)" if and only if \( \prod_{\varphi \in X \rightarrow Y} \varphi \) is a normal form of \( \prod_{\varphi \in X \rightarrow Y} \varphi \).

If \( s \) is an \( e_0 \)-sequence of elements from the universe of the relational system \( \mathfrak{U} \), then \( s \) satisfies the sequent \( X \rightarrow Y \) in \( \mathfrak{U} \) if and only if

\[ (\mathfrak{U}, s) = \prod_{\varphi \in X \rightarrow Y} \varphi : \prod_{\varphi \in X \rightarrow Y} \varphi \]

The notions which are defined in terms of satisfaction are extended in the natural way to sequents. Thus, e.g., we have that a formula \( \varphi \in L_{\text{mat}} \) is a valid formula if and only if the sequent \( 0 \rightarrow (\varphi) \) is a valid sequent.

3. Proof theory for \( L_{\text{mat}} \). In the axioms and rules of inference given below it is assumed that:

\( \varphi \) and \( \Phi \) are formulas of \( L_{\text{mat}} \),

\[ M, M', N \text{ and } N' \text{ are countable sets of formulas of } L_{\text{mat}}. \]

Structural Rules of Inference

1. Principal Structural Rule of Inference.

\[ M \rightarrow N \]
\[ M \cup M' \rightarrow N' \cup N. \]

2. Rule of Inference for Renaming Bound Variables. (*) If \( M \rightarrow N \) is a normal form of \( M \rightarrow N' \), then the following is a rule of inference:

\[ M' \rightarrow N' \]
\[ \rightarrow \rightarrow \]
\[ M \cup \langle (\varphi) \rangle \pi \rightarrow N. \]
\[ (\rightarrow \rightarrow) \]
\[ M \cup \langle (\varphi) \rangle \pi \rightarrow N. \]

Axioms

1. Logical Axioms.

\[ (\pi, \varphi) \rightarrow N; \pi \rightarrow N \cap \langle \varphi \rangle \]
\[ (\pi, \varphi) \rightarrow N; \]
\[ (\pi, \varphi) \rightarrow N; \]

Rules of Inference for the Logical Symbols

1. Rules of Inference for Negation.

\[ M \cup \langle \varphi \rangle \rightarrow N \]
\[ M \rightarrow N \cup \langle \neg \varphi \rangle \]
\[ M \rightarrow N \cup \langle \neg \varphi \rangle \]

2. Rules of Inference for Conjunction. If \( |I| < c_0 \) and \( \langle \varphi_i : i \in I \rangle \) \( \subseteq L_{\text{mat}} \), then the following two rules are rules of inference:

\[ \prod_{i \in I} \varphi_i \rightarrow N \]
\[ M \cup \langle \prod_{i \in I} \varphi_i \rangle \rightarrow N \]
\[ M \cup \langle \prod_{i \in I} \varphi_i \rangle \rightarrow N \]

3. Unrestricted Rule of Inference for the Quantifier. If \( x \) and \( y \) are free for \( v \) in \( \varphi \), then the following is a rule of inference:

\[ M \cup \langle \varphi[x] \rangle \rightarrow N \]
\[ M \cup \langle \forall y \varphi[y] \rangle \rightarrow N \]

(*) This rule of inference is needed because we have omitted the rule of inference usually known as a cut, i.e., from \( M \rightarrow N \cup \langle \varphi \rangle \) and \( M' \cup \langle \varphi \rangle \rightarrow N' \) to obtain \( M \cup M' \rightarrow N \cup N' \), and yet we allow a variable to occur both free and bound in a formula; cf. Kleene [7].
The following properties of $v \models w[\Gamma]$ are easily verified:

**Lemma 3.3.**

1. If $v \models w[\Gamma']$, then there exists a finite set $\Gamma' \subseteq \Gamma$ such that $v \models w[\Gamma']$.
2. If $v \models w[\Gamma]$ and $w \models z[\Gamma']$, then $v \models z[\Gamma]$.
3. If $v \models w[\Gamma]$ then $w \models v[\Gamma]$.
4. $v \models \theta[0]$.

**Lemma 3.4.** If for all $i < n$, $v_i \models g[i][\Gamma]$, then

1. $\models \Gamma \leftarrow (v_i = y_i)$, whenever $i < n$,
2. if $\varphi$ is an atomic formula, then
   
   $\models \Gamma \leftarrow (\varphi[\varphi[v_0, \ldots, v_{n-1}]]) \leftarrow (\varphi[n_0, \ldots, n_{n-1}]].$

Proof. It follows from the axioms and rules of inference for equality (noting that if $\pi = y \cdot \Gamma$, then $\Gamma \rightarrow (\pi = \pi) \rightarrow \Gamma$).

