A representation theorem for two-dimensional v^*-algebras

by

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The results presented here complete paper [5], where a full description of all at least three-dimensional v^*-algebras was given. For the terminology and notation used here, see [2] and [5]. In particular an algebra is said to be a v^*-*algebra* if it satisfies the following conditions:

1. *Each self-dependent element is an algebraic constant.*

2. *If the elements* $a_1, a_2, \ldots, a_n$ $(n \geqslant 1)$ *are independent and the elements* $a_1, a_2, \ldots, a_n, a_{n+1}$ *are dependent, then* a_{n+1} *is generated by* $a_1, a_2, \ldots, a_n$.

A simple description in terms of groups and semigroups of all one-dimensional v^*-algebras is contained in an expository paper [6]. G. Grätzer proved in [1] a representation theorem for two-dimensional v^*-algebras without non-trivial unary algebraic operations, i.e. for universal algebras independently generated by every two elements. His result is an analogue of assertion (ii) in [5], but the field $\mathcal{K}$ is replaced by a weaker algebraic structure, similar to the nearfield defined in terms of multiplication and subtraction.

The aim of the present paper is to prove a representation theorem for two-dimensional v^*-algebras with a non-trivial algebraic unary operation. The result we have obtained is rather unexpected. Namely, except one four-element algebra all two-dimensional v^*-algebras with a non-trivial algebraic unary operation have the same algebraic structure as at least three-dimensional ones.

Consider an algebra $\mathfrak{E} = (E; i, q^*)$, where E is a four-element set, the unary operation i is an involution without fixed points and the ternary symmetrical operation q^* is uniquely determined by the conditions $q^*(x, y, i(x)) = y$, $q^*(x, y, x) = x$. The algebra $\mathfrak{E}$ will be called *exceptional*. It is easy to prove that the involution i is the only non-trivial algebraic unary operation in the algebra $\mathfrak{E}$. Moreover, there is no binary algebraic operation in $\mathfrak{E}$ depending on every variable. Hence it follows that the elements $a, b \in E$ $(a \neq b)$ are independent if and only if $a \neq i(b)$. Furthermore, since the involution i has no fixed points, the algebra $\mathfrak{E}$ is generated by every pair of independent elements. Consequently, $\mathfrak{E}$ is a two-dimensional v^*-algebra.

It should be noted that the exceptional algebra $\mathfrak{E}$ can also be defined in terms of Boolean operations. Namely, $\mathfrak{E} = (E; i, q^*)$, where the set E is a four-element Boolean algebra, $i(x) = x'$,

$$q^*(x_1, x_2, x_3) = (x_1' \cap x_2' \cap x_3') \cup (x_1' \cap x_2 \cap x_3) \cup (x_1 \cap x_2' \cap x_3) \cup (x_1 \cap x_2 \cap x_3')$$

if all elements x_1, x_2, x_3 are different and

$$q^*(x_1, x_2, x_3) = (x_1 \cap x_2 \cap x_3) \cup (x_1' \cap x_2 \cap x_3) \cup (x_1 \cap x_2' \cap x_3) \cup (x_1 \cap x_2 \cap x_3')$$

in the opposite case.

We remind that two algebras defined on the same set are treated here as identical if they have the same classes of algebraic operations.

THEOREM. *Let $\mathfrak{A} = (A; \boldsymbol{F})$ be a two-dimensional v^*-algebra with a non-trivial algebraic unary operation. Then one of the following four cases holds:*

(i) *$\mathfrak{A}$ is the exceptional algebra $\mathfrak{E}$.*

(ii) *There is a field $\mathfrak{K}$ such that A is a linear space over $\mathfrak{K}$ and there exists a linear subspace A_0 of A such that the class of algebraic operations is the class of all operations f defined as*

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a\,,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathfrak{K}$ and $a \in A_0$.

(iii) *There is a field $\mathfrak{K}$ such that A is a linear space over $\mathfrak{K}$ and there exists a linear subspace A_0 of A such that the class of algebraic operations is the class of all operations f defined as*

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a\,,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathfrak{K}$, $\sum_{k=1}^{n} \lambda_k = 1$ and $a \in A_0$.

(iv) *There are a group $\mathfrak{G}$ of permutations of the set A and a subset A_0 of A containing all fixed points of permutations that are not the identical and invariant under all permutations from $\mathfrak{G}$ such that the class of algebraic operations is the class of all operations defined as*

$$f(x_1, x_2, \ldots, x_n) = g(x_j) \qquad (1 \leqslant j \leqslant n)$$

or

$$f(x_1, x_2, \ldots, x_n) = a\,,$$

where $g \in \mathfrak{G}$ and $a \in A_0$.

Before proving the Theorem we shall prove some lemmas. If $\mathfrak{A} = (A; \boldsymbol{F})$ is an algebra, then by $\mathfrak{A}^{(n)}$ we shall denote the algebra $(\boldsymbol{A}^{(n)}; \boldsymbol{F})$ of all n-ary algebraic operations in $\mathfrak{A}$ (see [2], p. 48). It is well known that $\mathfrak{A}^{(n)}$ is a v^*-algebra whenever $\mathfrak{A}$ is a v^*-algebra of dimension $\geqslant n$ (see [4]). In all further considerations we shall assume that the algebra $\mathfrak{A}$ is a two-dimensional v^*-algebra with a non-trivial algebraic unary operation.

The following lemma is a simple consequence of Theorem 1 in [4].

LEMMA 1. *Each non-constant algebraic unary operation is invertible. Moreover, the inverse operation is also algebraic.*

LEMMA 2. *If f is a binary algebraic operation depending on every variable and $c \in \boldsymbol{A}^{(0)}$, then the composition $f(x, c)$ is not a constant operation.*

Proof. Contrary to this let us suppose that

$$f(x, c) = c_0 \qquad (x \in A)\,, \tag{1}$$

where $c_0 \in \boldsymbol{A}^{(0)}$. Since the operation f depends on both variables, we infer that the operations f and $e_2^{(2)}$ treated as elements of the algebra $\mathfrak{A}^{(2)}$ are independent and, consequently, form a basis of $\mathfrak{A}^{(2)}$. Thus there exists an algebraic binary operation h such that $h(f(x, y), y) = x$. Setting $y = c$, we get, by (1), the equation $h(c_0, c) = x$ for all $x \in A$, which contradicts the assumption that the algebra $\mathfrak{A}$ is two-dimensional.

LEMMA 3. *Let $c \in \boldsymbol{A}^{(0)}$ and $f, g \in \boldsymbol{A}^{(2)}$. If*

$$f(x, c) = g(x, c) \tag{2}$$

and

$$f(x, x) = g(x, x) \tag{3}$$

for all $x \in A$, then $f = g$.

Proof. If both operations f and g depend only on one variable, then the equation $f = g$ is a simple consequence of (2) and (3). Suppose now that at least one of the operations f and g depends on every variable. Without loss of generality we may assume that the operation f depends on both variables. If the operations f and g treated as elements of the algebra $\mathfrak{A}^{(2)}$ are dependent, then the operation g is generated by the operation f, i.e.

$$g(x, y) = h_1(f(x, y)) \tag{4}$$

for an operation $h_1 \in \boldsymbol{A}^{(1)}$. Setting $y = c$ into (4) and taking into account (2), we get the equation $f(x, c) = h_1(f(x, c))$. By Lemma 2 the operation $f(x, c)$ is not constant. Thus, by Lemma 1, the last equation implies $h_1(x) = x$, which together with (4), gives the equation $f = g$.

If the operations f and g are independent in the algebra $\mathfrak{A}^{(2)}$ and, consequently, form a basis of $\mathfrak{A}^{(2)}$, then there exist operations $h_2, h_3 \in A^{(2)}$ such that

(5) $$h_2\big(f(x,y), g(x,y)\big) = f(x,c)$$

and

(6) $$h_3\big(f(x,y), g(x,y)\big) = f(y,y)\,.$$

Setting $y = c$ into (5), we get, in view of (2), the equation

$$h_2\big(f(x,c), f(x,c)\big) = f(x,c)\,.$$

By Lemma 2, the operation $f(x,c)$ is not constant. Consequently, by Lemma 1, the last equation implies $h_2(x,x) = x$. Hence and from (3) and (5) we get the equation

$$f(x,x) = h_2\big(f(x,x), g(x,x)\big) = f(x,c)\,,$$

which, shows in particular, that the operation $f(x,x)$ is not constant. Further, setting $y = x$ into (6), we obtain, by (3), the equation

$$h_3\big(f(x,x), f(x,x)\big) = f(x,x)\,.$$

Since $f(x,x)$ is not constant, the last equation implies $h_3(x,x) = x$. Hence and from (2) and (6) the equation

$$f(x,c) = h_3\big(f(x,c), f(x,c)\big) = h_3\big(f(x,c), g(x,c)\big) = f(c,c)$$

follows. Thus the operation $f(x,c)$ is constant. But this contradicts Lemma 2. The Lemma is thus proved.

