Multiple choice axioms and axioms of choice for finite sets

by

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Dedicated to Professor A. D. Wallace on the occasion of his Sixtieth Birthday

I. Introduction. In this paper we shall be concerned with the propositions FS_n (for definitions, see Section II) first considered by the author in [1], restricted versions of FS_n, their relation to the propositions $\langle n\rangle$ considered by Mostowski [5], Szmielew [7], and Sierpiński [6], and various other weakened versions of the axiom of choice. In particular we shall deduce some interesting decompositions of the axiom of choice into conjunctions of mutually independent propositions. We shall give a third proof of Theorem 8 (see [5] and [7]).

II. Definitions and terminology. If X and Y are any sets, $X\setminus Y$ denotes the relative complement of Y in X, $\mathfrak{P}(X)$ the set of all subsets of X, $\mathfrak{P}^*(X)$ the set of finite subsets of X, $\mathfrak{P}_k(X)$ the set of k-element subsets of X, and $|X|$ the cardinal of X.

We denote the set of positive integers by I, the set of all primes by $\mathfrak{I}$, $I\cup\{0\}$ by I_0.

If $m, n\in I_0$ we define (m, n) by

$$(m, n) = \begin{cases} \max\{m, n\} & \text{if } m = 0 \text{ or } n = 0\,, \\ \text{greatest common divisor of } m \text{ and } n & \text{otherwise}\,. \end{cases}$$

The multiple choice axiom FS_n, for $n\in I_0$, is defined as follows:

FS_n. *For every set X of non-empty sets there exists a function f such that for each $Y\in X$, $f(Y)$ is a finite non-empty subset of Y and $(|f(Y)|, n) = 1$.*

We note that with the above definitions, FS_0 *is the axiom of choice* and FS_1 says merely that $f(Y)$ *is finite and non-empty*.

The axioms FS_n first arose in connection with the theory of vector spaces (see [1]).

The proposition $\langle n\rangle$, $n\in I$, is defined as follows:

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$\langle n \rangle$. *If X is a set of n-element sets, then there is a function f such that for each $Y \in X$, $f(y) \in Y$.*

Many properties of $\langle n \rangle$ can be found in [5], [6], and [7].

If $N \subset I_0$, then by FS_N ($\langle N \rangle$) we mean the proposition that for each $n \in N$, FS_n ($\langle n \rangle$) holds.

If X is any set of non-empty (n-element) sets, then $\mathrm{FS}_n(X)$ ($\langle n \rangle(X)$) denotes the proposition that there exists a function for X which satisfies the conditions of FS_n ($\langle n \rangle$). We shall also have occasion to consider

FS_n^*. *For every set X of non-empty finite sets, $\mathrm{FS}_n(X)$.*

If $N \subset I_0$ then by FS_N^* we mean the proposition that for each $n \in N$, FS_n^* holds.

III. Some connections between FS_n and $\langle n \rangle$.

1. LEMMA. *If $n = p_1^{e_1} \cdot p_2^{e_2} \ldots p_k^{e_k}$ where the p_i are an increasing sequence of primes, $e_i \geqslant 1$, and $P = \{p_1, p_2, \ldots, p_k\}$, then $\mathrm{FS}_n(X)$ is effectively equivalent to $\mathrm{FS}_P(X)$.*

Proof. The implication $\mathrm{FS}_n(X)$ implies $\mathrm{FS}_p(X)$ follows from the number theoretical result that $(a, n) = 1$ implies $(a, d) = 1$ for all divisors d of n. It will be convenient to use the notation $d|n$ for d divides n.

To prove the reverse implication, let f_i be functions realizing $\mathrm{FS}_{p_i}(X)$, we must construct a function f which realizes $\mathrm{FS}_n(X)$.