**Definition 3.5.** A sequent $X \rightarrow Y$ is a "fundamental sequent" just in case that either (a) $X \rightarrow Y$ is an axiom or (b) there exists a finite set of equations $\Gamma' \subseteq X$ and a finite set of individual variables $x_i, y_i$ ($i < n$) such that $v_i \models g[i][\Gamma]$ for all $i < n$ and either (i) $x_i = y_i \cdot Y$ or (ii) for some atomic formula $\varphi$

$$\varphi[v_{n_0}, \ldots, v_{n_{n-1}}] \in X \text{ and } \varphi[y_0, \ldots, y_{n-1}] \not\in Y$$.

Using Lemma 3.4 we immediately obtain (by an application of the principal structural rule of inference) that

**Lemma 3.6.** If $S$ is a fundamental sequent, then $S \rightarrow P$ is provable.

**Definition 3.7.** If $P$ is a function and $S$ a sequent such that $P$ is an $P$-derivation of $S$ in which the rules of inference for equality are not applied and where $F$ is the set of fundamental sequents, then $P$ is a proof of $S$.

By $\models S$ we understand that there exists a proof of $S$. The following theorem can easily be proven by induction on the lengths of the domain of the proof and of the domain of the derivation respectively:

**Theorem 3.8.**

1. If $\models S$, then $\vdash S$.
2. If $\vdash S$, then $\models S$.

We will show in theorem 3.16 that the three notions: $\models S, \vdash S$ and $\models S$ are equivalent. The equivalence of $\models S$ with $\vdash S$ tells us that if we enlarge the set if axioms of $L_{\text{ax}}$ to include all the fundamental sequents, then we can omit the rules of inference for equality.

To show that $\models S$ then $\vdash S$ we proceed as follows. With every sequent $S$ we associate a function $Ta_S$ (called the tableau of $S$) such that

1. the domain of $Ta_S$ is a pseudo-tree,
2. the range of $Ta_S$ is a set
of sequents, (3) $TA_S(0) = S$, and (4) if $S$ is valid, then from $TA_S$ we can obtain a proof of $S$, while if $S$ is not valid then from $TA_S$ we can obtain a (countable) relational system in which $S$ is not valid. However, instead of defining $TA_S$ directly we first define an auxiliary function $TA_{0b}$ from which $TA_S$ is easily defined. We shall give an informal definition for $TA_{0b}$ (i.e. we define simultaneously the domain and the functional values) because the content of $TA_S$ more clearly expressed. The gist of the definition of $TA_S$ is as follows:

(a) the domain of $TA_S$ is to be a pseudo-tree,
(b) if $S \in DTa$ then $TA_S(s)$ is to be a 2-sequence such that $TA_S(s)$ and $TA_S(e)$ are finite sequences of sequences of type $\omega$ of substitution instances of subformulas occurring in a fixed normal form of $S$ ($TA_S(s)$ will then be defined by eliminating the structure of $TA_S(s)$ and $TA_S(e)$, i.e. by making them into sets of formulas),
(c) whenever $TA_S(s)$ is defined, then $TA_S(s^*\langle n \rangle): n < \omega \& s^*(n) \in DTa$ is defined in such a way that there exists a $TA_S(s^*(n)): n < \omega \& s^*(n) \in DTa$ — derivation of $TA_S(s)$ in which the rules of inference for equality are not applied, and this in turn is done by breaking down the formulas occurring in the first $|s|$ sequences of the sequences in $TA_S(s)$ and $TA_S(e)$.

**Definition 3.9.** If $S = X \multimap Y$ is a sequent, then $TA_S$ is defined as follows:

**PART 1 of Definition 3.9.** First we choose a normal form $S' = X \multimap Y$ of $S$. Then let $W$ be a set of individual variables such that $|W| = \omega$ and $W \cap VS(S') = \emptyset$. Then let $Z = W \cap FT(S')$ and finally let $\langle \rho \rangle_{ \in ccm}$ be an enumeration of $Z$ by (an enumeration we understand an enumeration without repetitions).

**PART 2 of Definition 3.9.** Let $S_b = \langle S_b(t) \rangle_{ \in ccm}$ and $T_b = \langle T_b(t) \rangle_{ \in ccm}$ be enumerations of $S'$ and $Y$ respectively. We “start” the definition of $TA$ (we omit the subscript $S$ because there is no risk of confusion) by letting $\langle 0 \rangle \in DTa$ and $TA\langle 0 \rangle = \langle S_b, T_b \rangle$.