LEMMA 4. *Let $A^{(0)} = \emptyset$ and let h be a non-trivial algebraic unary operation. If $f, g \in A^{(2)}$,*

(7) $$f\big(x, h(x)\big) = g\big(x, h(x)\big)$$

and

(8) $$f(x,x) = g(x,x)\,,$$

then $f = g$.

Proof. If the operations f and g are independent in the algebra $\mathfrak{A}^{(2)}$ and, consequently, form a basis of $\mathfrak{A}^{(2)}$, then there exist operations $h_1, h_2 \in A^{(2)}$ such that

(9) $$h_1\big(f(x,y), g(x,y)\big) = f(x,x)$$

and

(10) $$h_2\big(f(x,y), g(x,y)\big) = f(y,y)\,.$$

Setting $y = x$ into (9) and (10), we have, by (8), the equation

$$h_1\big(f(x,x), f(x,x)\big) = h_2\big(f(x,x), f(x,x)\big) = f(x,x)\,.$$

Since the operation $f(x,x)$ is not constant, the last equation, by Lemma 1, implies $h_1(x,x) = h_2(x,x) = x$. Hence and from (7), (9) and (10) it follows that

(11) $$f\big(h(x), h(x)\big) = h_2\Big(f\big(x, h(x)\big), f\big(x, h(x)\big)\Big) = f\big(x, h(x)\big) = h_1\Big(f\big(x, h(x)\big), f\big(x, h(x)\big)\Big) = f(x,x)\,.$$

By Lemma 1 the operation $f(x,x)$ is invertible. Consequently, equation (11) implies $h(x) = x$, which contradicts the assumption. Thus the operations f and g are dependent in the algebra $\mathfrak{A}^{(2)}$.

If f and g are dependent in the algebra $\mathfrak{A}^{(2)}$, then there exists an algebraic unary operation h_0 such that

(12) $$g(x,y) = h_0\big(f(x,y)\big)\,.$$

Setting $y = x$ into this equation and taking into account (8) we get the equation $f(x,x) = h_0\big(f(x,x)\big)$. Since $f(x,x)$ is not constant, we have the formula $h_0(x) = x$, which, by (12), implies the equation $f = g$. The Lemma is thus proved.

LEMMA 5. *Let $c \in A^{(0)}$ and $f, g \in A^{(3)}$. If*

(13) $$f(x,y,c) = g(x,y,c)$$

and

(14) $$f(x,x,x) = g(x,x,x)$$

for all $x, y \in A$, then $f = g$.

Proof. Set

$$f_1(x,y) = f(x,x,y)\,,\qquad g_1(x,y) = g(x,x,y)\,,$$
$$f_2(x,y) = f(y,x,y)\,,\qquad g_2(x,y) = g(y,x,y)\,,$$
$$f_3(x,y) = f(x,y,y)\,,\qquad g_3(x,y) = g(x,y,y)\,.$$

From (13) and (14) we get the equations $f_j(x,c) = g_j(x,c)$, $f_j(x,x) = g_j(x,x)$ $(j = 1,2,3)$. Hence, by Lemma 3, we obtain the equations $f_j = g_j$ $(j = 1,2,3)$. In other words, $f(x,y,z) = g(x,y,z)$ whenever at least two variables among x, y and z are equal.

Given a binary algebraic operation h we put

$$f_4(x,y) = f\big(h(x,y), x, y\big)\,,\qquad g_4(x,y) = g\big(h(x,y), x, y\big)\,,$$
$$f_5(x,y) = f\big(x, h(x,y), y\big)\,,\qquad g_5(x,y) = g\big(x, h(x,y), y\big)\,.$$

Since $f_4(x,x) = f_3\big(h(x,x), x\big)$, $g_4(x,x) = g_3\big(h(x,x), x\big)$, $f_5(x,x) = f_2\big(h(x,x), x\big)$ and $g_5(x,x) = g_2\big(h(x,x), x\big)$, we have the formula $f_j(x,x) = g_j(x,x)$ $(j = 4,5)$. Moreover, by (13), $f_j(x,c) = g_j(x,c)$ $(j = 4,5)$, which, by

Lemma 3, implies the equations $f_j = g_j$ $(j = 4, 5)$. Consequently, for each operation $h \in \boldsymbol{A}^{(2)}$ the equations

$$f(h(x, y), x, y) = g(h(x, y), x, y) \tag{15}$$

and

$$f(x, h(x, y), y) = g(x, h(x, y), y) \tag{16}$$

hold. In particular, taking a constant operation h we have, by (16),

$$f(x, c, y) = g(x, c, y) \quad (c \in \boldsymbol{A}^{(0)}) . \tag{17}$$

Further, given $h \in \boldsymbol{A}^{(2)}$ we put

$$f_6(x, y) = f(x, y, h(x, y)), \quad g_6(x, y) = g(x, y, h(x, y)) .$$

By (17) we have the equation

$$f_6(x, c) = f(x, c, h(x, c)) = g(x, c, h(x, c)) = g_6(x, c) .$$

Moreover, $f_6(x, x) = f_1(x, h(x, x)) = g_1(x, h(x, x)) = g_6(x, x)$. Thus, by Lemma 3, $f_6 = g_6$ and, consequently,

$$f(x, y, h(x, y)) = g(x, y, h(x, y)) . \tag{18}$$

Let a_1, a_2, a_3 be an arbitrary triplet of elements of A. Since each triplet of elements is dependent, one of the elements a_1, a_2, a_3 can be obtained by a binary algebraic operation from the remaining ones. Hence and from (15), (16) and (18) we get the equation $f(a_1, a_2, a_3) = g(a_1, a_2, a_3)$, which completes the proof.

LEMMA 6. *Suppose that either $\boldsymbol{A}^{(0)} \neq \emptyset$ or $\boldsymbol{A}^{(0)} = \emptyset$ and $\boldsymbol{A}^{(1)}$ contains at least two non-trivial operations. If $f, g \in \boldsymbol{A}^{(4)}$ and $f(x_1, x_2, x_3, x_4) = g(x_1, x_2, x_3, x_4)$ whenever $x_1 = x_2$ or $x_1 = x_3$, then $f = g$.*

Proof. First consider the case $\boldsymbol{A}^{(0)} \neq \emptyset$. Let c be an arbitrary element of $\boldsymbol{A}^{(0)}$ and

$$f_1(x, y, z) = f(z, x, c, y), \quad g_1(x, y, z) = g(z, x, c, y) .$$

Since

$$f_1(x, y, c) = f(c, x, c, y) = g(c, x, c, y) = g_1(x, y, c)$$

and

$$f_1(x, x, x) = f(x, x, c, x) = g(x, x, c, x) = g_1(x, x, x) ,$$

we infer, by Lemma 5, that $f_1 = g_1$ and, consequently,

$$f(z, x, c, y) = g(z, x, c, y) \tag{19}$$

for all $x, y, z \in A$ and $c \in \boldsymbol{A}^{(0)}$.

For any operation $h \in \boldsymbol{A}^{(3)}$ we put

$$f_2(x, y, z) = f(x, y, z, h(x, y, z)), \quad g_2(x, y, z) = g(x, y, z, h(x, y, z)),$$

$$f_3(x, y, z) = f(x, h(x, y, z), z, y), \quad g_3(x, y, z) = g(x, h(x, y, z), z, y) .$$

From (19) we get the equations

$$f_2(x, y, c) = f(x, y, c, h(x, y, c)) = g(x, y, c, h(x, y, c)) = g_2(x, y, c) ,$$

$$f_3(x, y, c) = f(x, h(x, y, c), c, y) = g(x, h(x, y, c), c, y) = g_3(x, y, c) .$$

Moreover, by the assumption,

$$f_2(x, x, x) = f(x, x, x, h(x, x, x)) = g(x, x, x, h(x, x, x)) = g_2(x, x, x)$$

and

$$f_3(x, x, x) = f(x, h(x, x, x), x, x) = g(x, h(x, x, x), x, x) = g_3(x, x, x) .$$

Hence, by Lemma 5, we obtain the equations $f_2 = g_2$ and $f_3 = g_3$.

