Multiple choice axioms and axioms of choice
for finite sets

by

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Dedicated to Professor A. D. Wallace
on the occasion of his Sixtieth Birthday

I. Introduction. In this paper we shall be concerned with the
propositions FS, (for definitions, see Section II) first considered by the
author in [1], restricted versions of FS, their relation to the propor-
tions 〈m〉 considered by Mostowski [5], Szmielew [7], and Sierpiński [6],
and various other weakened versions of the axiom of choice. In particular
we shall deduce some interesting decompositions of the axiom of choice
into conjunctions of mutually independent propositions. We shall give
a third proof of Theorem 8 (see [5] and [7]).

II. Definitions and terminology. If \( X \) and \( Y \) are any sets,
\( X \setminus Y \) denotes the relative complement of \( Y \) in \( X \), \( \mathbb{P}(X) \) the set of all
subsets of \( X \), \( \mathbb{P}^*(X) \) the set of finite subsets of \( X \), \( \mathbb{P}_n(X) \) the set of \( n \)-
element subsets of \( X \), and \( |X| \) the cardinal of \( X \).

We denote the set of positive integers by \( I \), the set of all primes
by \( P \), \( I \cup \{0\} \) by \( I_0 \).

If \( m, n \in I_0 \) we define \( (m, n) \) by

\[
(m, n) = \begin{cases} \max\{m, n\} & \text{if } m = 0 \text{ or } n = 0, \\
\text{greatest common divisor of } m \text{ and } n & \text{otherwise.}
\end{cases}
\]

The multiple choice axiom FS, for \( n \in I_0 \) is defined as follows:

FS, \( n \) For every set \( X \) of non-empty sets there exists a function \( f \) such
that for each \( Y \in X, f(Y) \) is a finite non-empty subset of \( Y \) and \(|f(Y)|, n \rangle = 1.

We note that with the above definitions, FS, is the axiom of choice
and FS, says merely that \( f(X) \) is finite and non-empty.

The axioms FS, first arose in connection with the theory of vector
spaces (see [1]).

The proposition 〈m〉, \( m \in I \), is defined as follows:

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assistance.
3. **Theorem.** If \( p \) is a prime and \( x \) is a set, then given a function \( f \) satisfying \( \langle p \rangle \langle \Psi_p(X) \rangle \) we can define effectively a function \( g \) satisfying \( \Psi_s(\Psi_p(X)) \).

**Proof.** According to 2 we have, for each integer \( q \in I \), an effectively determined function \( f_q \) defined on \( \Psi_{pq}(X) \) such that for \( y \in \Psi_{pq}(X) \), \( f_q(y) \) is a proper non-empty subset of \( y \). We now construct a function \( g \) satisfying \( \Psi_{pq}(\Psi_p(X)) \).

If \( y \in \Psi_p(X) \) and \( p \mid |y| \) then define \( g(y) = y \). To handle the situation when \( |y| = gp \), we first define, inductively auxiliary functions \( g_q \) satisfying \( \Psi_{pq}(\Psi_{pq}(X)) \). If \( q = 1 \), we take \( g_1 = f_1 \). Suppose that we have effectively defined \( g_q \) for \( q < r \). Let \( y \in \Psi_{pq}(X) \). Consider \( f_q(y) \), a proper non-empty subset of \( y \). If \( |f_q(y)| = p \), let \( g_q = f_q(y) \), otherwise \( |f_q(y)| = gp \) for some \( q < r \) and we define \( g_q(y) = f_q(f_q(y)) \). We now define \( g(y) = g_q(y) \) where \( |y| = gp \), and the construction is complete.

3*. **Theorem.** If \( p \) is any prime then \( \langle p \rangle \langle x \rangle \) implies \( \Psi_s \).

This theorem is an immediate consequence of 3.

4. **Lemma.** If \( k \) and \( m \) are positive integers then \( \langle km \rangle \langle x \rangle \) implies \( \langle x \rangle \).

A proof of this lemma may be found in [7] p. 79.

In view of 4, 3, and 1 we obtain

5. **Lemma.** If \( k \) and \( m \) have the same prime factors, then \( \langle km \rangle \langle x \rangle \) implies \( \Psi_s \).

6. **Theorem.** If \( X \) is a set, and \( n \) a positive integer, then the following are effectively equivalent:

- (1) For every prime \( p \leq n \), \( \langle p \rangle \langle \Psi_p(X) \rangle \).
- (2) For every prime \( p \leq n \), \( \Psi_{pq}(\Psi_{pq}(X)) \).
- (3) For every positive integer \( q \leq n \), \( \langle q \rangle \langle \Psi_q(X) \rangle \).
- (4) For every positive integer \( q \leq n \), \( \Psi_s(\Psi_p(X)) \).