**PART 3 of Definition 3.9.** Suppose that $TA\langle n_1, ..., n_{d-1} \rangle$ has already been defined ($d < \omega$) and that $TA\langle n_1, ..., n_{d-1} \rangle = \langle S, T \rangle$ where $S$ and $T$ are finite sequences of sequences of type $\omega$ of proper $Z$-substitution instances of subformulas occurring in $S'$. Then we define

1. $A = \bigcup \{S_b(t) : t \in S_b \cap DS_b\}$,
2. $B = \bigcup \{T_b(t) : t \in S_b \cap DT_b\}$, and
3. $\tilde{n} = \langle n_1, ..., n_{d-1} \rangle$.

**Case 1.** $d = 0$. Let $\langle \rho \rangle_{ \in ccm}$ be an enumeration of the negation formulas occurring in $A$ (note that $A$ is finite). Then $\tilde{n}^*\langle j \rangle \in DTa$ if and only if $j = 0$ and $TA\langle \tilde{n}^*\langle 0 \rangle \rangle = \langle S, T^*\langle \rho \rangle_{ \in ccm} \rangle$.

**Case 2.** $d = 6a + 1$. Corresponding case for the negation formulas in $B$.

**Case 3.** $d = 6a + 2$. Let $\langle \rho \rangle_{ \in ccm}$ be an enumeration of the conjunctions occurring in $A$. Then let $M$ be an $m$-sequence such that for each $p < m$, $M(p)$ is an enumeration of $(\rho_i \in I_b)$ into either a finite or an $\omega$-sequence.

Then $\tilde{n}^*\langle j \rangle \in DTa$ if and only if $j = 0$ and $TA\langle \tilde{n}^*\langle 0 \rangle \rangle = \langle S \cdot M, T \rangle$.

**Case 4.** $d = 6a + 3$. Let $\langle \rho \rangle_{ \in ccm}$ be an enumeration of the conjunctions occurring in $B$ (note then that for all $p < m$, $I_p \neq 0$). Then let $\langle \rho \rangle_{ \in ccm}$ be an enumeration of all possible $m$-sequences of formulas such that for all $p < m$ and $j < \mu$, $\rho_p \in (\rho_i \in I_p)$. Then $\tilde{n}^*\langle j \rangle \in DTa$ if and only if $j = 0$ and $TA\langle \tilde{n}^*\langle 0 \rangle \rangle = \langle S \cdot M, T \rangle$.

**Case 5.** $d = 6a + 4$. Let $\langle \rho \rangle_{ \in ccm}$ be an enumeration of the universally quantified formulas occurring in $A$. Then let $\langle \rho \rangle_{ \in ccm}$ be an enumeration of all possible $m$-sequences of formulas such that for all $p < m$ and $j < \mu$, $\rho_p \in (\rho_i \in I_p)$. Then $\tilde{n}^*\langle j \rangle \in DTa$ if and only if $j = 0$ and $TA\langle \tilde{n}^*\langle 0 \rangle \rangle = \langle S \cdot M, T \rangle$.

**Case 6.** $d = 6a + 5$. Let $\langle \rho \rangle_{ \in ccm}$ be an enumeration of the universally quantified formulas in $B$. Then let $I_1, ..., I_{m-1}$ be the first $m$ individual variables from the set $Z$ (see part 1 of the definition) which occur neither in the range of $S_b$, $t \in DS_b$ nor of $T_b$, $t \in DT_b$. Then $\tilde{n}^*\langle j \rangle \in DTa$ if and only if $j = 0$ and $TA\langle \tilde{n}^*\langle 0 \rangle \rangle = \langle S \cdot M, T^*\langle \rho \rangle_{ \in ccm} \rangle$.

Note that because $S'$ is a normal form of $S$ (i.e. no variable occurs both free and bound in $S'$) and because of the choice of the set $Z$ all the substitutions in cases 5 and 6 are proper $Z$-substitutions. Thus in all cases $TA\langle \tilde{n}^*\rangle$ (when defined) is again of the required form. Thus the definition of $TA_S$ is complete.

**Definition 3.10.** The tableau of a sequent $S$, in symbols: $TA_S$, is the function defined, as follows:
\[ \text{DT}_{A_0} = D_{T_{A_0} \cup \{0\}}, \]
\[ \text{TA}_0(0) = S, \]
\[ \text{if } x \in DT_{A_0}, \text{ then } \]
\[ \text{TA}_0(x) = \bigcup \{ \{ f : f \in Q \cap \text{TA}_0(x) \} | \rightarrow \{ f : f \in Q \cap \text{TA}_0(x) \} \} \]

The definition of \( \text{TA}_0 \) was chosen so that the following lemma holds.