Consequently, for any operation $h \in \boldsymbol{A}^{(3)}$ the equations

$$f(x, y, z, h(x, y, z)) = g(x, y, z, h(x, y, z)) \tag{20}$$

and

$$f(x, h(x, y, z), z, y)) = g(x, h(x, y, z), z, y) \tag{21}$$

hold.

Given a system a_1, a_2, a_3, a_4 of elements of A. If $a_3 \in \boldsymbol{A}^{(0)}$, then the equation $f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4)$ is a consequence of (19). Suppose that $a_3 \notin \boldsymbol{A}^{(0)}$. If a_4 is generated by a_1, a_2 and a_3, i.e. $a_4 = h(a_1, a_2, a_3)$, where $h \in \boldsymbol{A}^{(3)}$, then the equation $f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4)$ is a consequence of (20). Finally, if a_4 is not generated by a_1, a_2 and a_3, then a_3 and a_4 are independent and, consequently, form a basis of the algebra in question. Thus $a_2 = h(a_1, a_3, a_4)$, where $h \in \boldsymbol{A}^{(3)}$, and the equation $f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4)$ is a consequence of (21), which completes the proof in the case $\boldsymbol{A}^{(0)} \neq \emptyset$.

Suppose now that $\boldsymbol{A}^{(0)} = \emptyset$ and the class $\boldsymbol{A}^{(1)}$ contains at least two non-trivial operations. Given a non-trivial operation $h_0 \in \boldsymbol{A}^{(1)}$ and an operation $d \in \boldsymbol{A}^{(2)}$ we put

$$f_1(x, y) = f(y, h_0(x), x, d(x, y)), \quad g_1(x, y) = g(y, h_0(x), x, d(x, y)) .$$

By the assumption of the Lemma we have the equations

$$f_1(x, x) = f(x, h_0(x), x, d(x, x)) = g(x, h_0(x), x, d(x, x)) = g_1(x, x)$$

and

$$f_1(x, h_0(x)) = f(h_0(x), h_0(x), x, d(x, h_0(x))) = g(h_0(x), h_0(x), x, d(x, h_0(x)))$$
$$= g_1(x, h_0(x)) .$$

Since the operation h_0 is non-trivial, we infer, in view of Lemma 4, that $f_1 = g_1$ and, consequently,

$$f(y, h_0(x), x, d(x, y)) = g(y, h_0(x), x, d(x, y)) \tag{22}$$

for all non-trivial operations $h_0 \in A^{(1)}$ and all operations $d \in A^{(2)}$.

Further, for arbitrary operations $h \in A^{(1)}$ and $d \in A^{(2)}$ we put

$$f_2(x, y) = f(y, h(y), x, d(x, y)), \qquad g_2(x, y) = g(y, h(y), x, d(x, y)).$$

By the assumption of the Lemma we have the equation

$$f_2(x, x) = f(x, h(x), x, d(x, x)) = g(x, h(x), x, d(x, x)) = g_2(x, x). \tag{23}$$

Since the class $A^{(1)}$ contains at least two non-trivial operations, we can find a non-trivial operation $h_1 \in A^{(1)}$ such that the composition $h(h_1(x))$ is also non-trivial. Setting $y = h_1(x)$ and $h_0(x) = h(h_1(x))$ into (22) we obtain the equation

$$\begin{aligned} f_2(x, h_1(x)) &= f(h_1(x), h(h_1(x)), x, d(x, h_1(x))) \\ &= g(h_1(x), h(h_1(x)), x, d(x, h_1(x))) = g_2(x, h_1(x)). \end{aligned}$$

Hence and from (23), in virtue of Lemma 4, we get the equation $f_2 = g_2$. Consequently,

$$f(y, h(y), x, d(x, y)) = g(y, h(y), x, d(x, y)) \tag{24}$$

whenever $h \in A^{(1)}$ and $d \in A^{(2)}$.

Let h_1, h_2, h_3 and h_4 be operations from $A^{(1)}$. By Lemma 1 all these operations are invertible. Replacing in (24) $h(y)$ by $h_2(h_1^{-1}(y))$, $d(x, y)$ by $h_4(h_1^{-1}(y))$, x by $h_3(x)$ and setting $y = h_1(x)$ we get the equation

$$f(h_1(x), h_2(x), h_3(x), h_4(x)) = g(h_1(x), h_2(x), h_3(x), h_4(x)). \tag{25}$$

Given operations $d_1, d_2, d_3, d_4 \in A^{(2)}$ we put

$$f_3(x, y) = f(d_1(x, y), d_2(x, y), d_3(x, y), d_4(x, y)),$$
$$g_3(x, y) = g(d_1(x, y), d_2(x, y), d_3(x, y), d_4(x, y)).$$

From (25) it follows that $f_3(x, h(x)) = g_3(x, h(x))$ for operations $h \in A^{(1)}$. Thus, by Lemma 4, $f_3 = g_3$ and, consequently,

$$f(d_1(x, y), d_2(x, y), d_3(x, y), d_4(x, y)) = g(d_1(x, y), d_2(x, y), d_3(x, y), d_4(x, y))$$

for all binary algebraic operations d_1, d_2, d_3 and d_4. Since the algebra in question is two-dimensional, the last equation implies $f = g$. The Lemma is thus proved.

LEMMA 7. *If $A^{(0)} = \emptyset$ and $A^{(1)}$ contains exactly one non-trivial operation i, then there is no binary algebraic operation depending on every variable, the operation i is an involution without fixed points and A is a four-element set. Moreover, if a and b are independent elements of A, then $A = \{a, b, i(a), i(b)\}$.*

Proof. Since there is no self-dependent element in A, we have the inequality $i(x) \neq x$ for all $x \in A$. Moreover, $i(i(x)) = x$ and, consequently, the operation i is an involution without fixed points.

Further, for any operation $f \in A^{(2)}$ one of the following four cases holds:

$$f(x, x) = x, \qquad f(x, i(x)) = x, \tag{26}$$
$$f(x, x) = x, \qquad f(x, i(x)) = i(x), \tag{27}$$
$$f(x, x) = i(x), \qquad f(x, i(x)) = i(x), \tag{28}$$
$$f(x, x) = i(x), \qquad f(x, i(x)) = x = i(i(x)). \tag{29}$$

By Lemma 4 we have the equations $f(x, y) = x$ in case (26), $f(x, y) = y$ in case (27), $f(x, y) = i(x)$ in case (28) and $f(x, y) = i(y)$ in case (29). Thus there is no binary algebraic operation depending on both variables.

Let a and b be independent elements of A. Since the algebra in question is two-dimensional, the elements a and b generate the whole set A. Consequently, $A = \{a, b, i(a), i(b)\}$. From the independence of a and b it follows that $i(a) \neq b$ and $i(b) \neq a$. Thus the set A has four elements, which completes the proof.

LEMMA 8. *Suppose that $A^{(0)} = \emptyset$ and $A^{(1)}$ contains exactly one non-trivial operation. If $f, g \in A^{(4)}$ and $f(u_1, u_2, u_3, u_4) = g(u_1, u_2, u_3, u_4)$ whenever $u_j = x$ or y $(j = 1, 2, 3, 4;\ x, y \in A)$, then $f = g$.*

Proof. Let f and g satisfy the assumption of Lemma 8 and let i be the only non-trivial algebraic unary operation. Put

$$f_1(x, y) = f(x, x, y, i(x)), \qquad g_1(x, y) = g(x, x, y, i(x)),$$
$$f_2(x, y) = f(x, x, y, i(y)), \qquad g_2(x, y) = g(x, x, y, i(y)),$$
$$f_3(x, y) = f(x, y, i(x), i(y)), \qquad g_3(x, y) = g(x, y, i(x), i(y)).$$

From the assumption of the Lemma and the relation $i(i(x)) = x$ it follows that $f_j(x, x) = g_j(x, x)$ and $f_j(x, i(x)) = g_j(x, i(x))$ $(j = 1, 2, 3)$. Thus, by Lemma 4, $f_j = g_j$ $(j = 1, 2, 3)$ and, consequently,

$$f(x, x, y, i(x)) = g(x, x, y, i(x)),$$
$$f(x, x, y, i(y)) = g(x, x, y, i(y)),$$
$$f(x, y, i(x), i(y)) = g(x, y, i(x), i(y)).$$

Since the assumption of the Lemma is invariant under the permutation of variables in the operations f and g, the last equations imply the equation

(30) $$f(a_1, a_2, a_3, a_4) = g(a_1, a_2, a_3, a_4)$$

whenever $\langle a_1, a_2, a_3, a_4\rangle$ is a permutation of one of the systems $\langle x, x, y, i(x)\rangle$, $\langle x, x, y, i(y)\rangle$ and $\langle x, y, i(x), i(y)\rangle$ $(x, y \in A)$. Hence, by Lemma 7, it follows that equation (30) holds whenever the system $\langle a_1, a_2, a_3, a_4\rangle$ contains at least three different elements. Since equation (30) is assumed in the opposite case, we have the equation $f = g$. The Lemma is thus proved.