Let $y \in X$, and $y_0 = f_1(y)$; then either $(|y_0|, n) = 1$ in which case we define $f(y) = y_0$, or there is a least i, $1 \leqslant i \leqslant k$, such that $p_i | |y_0|$. In the latter case let $y_1 = f_i(y_0)$. Thus $y_1 \subset y_0 \subset y$ and $1 \leqslant |y_1| < |y_0| < \infty$ since $p_i | |y_0|$ and $(p_i, |y_1|) = 1$. We iterate this process, stopping only if we obtain a set $y_i \subset y$ such that $(|y_i|, n) = 1$. If the process has not terminated after l iterations we obtain: $y_l \subset y_{l-1} \subset \ldots \subset y_1 \subset y_0 \subset y$ and $1 \leqslant |y_l| < |y_{l-1}| < \ldots < |y_1| < |y_0| < \infty$. Thus, $l \leqslant |y_0|$, so this process must terminate, yielding a set $y_l \subset y$ for which $(|y_l|, n) = 1$. We define $f(y) = y_l$, and the desired function has been constructed.

1*. LEMMA. *Let $n = p_1^{e_1} \cdot p_2^{e_2} \ldots p_k^{e_k}$ where the p_i are an increasing sequence of primes, $e_i \geqslant 1$, and $P = \{p_1, p_2, \ldots, p_k\}$; then FS_n is equivalent to FS_P.*

Proof. This is an immediate consequence of 1.

COROLLARY. *If n and m are integers with the same prime factors then FS_n is equivalent to FS_m.*

We shall have occasion to use the following lemma.

2. LEMMA. *Let p be a prime, n a positive integer and X a set. Then, given a function f satisfying $\langle p \rangle(\mathfrak{P}_p(X))$, we can effectively define a function g on $\mathfrak{P}_{np}(X)$ such that for $y \in \mathfrak{P}_{np}(X)$; $g(y) \subset y$ and $1 \leqslant |g(y)| < |y|$.*

The proof of 2 is obtained by a simple modification of the proof of a lemma of A. Tarski for which proofs may be found in [5], Lemma 15, or [6], p. 99.

3. THEOREM. *If p is a prime and x is a set, then given a function f satisfying $\langle p \rangle(\mathfrak{P}_p(X))$ we can define effectively a function g satisfying $\mathrm{FS}_p(\mathfrak{P}^*(X))$.*

Proof. According to 2 we have, for each integer $q \in I$, an effectively determined function f_q defined on $\mathfrak{P}_{qp}(X)$ such that for $y \in \mathfrak{P}_{qp}(X)$, $f_q(y)$ is a proper non-empty subset of y. We now construct a function g satisfying $\mathrm{FS}_p(\mathfrak{P}^*(X))$.

If $y \in \mathfrak{P}^*(X)$ and $p \nmid |y|$ then define $g(y) = y$. To handle the situation when $|y| = qp$, we first define inductively auxiliary functions g_q satisfying $\mathrm{FS}_p(\mathfrak{P}_{qp}(X))$. If $q = 1$, we take $g_1 = f_1$. Suppose that we have effectively defined g_q for $q < r$. Let $y \in \mathfrak{P}_{rp}(X)$. Consider $f_r(y)$, a proper non empty subset of y. If $(|f_r(y)|, p) = 1$, let $g_r(y) = f_r(y)$, otherwise $|f_r(y)| = qp$ for some $q < r$ and we define $g_r(y) = f_q(f_r(y))$. We now define $g(y) = g_q(y)$ where $|y| = qp$, and the construction is complete.

3*. THEOREM. *If p is any prime then $\langle p \rangle$ implies FS_p^*.*

This theorem is an immediate consequence of 3.

4. LEMMA. *If k and n are positive integers then $\langle kn \rangle$ implies $\langle k \rangle$.*

A proof of this lemma may be found in [7] p. 79.

In view of 4, 3, and 1 we obtain

5. LEMMA. *If n and m have the same prime factors, then $\langle n \rangle$ implies FS_m^*.*

6. THEOREM. *If X is a set, and n a positive integer, then the following are effectively equivalent:*

(1) *For every prime $p \leqslant n$, $\langle p \rangle(\mathfrak{P}_p(X))$.*

(2) *For every prime $p \leqslant n$, $\mathrm{FS}_p(\mathfrak{P}^*(X))$.*

(3) *For every positive integer $q \leqslant n$, $\langle q \rangle(\mathfrak{P}_q(X))$.*

(4) *For every positive integer $q \leqslant n$, $\mathrm{FS}_q(P^*(X))$.*

Proof. The fact that (1) implies (2) follows from 3.