**Lemma 3.11.**

1. \( \text{DT}_{A_0} \) is a pseudo-tree.
2. \( \text{TA}_0(0) = S. \)
3. \( S \) is obtained from \( \text{TA}_0(\{0\}) \) by an application of the rule of inference for renaming the bound variables.
4. \( \text{If } x \in DT_{A_0}(\{0\}), \text{ then there exists } \alpha (\text{TA}_0(x'^{\alpha})): n < \omega \& \alpha \in \alpha \times \text{DT}_{A_0} \) \( \alpha \)-derivation of \( \text{TA}_0(x) \) in which neither the rule of inference for renaming the bound variables nor a rule of inference for equality are applied.

If \( F \) is a function whose domain is a pseudo-tree and whose range is a set of sequents, then \( (a) \) by a branch of \( F \) we understand \( F \) restricted to a branch of the domain of \( F \), \( b \) a sequent \( S \) occurs in \( F \) (or the occurrence of \( S \) in \( F \) at \( e \)) just in case that \( S \) occurs in the range of \( F \) (or \( F(e) = S \), and \( c \) an occurrence of \( S \) \( e \) above an occurrence of \( S \) in \( F \) just in case that \( F(e) = S \) and \( F(e') = S' \) for some \( e' \neq e \).

**Lemma 3.12.** If in each branch of \( \text{TA}_0 \) there occurs a fundamental sequent, then there is a proof of \( S \) (and hence \( S \) is provable).

**Proof.** By 3.11, 3.8 and 3.9.

**Lemma 3.13.** If \( B \) is a branch of \( \text{TA}_0 \), \( Z = \{ a_i : i < \omega \} \) is the set defined in part 1 of definition 3.9 (i.e. \( \text{TA}_0 \)), \( A \) is the union of the antecedents of the sequents occurring in \( B \) and \( S \) is the union of the succedents occurring in \( B \), then:

1. \( \neg \forall \varphi \in A, \varphi \in S \),
2. \( \neg \exists \varphi \in A, \varphi \in S \),
3. \( \{ \exists \forall \varphi \in A, \forall \varphi \in S \} \),
4. \( \{ \forall \varphi \in A, \forall \varphi \in S \} \)
5. \( \{ \forall \varphi \in A, \forall \varphi \in S \} \)
6. \( \{ \forall \varphi \in A, \forall \varphi \in S \} \)

If in addition \( B \) is a branch in which there occurs no fundamental sequent, then we have:

**Lemma 3.14.** If \( B \) is a branch of \( \text{TA}_0 \) in which there does not occur a fundamental sequent, \( Z = \{ a_i : i < \omega \} \), \( A \) and \( S \) are defined as in lemma 3.13, and \( I' \) is the set of equations occurring in \( A \), then:

\[ (1) \rightarrow (6) \text{ as in lemma 3.13,} \]
\[ (7) \neg \exists \alpha \in \mathbb{R}_S, \text{ then } \neg \exists \alpha \in \mathbb{R}_S, \]
\[ (8) \text{if } \varphi \in \mathbb{R}_S, \text{ then } \varphi \in \mathbb{R}_S, \]
\[ (9) \text{if } \varphi \text{ is an atomic formula, } n < \omega, \varphi_i \in \mathbb{R}_S \text{ for all } i < n \]
\[ \varphi(\varphi_0, \ldots, \varphi_{n-1}) \in A \text{, then } \varphi(\varphi_0, \ldots, \varphi_{n-1}) \in A \]
\[ \text{and } \neg \exists \varphi \in \mathbb{R}_S, \text{ then } \neg \exists \varphi \in \mathbb{R}_S. \]
4. The interpolation theorem for $\text{L}_{\text{max}}$.

**Theorem 4.0.** If $X \rightarrow Y$ is a valid sequent, then for every pair of partitions $(A_1, A_2)$ and $(B_1, B_2)$ of $X$ and $Y$ respectively there corresponds a formula $\phi$, called an "interpolating formula for $(A_1 \rightarrow B_1, A_2 \rightarrow B_2)$", such that

1. $A_1 \rightarrow B_1 \cup \{\phi\}$ and $A_2 \cup \{\phi\} \rightarrow B_2$ are provable,
2. $\text{FV}(\phi) \subseteq \text{FV}(A_1 \rightarrow B_1) \cap \text{FV}(A_2 \rightarrow B_2)$,
3. if the equality symbol occurs in $\phi$ then it occurs in $X \rightarrow Y$,
4. if a relational symbol has a positive (negative) occurrence in $\phi$, then it has a positive (negative) occurrence in $A_1 \rightarrow B_1$ and a positive (negative) occurrence in $A_2 \rightarrow B_2$.

Proof. Assume that $X \rightarrow Y$ is a valid sequent. Then by theorem 3.16 there exists a proof $P$ of $X \rightarrow Y$. Let $\mu$ be the length of $P$.

Part I: $\mu > 0$. Then $X \rightarrow Y$ is a fundamental sequent and for fundamental sequents the theorem is easily verified.