If $1 \leqslant k \leqslant n$, then $A^{(n,k)}$ will denote the subclass of the class $A^{(n)}$ consisting of all operations depending on at most k variables. Further, we shall denote by $\tilde{A}^{(n)}$ and $\tilde{A}^{(n,k)}$ respectively the subclasses of $A^{(n)}$ and $A^{(n,k)}$ consisting of all idempotent operations, i.e. operations f satisfying the condition $f(x, x, \ldots, x) = x$.

LEMMA 9. *If $A^{(3)} \neq A^{(3,1)}$, then $\tilde{A}^{(3)} \neq \tilde{A}^{(3,1)}$.*

Proof. First let us suppose that $A^{(2)} \neq A^{(2,1)}$. Let $f \in A^{(2)} \setminus A^{(2,1)}$. Since the operation f depends on both variables, the operations f and $e_2^{(2)}$ treated as elements of the algebra $\mathfrak{A}^{(2)}$ are independent and, consequently, form a basis of $\mathfrak{A}^{(2)}$. Thus there exists an operation $g \in A^{(2)}$ such that

(31) $$x = g\big(f(x, y), y\big).$$

Hence we get the equation $f(x, y) = f\Big(g\big(f(x, y), y\big), y\Big)$. Taking into account the independence of f and $e_2^{(2)}$, we have the equation

(32) $$x = f\big(g(x, y), y\big).$$

Put $h(x, y, z) = f\big(g(x, y), z\big)$. From (31) we obtain the equation

$$h\big(f(x, y), y, z\big) = f\Big(g\big(f(x, y), y\big), z\Big) = f(x, z),$$

which shows that the operation $h(x, y, z)$ depends on the variables x and z. Moreover, by (32), $h(x, x, x) = x$. Thus $h \in \tilde{A}^{(3)} \setminus \tilde{A}^{(3,1)}$, which completes the proof in the case $A^{(2)} \neq A^{(2,1)}$.

Now suppose that there exists an operation $f \in A^{(3)} \setminus A^{(3,1)}$ for which the operation $g(x) = f(x, x, x)$ is not constant. Then, by Lemma 1, $g^{-1} \in A^{(1)}$ and, consequently, the operation $h(x, y, z) = g^{-1}\big(f(x, y, z)\big)$ belongs to $\tilde{A}^{(3)} \setminus \tilde{A}^{(3,1)}$.

Finally, suppose that $\tilde{A}^{(2)} = \tilde{A}^{(2,1)}$ and $f(x, x, x)$ is a constant operation for all operations $f \in A^{(3)} \setminus A^{(3,1)}$. Since there is no binary algebraic operation depending on every variable, we have the equations $f(x, x, y) = f_1(x)$ or $f_1(y)$ and $f(x, y, x) = f_2(x)$ or $f_2(y)$, where $f_1, f_2 \in A^{(1)}$. Setting $y = x$ into these equations, we obtain the formula $f_1(x) = f_2(x) = c$, where $c \in A^{(0)}$. Thus, $f(x, x, y) = f(x, y, x) = c$, which, by Lemma 6, proves that the operation f is constant. But this contradicts the assumption $f \in A^{(3)} \setminus A^{(3,1)}$. Consequently, the last case never holds, which completes the proof.

LEMMA 10. *Suppose that $A^{(0)} = \emptyset$, $A^{(3)} \neq A^{(3,1)}$ and the class $A^{(1)}$ contains exactly one non-trivial operation. Then either $\mathfrak{A}$ is the exceptional algebra $\mathfrak{E}$ or the class $A^{(3)}$ contains exactly one non-trivial operation s and $s(x, x, y) = s(x, y, x) = s(y, x, x) = y$.*

Proof. By Lemma 9 the inequality $\tilde{A}^{(3)} \neq \tilde{A}^{(3,1)}$ holds. Let us suppose that the class $\tilde{A}^{(3)}$ contains a non-trivial operation f satisfying the equation

(33) $$f(x, x, y) = x.$$

Now we shall prove that there exists a ternary algebraic operation q^* satisfying the condition

(34) $$q^*(x, x, y) = q^*(x, y, x) = q^*(y, x, x) = x.$$

Since, by Lemma 7, the class $\tilde{A}^{(2)}$ consists of trivial operations, we have the equations $f(x, y, x) = x$ or y and $f(y, x, x) = x$ or y. If either $f(x, y, x) = x$ and $f(y, x, x) = y$ or $f(x, y, x) = y$ and $f(y, x, x) = x$, then, according to (33), either $f(x, y, z) = x$ or $f(x, y, z) = y$ whenever at least two variables among x, y, z are equal. Hence and from Lemma 8 it follows that either $f = e_1^{(3)}$ or $f = e_2^{(3)}$, which contradicts the assumption $f \in \tilde{A}^{(3)} \setminus \tilde{A}^{(3,1)}$. Thus, we have either

(35) $$f(x, y, x) = x, \quad f(y, x, x) = x$$

or

(36) $$f(x, y, x) = y, \quad f(y, x, x) = y.$$

In case (35) we put $q^* = f$. Further, in case (36) setting $q^*(x, y, z) = f\big(x, y, f(x, y, z)\big)$, we have, by (33) and (36), the equations

$$q^*(x, x, y) = f\big(x, x, f(x, x, y)\big) = f(x, x, x) = x,$$
$$q^*(x, y, x) = f\big(x, y, f(x, y, x)\big) = f(x, y, y) = x,$$
$$q^*(y, x, x) = f\big(y, x, f(y, x, x)\big) = f(y, x, y) = x,$$

which completes the proof of existence of the operation q^* satisfying condition (34). We note that, by Lemma 8, the operation q^* is uniquely determined by condition (34). Moreover, since condition (34) is invariant under permutations of variables, the operation q^* is symmetrical.

Let i be the only non-trivial unary algebraic operation. Since, by (34), $q^*\big(x, x, i(x)\big) = x$, we have either $q^*\big(x, y, i(x)\big) = x$ or $q^*\big(x, y, i(x)\big) = y$.

In the first case the equation $i(i(x)) = x$ and the symmetry of q^* would imply the equation

$$i(x) = q^*\big(i(x), y, i(i(x))\big) = q^*\big(i(x), y, x\big) = q^*\big(x, y, i(x)\big) = x,$$

which is impossible. Thus the equation

$$q^*\big(x, y, i(x)\big) = y \tag{37}$$

holds.

For each algebraic operation $g \in \boldsymbol{A}^{(n)}$ we have either $g(x, x, \ldots, x) = x$ or $g(x, x, \ldots, x) = i(x)$. In the first case $g \in \tilde{\boldsymbol{A}}^{(n)}$. Setting $g_0(x_1, x_2, \ldots, x_n) = i\big(g(x_1, x_2, \ldots, x_n)\big)$, we have $g_0 \in \tilde{\boldsymbol{A}}^{(n)}$ in the second case. Moreover, $g(x_1, x_2, \ldots, x_n) = i\big(g_0(x_1, x_2, \ldots, x_n)\big)$. Consequently, denoting by $\boldsymbol{F}$ the class of all algebraic operations g satisfying the condition $g(x, x, \ldots, x) = x$, we have the equation $A = (E;\ \{i\} \cup \boldsymbol{F})$. By Lemma 7 E is a four-element set and all binary operations from $\boldsymbol{F}$ are trivial. Denote by 0 and 1 a pair of elements of E and put $\mathfrak{A}_0 = (0, 1;\ \boldsymbol{F})$. By Lemma 8 for any pair $f, g \in \boldsymbol{F}$ the equation $f = g$ holds in the algebra $\mathfrak{A}$ if and only if it holds in the algebra $\mathfrak{A}_0$. Consequently,

$$\mathfrak{A} = (E;\ \{i\} \cup \boldsymbol{F}_0) \quad \text{if} \quad \boldsymbol{F}_0 \subset \boldsymbol{F} \text{ and } \mathfrak{A}_0 = (0, 1;\ \boldsymbol{F}_0). \tag{38}$$