**Proof.** The fact that (1) implies (2) follows from 3.

We now show that (2) implies (3). This will be done by induction on \( q \). It is clear that for \( q = 1 \) we can effectively construct a choice function on \( \Psi_p(X) \). Suppose that we have effectively constructed choice functions \( f_q \) on \( \Psi_{pq}(X) \) for \( q < r \). We now construct a choice function \( f_r \) on \( \Psi_{pq}(X) \). Let \( f \) be the least prime divisor of \( r \). Let \( g \) be a function realizing \( \Psi_s(\Psi_p(X)) \).

For \( y \in \Psi_p(X) \) define \( f_q(y) = f_r(f_r(y)) \) where \( q = |g(y)| < r \). It is clear that \( f \) is the desired function.

The implication (3) implies (1) is obvious. Thus (1), (2), and (3) are effectively equivalent. On the other hand, 1 yields the effective equivalence of (2) and (4).

6*. **Theorem.** Let \( N = \{ q \in I : q \leq n \} \), and \( P = \{ p \in I : p \leq n \} \); then the following are equivalent:

- (1) \( \langle p \rangle \langle x \rangle \)
- (2) \( \Psi_s \)

**Proof.**
Proof. The proof follows from 5, the fact that (1) and (8) are equivalent, and 7.

Theorem 8 was proved first by Szpilrajn [7] and, independently, by Mostowski who used entirely different methods. Mostowski also gave a condition (M) ([7], p. 160) which is necessary for the implication \(<N>\) implies \(\langle Q\rangle\). It is known that (S) is not necessary for the implication \(<N>\) implies \(\langle Q\rangle\). The proposition \(FS_{\alpha}\), in view of 8 and intuitive reasoning, appears to be weaker than \(<N>\), and it is unknown if (S) is necessary for the implication \(FS_{\alpha}\) implies \(\langle Q\rangle\).

IV. Some connections between \(FS_{\alpha}\) and the axiom of choice. We shall be concerned with the following axiom of countability:

(C) A countable union of disjoint finite sets is countable.

9. Theorem. The conjunction of (C) and \(FS_{\alpha}\) is equivalent to \(FS_{\beta}\).

Proof. It is clear that \(FS_{\alpha}\) implies (C) and \(FS_{\beta}\).

We suppose now that (C) and \(FS_{\beta}\) are valid. Let \(X\) be an arbitrary set of non-empty sets. We wish to construct a choice function for \(X\). For each \(p \in \beta\) let \(F_p\) be the set of all functions satisfying \(FS_{\beta}(\Psi_{\{X}\})\). By hypothesis, \(F_p \neq \emptyset\). Also by hypothesis there is a function \(g\) realizing \(FS_{\beta}(\bigcup F_p\; p \in \beta)\). Let \(F_p = g(F_p)\), then \(\bigcup F_p\; p \in \beta\) is a countable union of disjoint finite sets and hence countable by (C). If we fix on a particular counting of \(\bigcup F_p\; p \in \beta\), then in each \(F_p\) there is a least element, say \(f_p\). We shall construct a choice function \(f\) from the \(f_p\).

First we define an auxiliary function \(h\) as follows: For \(X \in X\), if \(|X| = 1\) define \(h(X) = y\) where \(y \in Y\). If \(|X| > 1\), but \(|X|\) is finite, define \(h(X) = f_p(X)\) where \(p\) is the least prime factor of \(|X|\). If \(|X|\) is infinite, define \(h(X) = f_p(X)\). It is clear that for any set \(Y \subseteq X\) there is a least integer \(n = n(Y)\) such that after \(n\) iteration of \(h\) applied to \(Y\) we obtain an element of \(Y\), i.e., \(h^n(Y) \subseteq Y\). Thus, if we define \(f(Y) = h^n(Y)\), we have the desired choice function.

The next proposition is of a metamathematical nature and shows that the hypothesis (C) cannot be deleted from 9. In the sequel we let \(\mathbb{S}\) stand for any one of a number of suitable set theories, for example that of Mostowski [4].

10. Theorem. If \(\mathbb{S}\) is consistent then \(FS_{\beta}\) does not imply \(FS_{\alpha}\).

Proof. From 8* we see that \(FS_{\alpha}\) is equivalent to \(FS_{\beta}\) and (7). The theorem now follows from a result of A. Lévy [3] (in our notation):

Theorem. If \(\mathbb{S}\) (or the system \(A, B, C\) of Gödel [2]) is consistent, then \(FS_{\beta}\) and (7) implies \(FS_{\alpha}\) is unprovable in \(\mathbb{S}\).