Part II: $\mu = 0$. Let $r$ be the root of $P$ (hence $P(r) = X \rightarrow Y$) and then let $U = \{r(x): p < \omega \text{ and } r(x) \in P(r)\}$. For each $c \in U$, let $P(c) = P(c) \{s: s \in \text{DP} \text{ and } c \subseteq s\}$; then $P(c)$ is a proof of the of the sequent $P(c)$ such that the length of $P(c)$ is strictly smaller than $\mu$. Thus we assume (induction hypothesis) that the theorem is true for all the sequents $P(c)$ where $c \in U$, and then we shall show that the theorem is also true for $P(r)$ (i.e. $X \rightarrow Y$).

Because of the definition of a proof (cf. definition 3.7) it follows that $X \rightarrow Y$ is the conclusion of a rule of inference, other than a rule of inference for equality, with $(P(c): c \in U)$ as the set of premises. We shall consider only the case when the rule applied is $\equiv \rightarrow$, the remaining cases being similar. In this case then $C = U$ and $P(c)$ must be of the form

$$A_1' \cup \phi \cup \{\{\forall x\} \Phi[x]\} \rightarrow B_1 \lor B_2$$

where either

(a) $A_1' = A_1$ and $A_2' \cup \{\forall x\} \Phi[x]\} = A_2$

or

(b) $A_1' \cup \{\forall x\} \Phi[x]\} = A_2$ and $A_2' = A_2$.

Suppose (a) (the case for (b) is analogous). Then by the induction hypothesis there exists an interpolating formula $\pi$ for $(A_1 \rightarrow B_1, A_2' \cup \{\Phi[x]\} \rightarrow B_2)$. Hence

(i) $A_1 \rightarrow B_1 \cup \{\pi\}$

(ii) $A_2' \cup \{\Phi[x]\} \cup \{\pi\} \rightarrow B_2$.

Hence from (ii) and $\equiv \rightarrow$, we obtain:

(iii) $A_1 \rightarrow B_1 \cup \{\Phi[x]\} \cup \{\pi\} \rightarrow B_2$.

that is:

$$\vdash A_1 \cup \{\forall x\} \Phi[x]\} \cup \{\pi\} \rightarrow B_2.$$
(ii) for all \( \mu < \eta \), \( P_\mu \) does not occur in \( \pi \),

(iii) if a relational symbol occurs in \( \pi \), then it occurs in \( \varphi \),

(iv) \( \{ \varphi \} \rightarrow (V_{\alpha_1} \ldots V_{\alpha_{\varphi}} \ldots V_{\alpha_{\varphi+1}}) \).

The proof of theorem 5.9 is essentially the same proof as in the case of \( I_{\text{hyp}} \) and is thus omitted.

A sentence \( \varphi \in I_{\text{hyp}} \) is preserved under homomorphism just in case that:

(i) for all structures \( \mathfrak{B} \), if \( \mathfrak{B} \models \varphi \), then \( \mathfrak{B} \models \psi \) whenever \( \mathfrak{B} \) is a homomorphic image of \( \mathfrak{Y} \). If we repeat (mutatis mutandis) the proof in Lyndon [10] we obtain:

**Theorem 5.1 (Homomorphism theorem).** If \( \varphi \) is a sentence of \( I_{\text{hyp}} \) then \( \varphi \) is preserved under homomorphisms if and only if \( \varphi \) is equivalent to a positive sentence.

As a final application of the interpolation theorem for \( I_{\text{hyp}} \) we give theorem 5.3 below. However we must first give some definitions.

Suppose that \( f \) is a function whose domain is the power set of \( \omega \) and whose range is included in the power set of \( \omega \); then \( f \) is an elementary function just in case that there exists a formula \( \varphi \in I_{\text{hyp}} \) which will then be called a description of \( f \) such that (i) \( FV(\varphi) = (\varsigma; \varsigma < 3) \), (ii) the only relational symbol occurring in \( \varphi \) is a binary relation symbol and (iii) for all \( R \subseteq \omega \),

\[
  f(R) = \langle (\varsigma\alpha)_{\varsigma<\varsigma}; (\varsigma\alpha)_{\varsigma<\varsigma} = \neg \varphi \rangle .
\]

Introducing the natural topology in the power sets of \( \omega \) and \( \omega^\omega \) it is clear what is meant by "if \( f \) is a Borel function". If \( \sigma \) is a permutation of \( \omega \) and \( R \subseteq \omega^\omega \), then we let \( o(R) = \langle \langle \sigma\alpha \rangle_{\varsigma<\varsigma} = \neg \varphi \rangle \). If \( f \) is a function with domain the power set of \( \omega^\omega \) and range consisting of subsets of \( \omega \), then \( f \) is an invariant function just in case that for all permutations \( \sigma \) of \( \omega \) and all \( R \subseteq \omega^\omega \), \( f(f(R)) = o(R) \). A set \( X \subseteq \mathcal{P}(\omega^\omega) \times \mathcal{P}(\omega^\omega) \) is an elementary set just in case that there exists a sentence \( \varphi \) of \( I_{\text{hyp}} \) such that the only relational symbols occurring in \( \varphi \) are a binary and ternary relational symbol and such that \( X = \langle (R, S); \langle R, S \rangle = \neg \varphi \rangle \). The notions of an invariant set and a Borel set are analogously defined.