Setting

$$q(x, y, z) = q^*\big(x, y, i(z)\big), \tag{39}$$

we have, by (37), $q \in \boldsymbol{F}$ and, by (34) and (37),

$$q(x, x, y) = x, \quad q(x, y, x) = q(y, x, x) = y,$$

which shows that the operation q coincides with the Post operation p in the algebra $\mathfrak{A}_0$ (see [3], p. 200, formula (6)). Consequently, $(0, 1;\ p)$ is a subsystem of the algebra $\mathfrak{A}_0$. Since all binary operations from $\boldsymbol{F}$ are trivial, the elements 0 and 1 are independent in the algebra $\mathfrak{A}_0$. Thus, by the representation theorem for two-element algebras in which all elements are independent ([3], p. 203), we have the equation $\mathfrak{A}_0 = (0, 1;\ p) = (0, 1;\ q)$. Hence and from (38) the equation $\mathfrak{A} = (E;\ i, q)$ follows. Since, by formula (39), the operation q is a composition of the operations i and q^*, the equation $\mathfrak{A} = (E;\ i, q^*)$ is true. Hence and from (34) and (37) it follows that the algebra A is exceptional, which completes the proof in the case of the existence of an operation f satisfying condition (33).

In the opposite case, by Lemma 7, the operation $f(x, x, y)$, being trivial, is equal to y whenever $f \in \tilde{\boldsymbol{A}}^{(3)} \setminus \tilde{\boldsymbol{A}}^{(3,1)}$. Let s be a non-trivial operation from $\tilde{\boldsymbol{A}}^{(3)}$. Then

$$s(x, x, y) = s(x, y, x) = s(y, x, x) = y,$$

which implies, by Lemma 8, that the class $\tilde{\boldsymbol{A}}^{(3)}$ contains exactly one non-trivial operation s. The Lemma is thus proved.

LEMMA 11. *Suppose that* $\boldsymbol{A}^{(0)} \neq \emptyset$, $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, $\mathfrak{A} \neq \mathfrak{E}$ *and the class* $\boldsymbol{A}^{(1)}$ *contains exactly one non-trivial operation. If* $f, g \in \boldsymbol{A}^{(4)}$ *and* $f(x_1, x_2, x_3, x_4) = g(x_1, x_2, x_3, x_4)$ *whenever* $x_1 = x_2$ *or* $x_1 = x_3$, *then* $f = g$.

Proof. Since either $f(x, x, x, x) = g(x, x, x, x) = x$ or $f(x, x, x, x) = g(x, x, x, x) = i(x)$, where, by Lemma 7, the operation i is an involution without fixed points, to prove the Lemma it suffices to consider the case of operations f and g from $\tilde{\boldsymbol{A}}^{(4)}$. Put

$$f_1(x, y, z) = f(x, y, z, x), \qquad g_1(x, y, z) = g(x, y, z, x),$$

$$f_2(x, y, z) = f(x, y, z, z), \qquad g_2(x, y, z) = g(x, y, z, z).$$

By the assumption we have the equations

$$f_j(x, x, y) = g_j(x, x, y), \quad f_j(x, y, x) = g_j(x, y, x) \quad (j = 1, 2). \tag{40}$$

Moreover, $f_j, g_j \in \tilde{\boldsymbol{A}}^{(3)}$ $(j = 1, 2)$. Suppose that at least one of the operations f_j, g_j is non-trivial. Without loss of generality we may assume that $f_j = s$, where, by Lemma 10, the operation s is the only non-trivial operation from $\tilde{\boldsymbol{A}}^{(3)}$. Moreover, by Lemma 10, $f_j(x, x, y) = f_j(x, y, x) = y$. Consequently, by (40), $g_j(x, x, y) = g_j(x, y, x) = y$. Hence it follows that the operation g_j is non-trivial, and consequently, equal to s. Thus $f_j = g_j$ whenever at least one of these operations os non-trivial.

Suppose now that both operations f_j and g_j are trivial, i.e. $f_j = e_k^{(3)}$ and $g_j = e_r^{(3)}$, where $1 \leqslant k \leqslant 3$ and $1 \leqslant r \leqslant 3$. From (40) we get the equations

$$e_k^{(3)}(x, x, y) = e_r^{(3)}(x, x, y) \tag{41}$$

and

$$e_k^{(3)}(x, y, x) = e_r^{(3)}(x, y, x). \tag{42}$$

Equation (41) holds if and only if either $k = r = 3$ or $1 \leqslant k \leqslant 2$ and $1 \leqslant r \leqslant 2$. Equation (42) holds if and only if either $k = r = 2$ or $k \neq 2$ and $r \neq 2$. Consequently, equations (41) and (42) hold if and only if $k = r$. Hence we get the equation $f_j = g_j$ whenever both operations f_j and g_j are trivial. Thus

$$f(x, y, z, x) = g(x, y, z, x) \quad \text{and} \quad f(x, y, z, z) = g(x, y, z, z).$$

Hence and from the assumption of the Lemma it follows that $f(u_1, u_2, u_3, u_4) = g(u_1, u_2, u_3, u_4)$ whenever $u_j = x$ or y $(j = 1, 2, 3, 4;\ x, y \in A)$. The Lemma is now a consequence of the Lemma 8.

LEMMA 12. *Suppose that either* $A^{(0)} \neq \emptyset$ *or* $A^{(0)} = \emptyset$ *and the class* $A^{(1)}$ *contains at least two non-trivial operations. If* $A^{(3)} \neq A^{(3,1)}$, *then there exists an operation* $s \in \tilde{A}^{(3)}$ *such that*

$$s(y, x, x) = s(x, y, x) = y\,.$$

Proof. If $A^{(3)} \neq A^{(3,1)}$, then, by Lemma 9, we have the inequality $\tilde{A}^{(3)} \neq \tilde{A}^{(3,1)}$.

First consider the case $\tilde{A}^{(2)} \neq \tilde{A}^{(2,1)}$. Let $f \in \tilde{A}^{(2)} \setminus \tilde{A}^{(2,1)}$. Of course, the operations f and $e_2^{(2)}$ treated as elements of the algebra $\mathfrak{A}^{(2)}$ are independent and, consequently, form a basis of $\mathfrak{A}^{(2)}$. Thus there exists an operation $g_1 \in A^{(2)}$ such that

$$x = g_1(y, f(x, y))\,. \tag{43}$$

Hence $f(x, y) = f(g_1(y, f(x, y)), y)$ and, by the independence of f and $e_2^{(2)}$,

$$x = f(g_1(y, x), y)\,. \tag{44}$$

Moreover, from (43) we obtain the equation

$$x = g_1(x, f(x, x)) = g_1(x, x)\,. \tag{45}$$

Further, taking into account the independence of the operations f and $e_1^{(2)}$ we can prove in the same way the existence of an operation $g_2 \in A^{(2)}$ such that

$$y = g_2(x, f(x, y))\,. \tag{46}$$

Hence $f(x, y) = f(x, g_2(x, f(x, y)))$ and, by the independence of f and $e_1^{(2)}$,

$$y = f(x, g_2(x, y))\,. \tag{47}$$

Moreover, by (46),

$$x = g_2(x, f(x, x)) = g_2(x, x)\,. \tag{48}$$

Setting $s(x, y, z) = f(g_1(z, x), g_2(z, y))$, we have, according to (44), (45), (47) and (48), the equations

$$s(y, x, x) = f(g_1(x, y), g_2(x, x)) = f(g_1(x, y), x) = y\,,$$
$$s(x, y, x) = f(g_1(x, x), g_2(x, y)) = f(x, g_2(x, y)) = y\,,$$

which completes the proof in the case $\tilde{A}^{(2)} \neq \tilde{A}^{(2,1)}$.

Suppose now that $\tilde{A}^{(2)} = \tilde{A}^{(2,1)}$. If for all operations $f \in \tilde{A}^{(3)} \setminus \tilde{A}^{(3,1)}$ the equation $f(x, x, y) = y$ holds, then, of course, $f(y, x, x) = f(x, y, x) = y$ and, consequently, each operation from $\tilde{A}^{(3)} \setminus \tilde{A}^{(3,1)}$ satisfies the assertion of the Lemma.