We now show that (2) implies (3). This will be done by induction on q. It is clear that for $q = 1$ we can effectively construct a choice function on $\mathfrak{P}_1(X)$. Suppose that we have effectively constructed choice functions f_q on $\mathfrak{P}_q(X)$ for $q < r \leqslant n$. We now construct a choice function f_r on $\mathfrak{P}_r(X)$. Let p be the least prime divisor of r. Let g be a function realizing $\mathrm{FS}_p(\mathfrak{P}^*(X))$. For $y \in \mathfrak{P}_r(X)$ define $f_r(y) = f_q(g(y))$ where $q = |g(y)| < r$. It is clear that f is the desired function.

The implication (3) implies (1) is obvious. Thus (1), (2) and (3) are effectively equivalent. On the other hand, 1 yields the effective equivalence of (2) and (4).

6*. THEOREM. *Let $N = \{q \in I\colon q \leqslant n\}$, and $P = \{p \in \mathfrak{I}\colon p \leqslant n\}$; then the following are equivalent:*

(1) $\langle P \rangle$;

(2) FS_P^*;

(3) $\langle N\rangle$;

(4) FS_N^*.

The proof of this theorem is immediate from 6.

The equivalence of (1) and (3) is also a simple corollary of the known sufficient conditions for implications of the type "$\langle Z\rangle$ implies $\langle m\rangle$" where $Z \subset I$ and $m \in I$ ([5] Theorem II, p. 149, [7] Theorem 2, p. 79). We shall obtain these conditions later.

It is convenient to define the condition (Σ) for a given $q \in I$ and $N \subset I$.

(Σ) *If* $q = \sum_{i=1}^{k} q_i$ *where* $q_i > 1$, *then there is an integer* $n \in N$ *such that* $(n, q_i) > 1$ *for at least one* q_i.

It is not difficult to see that (Σ) is equivalent to the following condition (S) of Mostowski.

(S) *If* $q = \sum_{i=1}^{k} p_i$ *where the* p_i *are primes, then for at least one* p_i, *there is a multiple of* p_i *in* N.

7. THEOREM. *If* N *and* q *satisfy* (Σ) *and for each* $n \in N$, f_n *realizes* $FS_n(\mathfrak{P}^*(X))$, *then we can effectively realize* $\langle q\rangle(\mathfrak{P}_q(X))$.

Proof. In virtue of 1 we know that if P is the set of all prime factors of all the integers in N, then for each $p \in P$, we can effectively define a function f_p which satisfies $FS_p(\mathfrak{P}^*(X))$. Thus the proof will be complete if we can effectively define a function g_q realizing $\langle q\rangle(\mathfrak{P}_q(X))$ given the functions f_p.

If $q = 1$, it is clear that we can define g_q. Suppose, inductively, that the proposition is true for all q such that $q < r$ and P and q satisfy (Σ). We now suppose that r and P satisfy (Σ) and construct a function g_r realizing $\langle r\rangle(\mathfrak{P}_r(X))$. By (Σ) we know that there is a prime $p \in P$ such that $p|r$; let p_0 be the least such prime. Let $y \in \mathfrak{P}_r(X)$ and $y_1 = f_{p_0}(y)$. Consider the following decomposition:

$$y = y_1 \cup (y \setminus y_1) .$$

Since $r = |y_1| + |y \setminus y_1|$, we see that one of $|y_1|$ and $|y \setminus y_1|$ must satisfy (Σ) for, if neither did, then r would also fail to satisfy (Σ). If $|y_1|$ satisfies (Σ), then define $g(y) = f_q(y_1)$ where $q = |y_1| < r$. If $|y_1|$ does not satisfy (Σ), then $|y \setminus y_1|$ does. Define $g(y) = f_q(y \setminus y_1)$ where $q = |y \setminus y_1|$. This can be done since y_1 is a non-empty proper subset of y and hence in either case $1 \leqslant q < r$.

7*. THEOREM. *If* q *is a positive integer and* N *is a set of positive integers which satisfy* (Σ), *then* FS_N *implies* $\langle q\rangle$.