Before the author had shown the interpolation theorem for \( I_{\text{hyp}} \), D. S. Scott had shown that: if the interpolation theorem is true for \( I_{\text{hyp}} \) then a set is invariant and Borel if and only if it is an elementary set (1) (and in fact D. S. Scott's result was a motivation for studying the interpolation theorem for \( I_{\text{hyp}} \)). If we slightly modify the proof of Scott's (unpublished) result, in particular if we use the definability theorem instead of the interpolation theorem, we obtain that the elementary functions are exactly the invariant Borel functions. First we show the following lemma.

**Lemma 5.3.** To every Borel set \( X \subseteq \mathcal{P}(\omega^\omega) \times \mathcal{P}(\omega^\omega) \) there corresponds a sentence \( \varphi \in I_{\text{hyp}} \) such that (i) the relational symbols occurring in \( \varphi \) are two binary relational symbols, \( \text{a ternary relation symbol}, \) and for each \( n < \omega \), a unary relation symbol \( P_n \) and (ii)

\[
  X = \langle (R, S); \langle \omega, <, \varsigma, \varsigma \rangle, (R, S)_{\varsigma \varsigma \varsigma} = \neg \varphi \rangle .
\]

Proof. Let \( \Phi \) be the sentence of \( I_{\text{hyp}} \) which characterizes (up to isomorphism) the relational system \( \langle \omega, <, \varsigma, \varsigma \rangle_{\varsigma \varsigma \varsigma} \). Note that the elements of the subbase for the topology for \( \mathcal{P}(\omega^\omega) \times \mathcal{P}(\omega^\omega) \) are of form either \( B_{\Phi} = \langle (R, S); \langle \varsigma, j \rangle \in \mathcal{R} \rangle \), or \( B_{\Phi} = \langle (R, S); \langle \varsigma, j \rangle \in \mathcal{R} \rangle \) or their complements. Now it is clear that \( B_{\Phi} = \langle (R, S); \langle \omega, <, \varsigma, \varsigma \rangle, (R, S)_{\varsigma \varsigma \varsigma} = \neg \varphi \rangle \) and similarly for \( B_{\Phi} \) or their complements. But the Borel sets are built up from these sets by means of countable (propositional) operations. Thus a simple induction on the rank of the Borel set completes the proof of the lemma.

**Theorem 5.3.** If \( f \) is a function whose domain is the power set of \( \omega \) and whose range is included in the power set of \( \omega \), then \( f \) is an elementary function if and only if \( f \) is an invariant Borel function (1).

Proof. Let \( \Phi \) be the sentence of \( I_{\text{hyp}} \) which characterizes the relational system \( \langle \omega, <, \varsigma, \varsigma \rangle_{\varsigma \varsigma \varsigma} \). The proof is completed in 3 parts.

**Part I.** If \( f \) is elementary, then \( f \) is an invariant and Borel function. Suppose that \( f \) is elementary. Then let \( \varphi \) be the description of \( f \). It is clear, because of the way satisfaction is defined, that \( f \) is then invariant.

**Part II.** If \( f \) is a Borel function, then the graph of \( f \) (i.e. \( \langle (R, f(R))^\omega \rangle \)) is a Borel set.

For a proof of the above, see Kuratowski [8], p. 291.

**Part III.** If \( f \) is an invariant Borel function whose domain is the power set of \( \omega \), then \( f \) is an elementary function.

By part II, the graph of \( f \) is a Borel set, and hence by lemma 5.2 there exists a sentence \( \psi \) such that:

\[
  \langle (R, f(R)); \langle R, \omega \rangle = \langle (R, S); \langle \omega, <, \varsigma, \varsigma \rangle, (R, S)_{\varsigma \varsigma \varsigma} = \neg \varphi \rangle .
\]

Then let \( \psi \) be the formula obtained from \( \varphi \land \Phi \) by replacing the ternary relational symbol \( Q \) by a new ternary relational symbol \( Q' \), the relation