Finally, let us assume that there exists an operation $s \in \tilde{A}^{(3)} \setminus \tilde{A}^{(3,1)}$ for which $s(x, x, y) \neq y$. Since $\tilde{A}^{(2)} = \tilde{A}^{(2,1)}$, we have the equation

$$s(x, x, y) = x\,. \tag{49}$$

If either

$$s(y, x, x) = x \tag{50}$$

or

$$s(x, y, x) = x\,. \tag{51}$$

then $s(x_1, x_2, x_3) = x_2$ in the case (50) whenever $x_2 = x_1$ or $x_2 = x_3$ and $s(x_1, x_2, x_3) = x_1$ in the case (51) whenever $x_1 = x_2$ or $x_1 = x_3$. Hence and from Lemma 6 it follows that $s = e_2^{(3)}$ in the case (50) and $s = e_1^{(3)}$ in the case (51). But this contradicts the assumption $s \in \tilde{A}^{(3)} \setminus \tilde{A}^{(3,1)}$. Thus $s(y, x, x) = s(x, y, x) = y$, which completes the proof.

LEMMA 13. *Suppose that the algebra* $\mathfrak{A}$ *is not exceptional. Then for every operation* $s \in \tilde{A}^{(3)}$ *satisfying the condition*

$$s(y, x, x) = s(x, y, x) = y \tag{52}$$

the following equations hold:

$$s(x_1, x_2, x_3) = s(x_2, x_1, x_3)\,, \tag{53}$$

$$f(s(x_1, x_2, x_3), x_3) = s(f(x_1, x_3), f(x_2, x_3), x_3) \quad \textit{for any} \quad f \in A^{(2)}, \tag{54}$$

$$f(x_1, x_2, x_3) = s(f(x_1, x_1, x_3), f(x_1, x_2, x_1), x_1) \quad \textit{for any} \quad f \in \tilde{A}^{(3)} \tag{55}$$

and

$$s(s(x_1, x_2, x_3), x_4, x_3) = s(x_1, s(x_2, x_4, x_3), x_3)\,. \tag{56}$$

Proof. From formula (52) it follows that equation (53) holds whenever $x_3 = x_1$ or $x_3 = x_2$. Thus, by Lemmas 6 and 11, it holds for all $x_1, x_2, x_3 \in A$. Further, by (52), for any operation $f \in \tilde{A}^{(2)}$ we have the equations

$$f(s(x_1, x_2, x_1), x_1) = f(x_2, x_1)\,,$$
$$s(f(x_1, x_1), f(x_2, x_1), x_1) = f(x_2, x_1)\,,$$
$$f(s(x_1, x_2, x_2), x_2) = f(x_1, x_2)\,,$$
$$s(f(x_1, x_2), f(x_2, x_2), x_2) = f(x_1, x_2)\,,$$

which show that (54) holds whenever $x_3 = x_1$ or $x_3 = x_2$. Hence, by Lemmas 6 and 11, it holds for all $x_1, x_2, x_3 \in A$.

Taking into account formula (52) for any operation $f \in \tilde{A}^{(3)}$ we have the equations

$$f(x_2, x_2, x_3) = s(f(x_2, x_2, x_3), f(x_2, x_2, x_2), x_2)\,,$$
$$f(x_3, x_2, x_3) = s(f(x_3, x_3, x_3), f(x_3, x_2, x_3), x_3)\,,$$

which show that (55) holds whenever $x_1 = x_2$ or $x_1 = x_3$. Hence, by Lemmas 6 and 11, it follows that it holds for all $x_1, x_2, x_3 \in A$.

Finally, from the equations

$$s\big(s(x_1, x_2, x_2), x_4, x_2\big) = s(x_1, x_4, x_2),$$
$$s\big(x_1, s(x_2, x_4, x_2), x_2\big) = s(x_1, x_4, x_2),$$
$$s\big(s(x_1, x_2, x_4), x_4, x_4\big) = s(x_1, x_2, x_4),$$
$$s\big(x_1, s(x_2, x_4, x_4), x_4\big) = s(x_1, x_2, x_4)$$

it follows that (56) holds whenever $x_3 = x_2$ or $x_3 = x_4$, which, by Lemmas 6 and 11, implies equation (56) for all $x_1, x_2, x_3, x_4 \in A$. The Lemma is thus proved.

In the sequel we shall denote by $\mathcal{K}$ the class $\tilde{A}^{(2)}$. Elements of $\mathcal{K}$ will be denoted by small Greek letters: $\lambda, \mu, \nu, \ldots$

LEMMA 14. *Suppose that $\mathfrak{A}$ is not the exceptional algebra. If $A^{(3)} \neq A^{(3,1)}$, then $\mathcal{K}$ is a field with respect to the operations*

$$(\lambda+\mu)(x, y) = s\big(\lambda(x, y), \mu(x, y), y\big), \tag{57}$$

$$(\lambda\cdot\mu)(x, y) = \lambda\big(\mu(x, y), y\big), \tag{58}$$

where s is a ternary algebraic operation satisfying the condition $s(y, x, x) = s(x, y, x) = y$.

Proof. First of all we note that the existence of an operation s follows from Lemmas 10 and 12.

We define the zero-element and the unit element by the following formulas: $0(x, y) = y$, $1(x, y) = x$. Obviously, $0 \neq 1$ and for every $\lambda \in \mathcal{K}$

$$(\lambda+0)(x, y) = s\big(\lambda(x, y), y, y\big) = \lambda(x, y),$$
$$(\lambda\cdot 1)(x, y) = \lambda(x, y) = (1\cdot\lambda)(x, y),$$

which implies $\lambda+0 = \lambda$ and $\lambda\cdot 1 = 1\cdot\lambda = \lambda$.

The formula $\lambda\cdot(\mu\cdot\nu) = (\lambda\cdot\mu)\cdot\nu$ $(\lambda, \mu, \nu \in \mathcal{K})$ is a direct consequence of (58).

Given $\lambda \in \mathcal{K}$, we put $(-\lambda)(x, y) = s\big(y, y, \lambda(x, y)\big)$. Setting $f = s$ into (55) and taking into account (53), we get the formula

$$s(x_1, x_2, x_3) = s\big(s(x_1, x_1, x_3), x_2, x_1\big) = s\big(x_2, s(x_1, x_1, x_3), x_1\big).$$

Hence the equation

$$\big(\lambda+(-\lambda)\big)(x, y) = s\Big(\lambda(x, y), s\big(y, y, \lambda(x, y)\big), y\Big)$$
$$= s\big(y, \lambda(x, y), \lambda(x, y)\big) = y = 0(x, y)$$

follows. Thus $\lambda+(-\lambda) = 0$.

Let $\lambda \neq 0$, i.e. let $\lambda(x, y)$ depend on the variable x. Then the operations $\lambda(x, y)$ and $0(x, y)$ treated as elements of the algebra $\mathfrak{A}^{(2)}$ are independent and, consequently, form a basis of $\mathfrak{A}^{(2)}$. Thus there is an operation $\lambda^{-1} \in A^{(2)}$ such that

$$x = \lambda^{-1}\big(\lambda(x, y), y\big). \tag{59}$$

Setting $y = x$ into the last equation we obtain the formula $x = \lambda^{-1}(x, x)$, which shows that $\lambda^{-1} \in \mathcal{K}$. Moreover, from (59) we get the equation $\lambda(x, y) = \lambda\Big(\lambda^{-1}\big(\lambda(x, y), y\big), y\Big)$, which, by the independence of $\lambda(x, y)$ and $0(x, y)$ implies

$$x = \lambda\big(\lambda^{-1}(x, y), y\big).$$

This equation and (59) can be written in the form $\lambda^{-1}\cdot\lambda = \lambda\cdot\lambda^{-1} = 1$.