The proof of this theorem is immediate from 7.

8. THEOREM. *If* q *is any positive integer and* N *any set of positive integers satisfying* (S), *then* $\langle N\rangle$ *implies* $\langle q\rangle$.

Proof. The proof follows from 5, the fact that (Σ) and (S) are equivalent, and 7*.

Theorem 8 was proved first by Szmielew [7] and, independently, by Mostowski who used entirely different methods. Mostowski also gave a condition (M) ([7], p. 160) which is necessary for the implication $\langle N\rangle$ implies $\langle q\rangle$. It is known that (S) is not necessary for the implication $\langle N\rangle$ implies $\langle q\rangle$. The proposition FS_N^*, in view of 5 and intuitive reasoning, appears to be weaker than $\langle N\rangle$, and it is unknown if (S) is necessary for the implication FS_N^* implies $\langle q\rangle$.

IV. Some connections between FS_n and the axiom of choice. We shall be concerned with the following axiom of countability:

(C) *A countable union of disjoint finite sets is countable.*

9. THEOREM. *The conjunction of* (C) *and* $FS_{\mathfrak{I}}$ *is equivalent to* FS_0.

Proof. It is clear that FS_0 implies (C) and $FS_{\mathfrak{I}}$.

We suppose now that (C) and $FS_{\mathfrak{I}}$ are valid. Let X be an arbitrary set of non-empty sets. We wish to construct a choice function for X. For each $p \in \mathfrak{I}$ let $\mathcal{F}_p$ be the set of all functions satisfying $FS_p(\mathfrak{P}(\bigcup X))$. By hypothesis, $\mathcal{F}_p \neq \emptyset$. Also by hypothesis there is a function g realizing $FS_2(\bigcup \{\mathcal{F}_p\colon p \in \mathfrak{I}\})$. Let $F_p = g(\mathcal{F}_p)$, then $\bigcup \{F_p\colon p \in \mathfrak{I}\}$ is a countable union of disjoint finite sets and hence countable by (C). If we fix on a particular counting of $\bigcup \{F_p\colon p \in \mathfrak{I}\}$, then in each F_p there is a least element, say f_p. We shall construct a choice function f from the f_p.

First we define an auxiliary function h as follows: For $Y \in X$, if $|Y| = 1$ define $h(Y) = y$ where $y \in Y$. If $|Y| > 1$, but $|Y|$ is finite, define $h(Y) = f_p(Y)$ where p is the least prime factor of $|Y|$; if Y is infinite, define $h(Y) = f_2(Y)$. It is clear that for any set $Y \in X$ there is a least integer $n = n(Y)$ such that after n iteration of h applied to Y we obtain an element of Y, i.e., $h^n(Y) \in Y$. Thus, if we define $f(Y) = h^n(Y)$, we have the desired choice function.

The next proposition is of a metamathematical nature and shows that the hypothesis (C) cannot be deleted from 9. In the sequel we let $\mathfrak{S}$ stand for any one of a number of suitable set theories, for example that of Mostowski [4].

10. THEOREM. *If* $\mathfrak{S}$ *is consistent then* FS_I *does not imply* FS_0.

Proof. From 6* we see that FS_I is equivalent to FS_1 and $\langle I\rangle$. The theorem now follows from a result of A. Lévy [3] (in our notation):

THEOREM. *If* $\mathfrak{S}$ (or the system A, B, C of Gödel [2]) *is consistent, then* FS_1 *and* $\langle I\rangle$ *implies* FS_0 *is unprovable in* $\mathfrak{S}$.

In order to show the independence of FS_n from the axioms of $\mathfrak{S}$ we need the following two lemmas.

11. Lemma. *If every set can be linearly ordered, then* FS_n^* *holds.*

The truth of 11 was first observed by Kuratowski. The proof is not difficult and may be found in Sierpiński [6], p. 412.

12. Lemma. *The conjunction of* FS_1 *and* FS_0^* *is equivalent to* FS_0.

Proof. Obviously FS_0 implies both FS_1 and FS_0^*. On the other hand, FS_1 says that given a set X of non-empty sets there is a function f such that for $Y \in X$, $f(Y) \in \mathfrak{P}^*(Y)$, while FS_0^* says there is a choice function g on $\mathfrak{P}^*(\bigcup X)$. Since $\mathfrak{P}^*(Y) \subset \mathfrak{P}^*(\bigcup X)$ for each $Y \in X$, it follows that $g(f)$ is the desired choice function.