(1) C. Rydl-Narkiewicz has shown the converse.
symbol corresponding to \(<\) by a new binary relation symbol and also replacing, for every \(n < \omega\), the unary relational symbol \(P_n\) by a new unary relational symbol \(P'_n\). Then because \(f\) is an invariant function and \(\Phi\) characterizes (up to isomorphism) the relational system \(\langle \omega, \langle, (n)\rangle_{n < \omega} \rangle\) we then obtain from (a) that

\[
| \Phi \land \exists \varphi \land \Phi \rightarrow (\forall v_0)(\forall v_2)(\forall v_3) Q_{\varphi}(v_0, v_2) v_3) Q_{\varphi}(v_0, v_3) .
\]

Thus applying the definability theorem (Theorem 5.0) we obtain that there exists a formula \(\pi\) such that \(\forall V_{\langle n \rangle} (\pi) \subseteq \{v_3: i < 3\}\), the only relational symbol occurring in \(\pi\) is a binary relational symbol and such that

\[
| \Phi \land \exists \varphi \land \Phi \rightarrow (\forall v_0)(\forall v_2)(\forall v_3) (\pi \rightarrow Q_{\varphi}(v_0, v_2, v_3)) .
\]

But because the domain of \(f\) is the whole of the power set of \(\omega\) we also have that

\[
| \Phi \land \exists \varphi \land \Phi \rightarrow (\forall v_0)(\forall v_2)(\forall v_3) (\pi \rightarrow Q_{\varphi}(v_0, v_2, v_3)) .
\]

Because suppose that (d) were false, then there would exist \(R \subseteq \omega^2\) such that

\[
\langle \omega, \langle, (n)\rangle_{n < \omega} , R, 3 \rangle_{n < \omega} | \Phi \land \exists \varphi \land \Phi \rightarrow (\forall v_0)(\forall v_2)(\forall v_3) (\pi \rightarrow Q_{\varphi}(v_0, v_2, v_3)) .
\]

From (a) we then obtain that

\[
| \Phi \land \exists \varphi \land \Phi \rightarrow (\forall v_0)(\forall v_2)(\forall v_3) (\pi \rightarrow Q_{\varphi}(v_0, v_2, v_3)) .
\]

From (a) (d) we obtain that

\[
\langle \omega, \langle, (n)\rangle_{n < \omega} , R, 3 \rangle_{n < \omega} | \Phi \land \exists \varphi \land \Phi \rightarrow (\forall v_0)(\forall v_2)(\forall v_3) (\pi \rightarrow Q_{\varphi}(v_0, v_2, v_3)) .
\]

From (a) and (i), because \(Q\) does not occur in \(\pi\), we obtain that \(R \subseteq \omega\) and then (ii) and (ii) would contradict each other. Thus combining (c) with (d) we have that

\[
| \Phi \land \exists \varphi \land \Phi \rightarrow (\forall v_0)(\forall v_2)(\forall v_3) (\pi \rightarrow Q_{\varphi}(v_0, v_2, v_3)) .
\]

6. Model-theoretical versions of the Interpolation theorem for \(L_{\omega\omega}\).

**Definition 6.1.** If \(K\) is a class of relational systems of type \(\nu\), then \(K\) is an "elementary class" in symbols: \(K \in EC(L_{\omega\omega})\) ("elementary class in the wide sense" in symbols: \(K \in EC_c(L_{\omega\omega})\) just in case that there exists a sentence \(\varphi \in L_{\omega\omega}\) of the form \(\langle \nu, \circ \rangle \subseteq \mathcal{M}\) such that the relational symbols occurring in \(\varphi\) correspond to the similarity type \(\nu\) and such that \(K\) is the class of all models of \(\varphi\) [of \(\Gamma\)].

**Theorem 6.2.** If \(K, K'\) are classes of relational systems of type \(\nu\) such that \(K \subseteq K' = 0\) and such that \(K \in EC(L_{\omega\omega})\), then there exists a class \(N\) of relational systems of type \(\nu\) such that \(K \subseteq N\) and \(N \in EC(L_{\omega\omega})\).

In the case of \(L_{\omega\omega}\), the corresponding theorem is also true for \(EC_c\)-classes (cf. Keisler [6]). However in the case of \(L_{\omega\omega}\) it is not so, in fact in we have the following:

**Theorem 6.3.** The classes \(K_1, K_2\) are isomorphic to \(\langle \alpha_1, \varepsilon_1 \rangle\), \(\langle \alpha_2, \varepsilon_2 \rangle\), \(K_1 \in EC(L_{\omega\omega})\), \(K_2 \in PC(L_{\omega\omega})\), \(K_1 \cap K_2 = 0\) and such that there does not exist a class \(N\) which is closed under \(L_{\omega\omega}\)-elementary equivalence and \(K_1 \subseteq N\) and \(K_2 \subseteq N\). (*)