In order to show the independence of \(FS_{\alpha}\) from the axioms of \(\mathbb{S}\) we need the following two lemmas.
11. Lemma. If every set can be linearly ordered, thenabelle holds.
The truth of 11 was first observed by Kuratowski. The proof is not
difficult and may be found in Sierpiński [5], p. 412.

12. Lemma. The conjunction of F Sn and F$^n$ is equivalent to FSn.
Proof. Obviously FSn implies both FSn and F$^n$. On the other hand,
F$^n$ says that given a set X of non-empty sets there is a function f
such that for $Y \in X$, $f(Y) \in \psi^*(Y)$, while FSn says that there is a choice function g
on $\psi^*(\{Y\})$. Since $\psi^*(X) \subseteq \psi^*(\{Y\})$ for each $Y \in X$, it follows that
if g(f) is the desired choice function.

We are now in a position to prove

13. Theorem. If $\kappa$ is consistent, then the axioms of $\kappa$ do not imply FSn
for any $n \in \kappa$.
Proof. Since for each $n \in \kappa$, FSn implies FSn, it is sufficient to prove that
FSn is independent of the axioms of $\kappa$. In view of Lemmas 12 and 13 we see that
FSn, together with the supposition that every set can be
linearly ordered, implies the axiom of choice. Since Mostowski [4] has
shown that the axioms of $\lbrace \kappa \rbrace$, together with the principle of linear ordering
do not imply the axiom of choice, it is clear that they cannot imply FSn.

It follows that the axioms of $\kappa$ alone certainly cannot imply FSn.

It is clear from the above independence results that FSn is independent of
the conjunction of the axioms of $\kappa$ and F$^n$, and conversely that F$^n$
is independent of the conjunction of the axioms of $\kappa$ and FSn. In view of
this independence 13 gives a nice decomposition of the axiom of choice
into independent, heuristically complementary statements. The feeling
that the statements are complementary is strengthened by the fact that
if FSn is replaced by $P$, the new conjunction does not imply the axiom of
choice.

In view of it would be interesting to know if there is a prime $p \neq \#$
such that F$^n_p$ implies F$^n$, where $P = \sigma(p)$.

References
[4] A. Mostowski, Über die Unabhängigkeit des Wohlordnungssatzes von Ordnungs-
[6] W. Sierpiński, Cardinal and ordinal numbers, Monografie Matematyczne 34,
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An interpolation theorem for denumerably long formulas

by

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O. Introduction. Lω is the first-order language obtained by
modifying the usual formation rules for the finite first-order formulas
so that conjunctions and disjunctions of less than $\beta$ individual variables are allowed
(thus in particular, Lω is the usual (finite) first-order language) (1). We say that
the interpolation theorem is true for Lω if it is true that for all formulas $\varphi, \psi$ of Lω
if the implication $\varphi \Rightarrow \psi$ is valid, then there exists a formula $\pi$ of Lω
such that (1) $\varphi \Rightarrow \pi$ and $\pi \Rightarrow \psi$ are valid, (2) if a variable occurs free in $\pi$,
then it occurs free both in $\varphi$ and in $\psi$, and (3) if a relational symbol occurs (occurs positively, occurs negatively in $\pi$,
then it occurs (occurs positively, occurs negatively) both in $\varphi$ and in $\psi$.

The interpolation theorem is known to be true for Lω (see Craig [3]
and Lyndon [10]) and, whenever $\beta$ is an inaccessible cardinal, for Lω (see Mac
Kern-Takeuti [11]) (2). In this paper we show that the interpolation
theorem is true for Lω.

The interpolation theorem is obtained as a consequence of the completeness of a Gentzen type formalization for Lω. The essential
property of our rules of inference, in addition to the usual subformula
property associated with Gentzen type systems, is that the number of
variables that occur free in the premise(s) do not occur free in the conclusion (i.e. are quantified out) is always finite (3).

1 The results in this paper form part of the results contained in a dissertation
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(1) These kinds of infinitary languages (i.e. infinitary languages Lω where a
need not be equal to $\beta$) have been studied by C. E. Karp [4].

2 Added in proof, J. Malitz has shown that the interpolation theorem for

3 E. Weyl [3] has obtained a formalization for Lω which has the subformula
property, however his formalization does not have the property mentioned above
and thus it is unsuitable for deriving the interpolation theorem (the author did not learn