Taking into account assertions (53), (54) and (56) of Lemma 13, we have the equations

$$(\lambda+\mu)(x, y) = s\big(\lambda(x, y), \mu(x, y), y\big) = s\big(\mu(x, y), \lambda(x, y), y\big) = (\mu+\lambda)(x, y).$$

$$\big((\lambda+\mu)+\nu\big)(x, y) = s\Big(s\big(\lambda(x, y), \mu(x, y), y\big), \nu(x, y), y\Big)$$
$$= s\Big(\lambda(x, y), s\big(\mu(x, y), \nu(x, y), y\big), y\Big) = \big(\lambda+(\mu+\nu)\big)(x, y),$$

$$\big(\lambda\cdot(\mu+\nu)\big)(x, y) = \lambda\Big(s\big(\mu(x, y), \nu(x, y), y\big), y\Big)$$
$$= s\Big(\lambda\big(\mu(x, y), y\big), \lambda\big(\nu(x, y), y\big), y\Big) = (\lambda\cdot\mu+\lambda\cdot\nu)(x, y),$$

which imply

$$\lambda+\mu = \mu+\lambda, \quad (\lambda+\mu)+\nu = \lambda+(\mu+\nu)$$

and

$$\lambda\cdot(\mu+\nu) = \lambda\cdot\mu+\lambda\cdot\nu \quad \text{for every} \quad \lambda, \mu, \nu \in \mathcal{K}.$$

Finally, the following equation is a direct consequence of the definitions (57) and (58)

$$\big((\mu+\nu)\cdot\lambda\big)(x, y) = s\Big(\mu\big(\lambda(x, y), y\big), \nu\big(\lambda(x, y), y\big), y\Big) = (\mu\cdot\lambda+\nu\cdot\lambda)(x, y).$$

Thus $(\mu+\nu)\cdot\lambda = \mu\cdot\lambda+\nu\cdot\lambda$ for every $\lambda, \mu, \nu \in \mathcal{K}$, which completes the proof.

LEMMA 15. *If the algebra $\mathfrak{A}$ is not exceptional and $A^{(3)} \neq A^{(3,1)}$, then A is a linear space over $\mathcal{K}$ with respect to the operations*

$$x+y = s(x, y, \Theta) \quad (x, y \in A),$$
$$\lambda\cdot x = \lambda(x, \Theta) \quad (\lambda \in \mathcal{K},\ x \in A),$$

where Θ is an element of $A^{(0)}$ if $A^{(0)} \neq \emptyset$ and is an arbitrary element of A if $A^{(0)} = \emptyset$ and s is a ternary algebraic operation satisfying the condition $s(y, x, x) = s(x, y, x) = y$.

Proof. The element Θ is the zero-element of A. In fact, $x+\Theta = s(x,\Theta,\Theta) = x$ for every $x \in A$. Further, we have, in virtue of Lemma 13, the equations

$$x+y = s(x,y,\Theta) = s(y,x,\Theta) = y+x,$$
$$(x+y)+z = s\big(s(x,y,\Theta),z,\Theta\big) = s\big(x,s(y,z,\Theta),\Theta\big) = x+(y+z),$$
$$\lambda\cdot(x+y) = \lambda\big(s(x,y,\Theta),\Theta\big) = s\big(\lambda(x,\Theta),\lambda(y,\Theta),\Theta\big) = \lambda\cdot x+\lambda\cdot y$$

for any $x,y,z \in A$ and $\lambda \in \mathcal{K}$. Moreover, we have the equations

$$\lambda\cdot(\mu\cdot x) = \lambda\big(\mu(x,\Theta),\Theta\big) = (\lambda\cdot\mu)\cdot x,$$
$$1\cdot x = x,$$
$$(\lambda+\mu)\cdot x = s\big(\lambda(x,\Theta),\mu(x,\Theta),\Theta\big) = \lambda\cdot x+\mu\cdot x$$

for any $x \in A$ and $\lambda,\mu \in \mathcal{K}$. Hence, setting $-x = (-1)\cdot x$, we get the equation $x+(-x) = 0\cdot x = \Theta$. The Lemma is thus proved.

LEMMA 16. *If the algebra $\mathfrak{A}$ is not exceptional and $A^{(3)} \neq A^{(3,1)}$, then the class $\bar{A}^{(3)}$ consists of all operations of the form*

$$g(x_1,x_2,x_3) = \lambda_1x_1+\lambda_2x_2+\lambda_3x_3,$$

where $\lambda_1,\lambda_2,\lambda_3 \in \mathcal{K}$ and $\lambda_1+\lambda_2+\lambda_3 = 1$.

Proof. First we shall prove the formulas

(60) $$\lambda(y,x) = (1-\lambda)(x,y),$$

(61) $$\lambda(x,y) = \lambda\cdot x+(1-\lambda)\cdot y$$

for any operation $\lambda \in \mathcal{K}$. Setting $f(x_1,x_2,x_3) = \lambda(x_2,x_3)$ into formula (55) of the Lemma 13, we infer that

(62) $$\lambda(x_2,x_3) = s\big(\lambda(x_1,x_3),\lambda(x_2,x_1),x_1\big).$$

Replacing in this formula x_2 and x_3 by x, x_1 by y we obtain the equation

$$x = s\big(\lambda(y,x),\lambda(x,y),y\big).$$

Hence, according to the definition of the unit element and addition in $\mathcal{K}$, we get equation (60). Further, setting $x_1 = \Theta$ into (62) and replacing x_2 by x and x_3 by y, we infer that

$$\lambda(x,y) = s\big(\lambda(\Theta,y),\lambda(x,\Theta),\Theta\big) = s\big(\lambda(x,\Theta),(1-\lambda)(y,\Theta),\Theta\big)$$
$$= \lambda\cdot x+(1-\lambda)\cdot y,$$

which completes the proof of (61).

Given $\lambda_1,\lambda_2,\lambda_3 \in \mathcal{K}$ with $\lambda_1+\lambda_2+\lambda_3 = 1$, we put

$$h_1(x_1,x_2,x_3) = s\big(\lambda_1(x_1,x_2),\lambda_3(x_3,x_2),x_2\big),$$
$$h_2(x_1,x_2,x_3) = s\big(\lambda_2(x_2,x_1),\lambda_3(x_3,x_1),x_1\big).$$

Of course,

$$h_1(x_2,x_2,x_3) = \lambda_3(x_3,x_2) = h_2(x_2,x_2,x_3)$$

and, by (60),

$$h_1(x_3,x_2,x_3) = s\big(\lambda_1(x_3,x_2),\lambda_3(x_3,x_2),x_2\big) = (\lambda_1+\lambda_3)(x_3,x_2)$$
$$= (1-\lambda_2)(x_3,x_2) = \lambda_2(x_2,x_3),$$
$$h_2(x_3,x_2,x_3) = s\big(\lambda_2(x_2,x_3),\lambda_3(x_3,x_3),x_3\big) = s\big(\lambda_2(x_2,x_3),x_3,x_3\big)$$
$$= \lambda_2(x_2,x_3).$$

Consequently, $h_1(x_1,x_2,x_3) = h_2(x_1,x_2,x_3)$ whenever $x_1 = x_2$ or $x_1 = x_3$, which, by Lemmas 6 and 11, implies the equation $h_1 = h_2$. Thus

(63) $$s\big(\lambda_1(x_1,x_2),\lambda_3(x_3,x_2),x_2\big) = s\big(\lambda_2(x_2,x_1),\lambda_3(x_3,x_1),x_1\big).$$

Further, put

(64) $$h(x_1,x_2,x_3,x_4) = s\Big(\lambda_1(x_1,x_4),s\big(\lambda_2(x_2,x_4),\lambda_3(x_3,x_4),x_4\big),x_4\Big).$$

Obviously, the operation h is algebraic. Moreover, by (56),

$$h(x_1,x_2,x_3,x_1) = s\Big(x_1,s\big(\lambda_2(x_2,x_1),\lambda_3(x_3,x_1),x_1\big),x_1\Big)$$
$$= s\Big(s\big(x_1,\lambda_2(x_2,x_1),x_1\big),\lambda_3(x_3,x_1),x_1\Big)$$
$$= s\big(\lambda_2(x_2,x_1),\lambda_3(x_3,x_1),x_1\big)$$

and

$$h(x_1,x_2,x_3,x_2) = s\Big(\lambda_1(x_1,x_2),s\big(x_2,\lambda_3(x_3,x_2),x_2\big),x_2\Big)$$
$$= s\Big(s\big(\lambda_1(x_1,x_2),x_2,x_2\big),\lambda_3(x_3,x_2),x_2\Big)$$
$$= s\big(\lambda_1(x_1,x_2),\lambda_3(x_3,x_2),x_2\big).$$

Hence and from (63) we get the equation

$$h(x_1,x_2,x_3,x_1) = h(x_1,x_2,x_3,x_2).$$

Thus the equation $h(x_1,x_2,x_3,x_4) = h(x_1,x_2,x_3,x_2)$ holds whenever $x_4 = x_1$ or $x_4 = x_2$, which, by Lemmas 6 and 11, implies that the operations $h(x_1,x_2,x_3,x_4)$ and $h(x_1,x_2,x_3,x_2)$ are identical. Consequently, the operation $h(x_1,x_2,x_3,x_4)$ does not depend on the variable x_4. Thus, by (64) and Lemma 15,

$$h(x_1,x_2,x_3,x_4) = s\Big(\lambda_1(x_1,\Theta),s\big(\lambda_2(x_2,\Theta),\lambda_3(x_3,\Theta),\Theta\big),\Theta\Big) = \lambda_1x_1+\lambda_2x_2+\lambda_3x_3,$$

which shows that the operation $\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3$ is algebraic. Since $\lambda_1 x+\lambda_2 x+\lambda_3 x=(\lambda_1+\lambda_2+\lambda_3)x=x$, it belongs to $\tilde{A}^{(3)}$.