We are now in a position to prove

13. Theorem. *If* $\mathfrak{S}$ *is consistent, then the axioms of* $\mathfrak{S}$ *do not imply* FS_n *for any* $n \in X$.

Proof. Since for each $n \in I$, FS_n implies FS_1, it is sufficient to prove that FS_1 is independent of the axioms of $\mathfrak{S}$. In view of Lemmas 12 and 13 we see that FS_1, together with the supposition that every set can be linearly ordered, implies the axiom of choice. Since Mostowski [4] has shown that the axioms of $\mathfrak{S}$, together with the principle of linear ordering do not imply the axiom of choice, it is clear that they cannot imply FS_1. It follows that the axioms of $\mathfrak{S}$ alone certainly cannot imply FS_1.

It is clear from the above independence results that FS_1 is independent of the conjunction of the axioms of $\mathfrak{S}$ and FS_0^*, and conversely that FS_0^* is independent of the conjunction of the axioms of $\mathfrak{S}$ and FS_1. In view of this independence 12 gives a nice decomposition of the axiom of choice into independent, heuristically complementary statements. The feeling that the statements are complementary is strengthened by the fact that if FS_0^* is replaced by $\langle I \rangle$ the new conjunction does not imply the axiom of choice.

In view of 9 it would be interesting to know if there is a prime $p \in \mathfrak{T}$ such that $FS_{\mathfrak{T}}$ implies FS_P where $P = \mathfrak{T} \setminus \{p\}$.

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An interpolation theorem for denumerably long formulas *

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0. Introduction. $\mathbf{L}_{\alpha\beta}$ is the first-order language obtained by modifying the usual formation rules for the finite first-order formulas so that conjunctions and disjunction of less that α formulas and quantifications over fewer than β individual variables are allowed (thus in particular, $\mathbf{L}_{\omega\omega}$ is the usual (finite) first-order language) (1). We say that the interpolation theorem is true for $\mathbf{L}_{\alpha\beta}$ just in case that for all formulas φ, Φ of $\mathbf{L}_{\alpha\beta}$ if the implication $\varphi \to \Phi$ is valid, then there exists a formula π of $\mathbf{L}_{\alpha\beta}$ such that (1) $\varphi \to \pi$ and $\pi \to \Phi$ are valid, (2) if a variable occurs free in π, then it occurs free both in φ and in Φ, and (3) if a relational symbol occurs (occurs positively, occurs negatively) in π, then it occurs (occurs positively, occurs negatively) both in φ and in Φ. The interpolation theorem is known to be true for $\mathbf{L}_{\omega\omega}$ (see Craig [2] and Lyndon [10]) and, whenever θ is an inaccessible cardinal, for $\mathbf{L}_{\theta\theta}$ (see Maehara-Takeuti [11])**. In this paper we show that the interpolation theorem is true for $\mathbf{L}_{\omega_1\omega}$.

The interpolation theorem is obtained as a consequence of the completeness of a Gentzen type formalization for $\mathbf{L}_{\omega_1\omega}$. The essential property of our rules of inference, in addition to the usual subformula property associated with Gentzen type systems, is that the number of variables that occur free in the premise(s) and do not occur free in the conclusion (i.e. are quantified out) is always finite (2).

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(1) These kinds of infinitary languages (i.e. infinitary languages $\mathbf{L}_{\alpha\beta}$ where α need not be equal to β) have been studied by C. R. Karp [4].

** Added in proof. J. Malitz has shown that the interpolation theorem for $\mathbf{L}_{\theta\theta}$ is false, cf. Notices of Amer. Math. Soc. 12(1965), p. 379.

(2) E. Engeler [3] has obtained a formalization for $\mathbf{L}_{\omega_1\omega}$ which has the subformula property, however his formalization does not have the property mentioned above and thus it is unsuitable for deriving the interpolation theorem (the author did not learn