**Proof.** That \(K_1 \in EC(L_{\omega\omega})\) and \(K_1 \cap K_2 = 0\) is immediate. To show that \(K_1 \in PC(L_{\omega\omega})\), we first note that \(\langle \alpha_1, \varepsilon_1 \rangle\) can be characterized up to isomorphism as a relational system \(\langle A, E \rangle\) such that (i) \(E\) is a linear ordering of \(A\), (ii) for every \(\mu < \varepsilon_1\), there exists an initial segment of \(\langle \mu, \varepsilon_1 \rangle\) and (iii) for every \(\alpha \in A\), the set of \(E\)-predecessors of \(\alpha\) is countable. Condition (ii) can be expressed in the following form: (iii) there exists a ternary relation \(E\) on \(A\) such that for each \(\alpha \in A\), \((\alpha, \beta, \gamma): (\alpha, \beta, \gamma) \in E\) is a function whose domain is (isomorphic to) \(\omega\) and whose range is the set of \(E\)-predecessors of \(\alpha\). Since every countable ordinal can be characterized by a single sentence of \(L_{\omega\omega}\) (cf. [9] and [13]), it follows then from (i), (ii) and (iii) that \(K_1 \in PC(L_{\omega\omega})\). Suppose next that \(N\) is a class such that:

(*) \(K_1 \subseteq N\) and \(K_2 \cap N = 0\).

Then \(\langle \alpha_1, \varepsilon_1 \rangle\) \(\in N\) and all linear orderings in \(N\) must be well-orderings. Hence since \(\alpha_1\) and \(\alpha_1 \circ \alpha_2\) (considered as linear orderings and where \(\circ\) is the order type of the negative integers) are \(L_{\omega\omega}\)-elementarily equivalent (cf. [5] and [9]), it follows that if \(N\) satisfies the condition (*) then \(N\) cannot be closed under \(L_{\omega\omega}\)-elementary equivalence (and in particular \(N\) cannot be an elementary class).

(*) Two relational systems \(\mathfrak{A}\) and \(\mathfrak{B}\) of the same similarity type \(\nu\) are \(L_{\omega\omega}\)-elementarily equivalent just in case that for every sentence \(\varphi \in L_{\omega\omega}\) such that the relational symbols occurring in \(\varphi\) correspond to the similarity type \(\nu\), \(\mathfrak{A} \models \varphi\) if and only if \(\mathfrak{B} \models \varphi\).
References


On a Freedman's problem

by

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Given a compact and metrizable space \( X \), let us consider the space \( 2^X \) of all closed subsets of \( X \) (cf. [1], p. 106) and an arbitrary fixed \( Z \times 2^X \). Let \( 2^Z \subset 2^X \) denote the set of all homeomorphs of \( Z \) contained in \( X \). D. Freedman has conjectured that \( 2^Z \) is always a Borel set in \( 2^X \) and, in fact, for the case where \( X \) is the Cantor dyadic set this was proved by D. Scott ([2], pp. 126-128). Our aim now is to prove the general statement for an arbitrary \( X \). This is based on a refinement of the method of [3] (1).

Theorem 1. If \( F \) is an arbitrary group of automorphisms of a separable topological space \( G \) admitting a complete metrization (2) and a continuous map \( \varphi \) from \( G \) into a metric space satisfies the condition (3)

\[
\{ (g, \varphi(g) - \varphi(g_0)) : g_0 \in G \} = \{ (g, f \in F) : g \in G \},
\]

then \( \varphi(G) \) is an absolutely Borel set (i.e. every homeomorph of \( \varphi(G) \) in any metric space is Borel).

Proof. The decomposition of \( G \) given by \( F_\varphi = \{ (g, f \in F) \} \) is open in the sense of [3] since

\[
\{ g : F_\varphi \cap U \neq 0 \} = \bigcup_{i \in I} \{ g : F_i \neq U \}.
\]

Let \( S \) be a Borel selector given by the Lemma (see [3], p. 129). The continuous mapping \( \varphi \) is one-to-one on \( S \) and \( \varphi(S) = \varphi(G) \) (\( S \) is a selector), whence \( \varphi(G) \) is an absolutely Borel set (cf. [1], p. 396).

Theorem 2. The set \( 2^Z \) (introduced at the beginning) is Borel.

Proof. The set \( G \) of all homeomorphs of \( Z \) into \( X \) is a \( \mathcal{G}_\delta \) set in the space \( X^Z \) of all continuous maps of \( Z \) into \( X \) with the topology

(1) (2) and (3) give information on other topics similar to those presented in this note.
(2) Let us recall that every \( G_\delta \) set in a complete metric space always admits a complete metrization topologically equivalent to the original one ([1], p. 316), e.g. the set \( N \) of [2] is such a set in the space \( X^\omega \) of [2].