Given an operation $g\in\tilde{A}^{(3)}$, we put

$$\lambda_1(x,y)=g(x,y,y),\qquad \lambda_2(x,y)=g(y,x,y),\qquad \lambda_3=1-\lambda_1-\lambda_2 \tag{65}$$

and

$$g_0(x_1,x_2,x_3)=\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3. \tag{66}$$

From the preceding reasoning it follows that the operation g_0 is algebraic. Moreover, by (61), (65) and (66),

$$g_0(x_1,x_2,x_1)=(1-\lambda_2)x_1+\lambda_2 x_2=\lambda_2(x_2,x_1)=g(x_1,x_2,x_1),$$
$$g_0(x_1,x_2,x_2)=\lambda_1 x_1+(1-\lambda_1)x_2=\lambda_1(x_1,x_2)=g(x_1,x_2,x_2).$$

Consequently, the equation $g(x_1,x_2,x_3)=g_0(x_1,x_2,x_3)$ holds whenever $x_3=x_1$ or $x_3=x_2$. Hence, by Lemmas 6 and 11, we get the equation $g=g_0$, which, in view of (66), completes the proof.

LEMMA 17. *Suppose that $\mathfrak{A}$ is not the exceptional algebra and $A^{(3)}\neq A^{(3,1)}$. Then there is a linear subspace A_0 of A such that the class $A^{(3)}$ consists of all operations of the form*

$$g(x_1,x_2,x_3)=\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3+a, \tag{67}$$

where $a\in A_0$, $\lambda_1,\lambda_2,\lambda_3$ are arbitrary elements of $\mathcal{K}$ if $A^{(0)}\neq\emptyset$ and $\lambda_1+\lambda_2+\lambda_3=1$ if $A^{(0)}=\emptyset$.

Proof. Put

$$A_0=\{f(\Theta):\ f\in A^{(1)}\}.$$

The set A_0 is a linear subspace of A. In fact, consider an arbitrary pair f_1,f_2 of operations from $A^{(1)}$ and an arbitrary pair λ_1,λ_2 of elements of $\mathcal{K}$. By Lemma 16 the operation

$$h(x_1,x_2,x_3)=\lambda_1 x_1+\lambda_2 x_2+(1-\lambda_1-\lambda_2)x_3$$

belongs to $\tilde{A}^{(3)}$. Consequently, the operation

$$f_3(x)=h\big(f_1(x),f_2(x),x\big)$$

belongs to $A^{(1)}$. Since $f_3(\Theta)=\lambda_1 f_1(\Theta)+\lambda_2 f_2(\Theta)$, the set A_0 is a linear subspace of A.

By Lemma 16 the operation h_0 defined by the formula

$$h_0(x_1,x_2,x_3)=x_1-x_2+x_3 \tag{68}$$

belongs to $\tilde{A}^{(3)}$. Given $f\in A^{(1)}$, we put

$$\lambda(x_1,x_2)=h_0\big(f(x_1),f(x_2),x_2\big)=f(x_1)-f(x_2)+x_2. \tag{69}$$

Obviously, $\lambda(x,x)=x$ and, consequently, $\lambda\in\mathcal{K}$. By the definition of scalar-multiplication in A we have $\lambda(x,\Theta)=\lambda\cdot x$. On the other hand, from (69) we get the equation

$$\lambda(x,\Theta)=f(x)-f(\Theta).$$

Thus $f(x)=\lambda\cdot x+f(\Theta)$. Consider the case $A^{(0)}=\emptyset$. If $\lambda\neq 1$, then, by Lemma 16, the operation

$$f_0(x_1,x_2)=(1-\lambda)^{-1}x_1-\lambda(1-\lambda)^{-1}x_2$$

is algebraic. Thus the composition $f_0\big(f(x),x\big)$ is an algebraic operation. But this composition is equal to $(1-\lambda)^{-1}f(\Theta)$, which contradicts the assumption $A^{(0)}=\emptyset$. Consequently, if $A^{(0)}=\emptyset$, then each unary algebraic operation f satisfies the equation $f(x)=x+f(\Theta)$.

Let $g\in A^{(3)}$ and h_0 be defined by formula (68). Setting $f_1(x)=g(x,x,x)$ and

$$g_1(x_1,x_2,x_3)=h_0\big(g(x_1,x_2,x_3),f_1(x_1),x_3\big)=g(x_1,x_2,x_3)-f_1(x_1)+x_3, \tag{70}$$

we infer that the operation g_1 is algebraic. Moreover, $g_1(x,x,x)=x$ and, consequently, $g_1\in\tilde{A}^{(3)}$. By Lemma 16 there are elements $\lambda_1,\lambda_2,\lambda_3\in\mathcal{K}$ such that $\lambda_1+\lambda_2+\lambda_3=1$ and

$$g_1(x_1,x_2,x_3)=\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3. \tag{71}$$

Since $f_1(x)=\lambda\cdot x+f_1(\Theta)$, where $\lambda\in\mathcal{K}$ and $\lambda=1$ if $A^{(0)}=\emptyset$, we have, by virtue of (70) and (71), the equation

$$g(x_1,x_2,x_3)=(\lambda_1+\lambda)x_1+\lambda_2 x_2+(\lambda_3-1)x_3+f_1(\Theta).$$

Moreover, in the case $A^{(0)}=\emptyset$ the sum of coefficients is equal to 1. Thus each ternary algebraic operation is of the form described by the assertion of the Lemma.

If $A^{(0)}\neq\emptyset$, then, by virtue of the relation $\Theta\in A^{(0)}$, we have the equation $A_0=A^{(0)}$. Moreover, the addition and the scalar-multiplication in A are, by definition, algebraic operations. Hence it follows that each operation (67) is algebraic.

Suppose that $A^{(0)}=\emptyset$. Let $f\in A^{(1)}$, $\lambda_1+\lambda_2+\lambda_3=1$ $(\lambda_1,\lambda_2,\lambda_3\in\mathcal{K})$. By Lemma 16 the operation

$$g_0(x_1,x_2,x_3)=\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3$$

is algebraic. Moreover, $f(x)=x+f(\Theta)$. Thus the composition

$$f\big(g_0(x_1,x_2,x_3)\big)=\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3+f(\Theta)$$

is algebraic. The Lemma is thus proved.

Proof of the Theorem. Suppose that $A^{(3)} = A^{(3,1)}$. Applying Narkiewicz's theorem ([4], p. 338, Theorem II for $n = 2$) to the algebra $\mathfrak{A}$ we infer that there exist a group $\mathfrak{G}$ of transformations of the set A and a subset $A_0 \subset A$ containing all fixed points of the transformations that are not the identical and invariant under all transformations from $\mathfrak{G}$ such that $A^{(3)}$ consists of all operations defined as

$$f(x_1, x_2, x_3) = g(x_j) \qquad (j = 1, 2, 3),$$

$$f(x_1, x_2, x_3) = a,$$

where $g \in \mathfrak{G}$ and $a \in A_0$.

If $A^{(3)} \neq A^{(3,1)}$ and the algebra $\mathfrak{A}$ is not exceptional, then the class $A^{(3)}$ is completely described by Lemma 17. Hence it follows that if $\mathfrak{A} \neq \mathfrak{C}$, then the algebra $\mathfrak{A}_0^{(3)} = (A^{(3)}, A^{(3)})$ is a three-dimensional v^*-algebra. Since the algebras $\mathfrak{A}^{(3)}$ and $\mathfrak{A}_0^{(3)}$ have identical ternary algebraic operations, the algebra $\mathfrak{A}^{(3)}$ is also a three-dimensional v^*-algebra (see [4], p. 338). Now our theorem is a direct consequence of the representation theorem for v^*-algebras of dimension $\geqslant 3$ (see [5]), because $\mathfrak{A}$ is isomorphic to $\mathfrak{A}^{(2)}$ and $\mathfrak{A}^{(2)}$ is a subalgebra of $\mathfrak{A}^{(3)}$.

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Reçu par la Rédaction le 7. 12. 1964